Homotopy Theory in Type Theory

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Joint work with Eric Finster, Kuen-Bang Hou (Favonia), Michael Shulman

Homotopy Theory

A branch of topology,
the study of spaces and continuous deformations

[image from wikipedia]

Homotopy

Deformation of one path into another

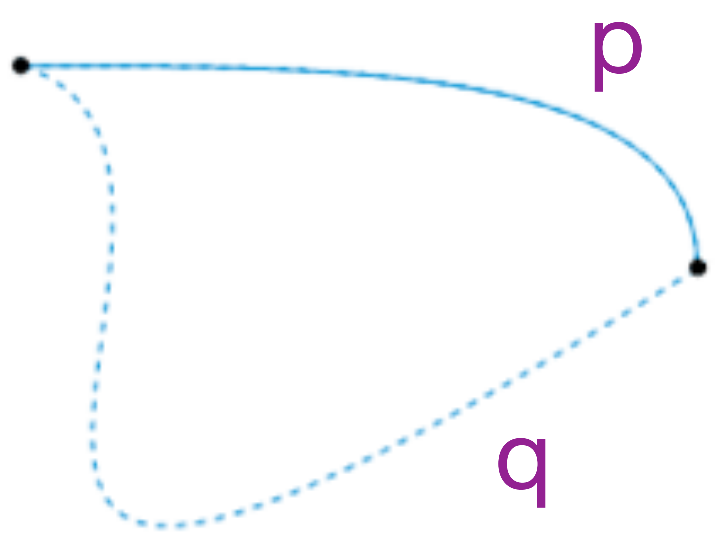
p

q

[image from wikipedia]

Homotopy

Deformation of one path into another



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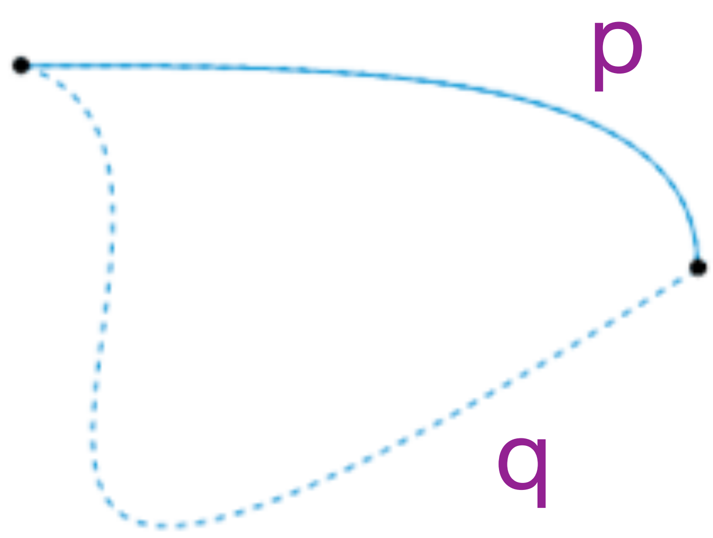

= 2-dimensional *path between paths*

[image from wikipedia]

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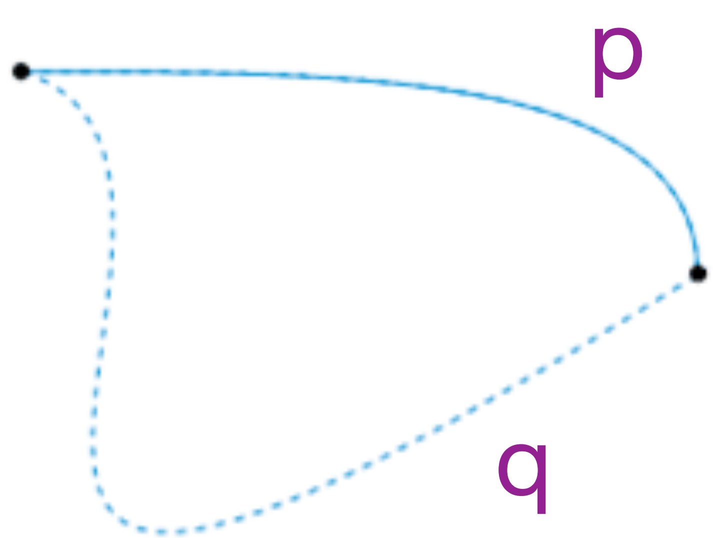

= 2-dimensional *path between paths*

Homotopy theory is the study of spaces by way of their paths, homotopies, homotopies between homotopies, ….

[image from wikipedia]

Homotopy groups

k[th] homotopy group

n-dimensional sphere

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

[image from wikipedia]

Type Theory

An alternative to set theory, organized around *types:*

- Basic data types ($\mathbb{N}$, $\mathbb{Z}$, booleans, lists, …)
- Functions

```
double : ℕ → ℕ
double 0 = 0
double (n +1) = double n + 2
```

- Unifies sets and logic

Propositions as Types

1. A proposition is represented by a type
2. A proof is represented by an element of that type

$$\forall x : \mathbb{N}.\ \mathrm{double}(x) = 2 * x$$

***type* of proofs of equality**

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proof by case analysis represented by a function defined by cases

Type are sets?

Traditional view:

type theory	**set theory**
`<element> : <type>`	$x \in S$
`<elem1> = <elem2>`	$x = y$

In set theory, an equation is a *proposition*:
we don't ask *why* 1+1=2

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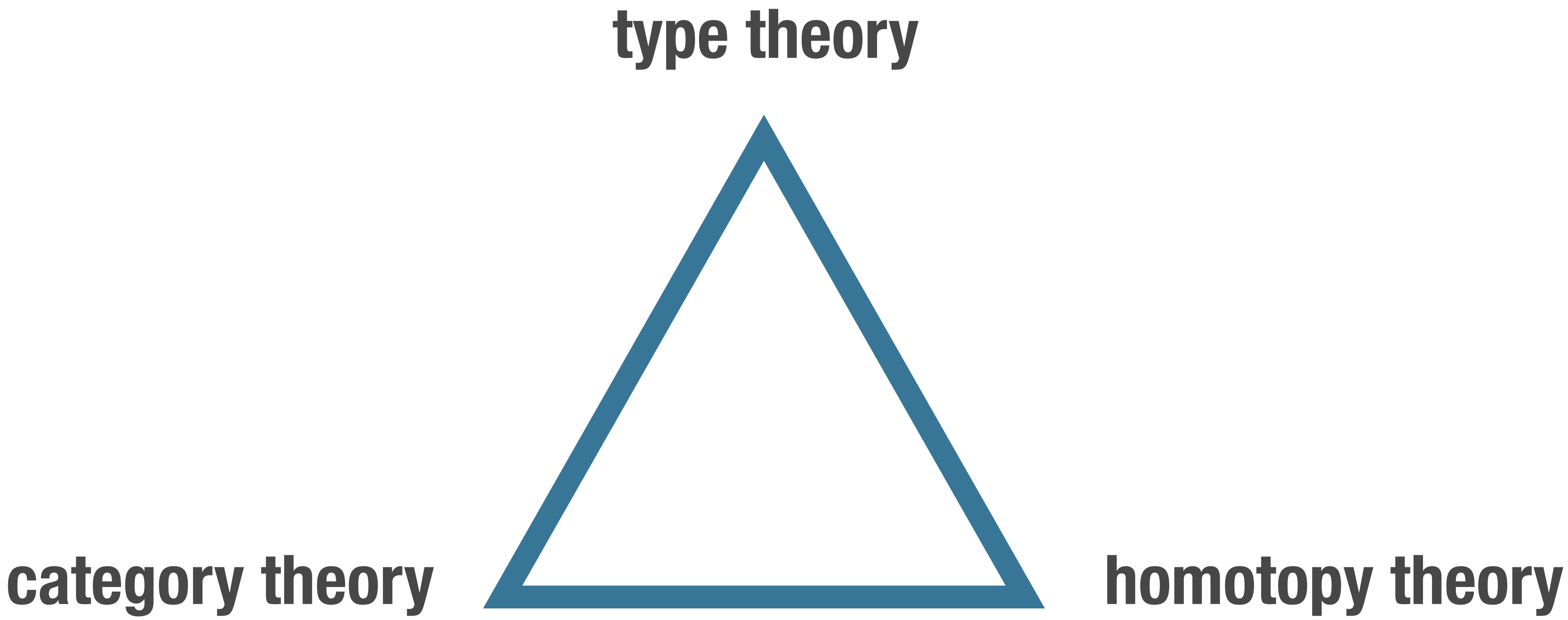
In type theory, an equation has a <proof>

Homotopy Type Theory



Types are ∞-groupoids

[Hofmann,Streicher,Awodey,Warren,Voevodsky
Lumsdaine,Gambino,Garner,van den Berg]

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set theory

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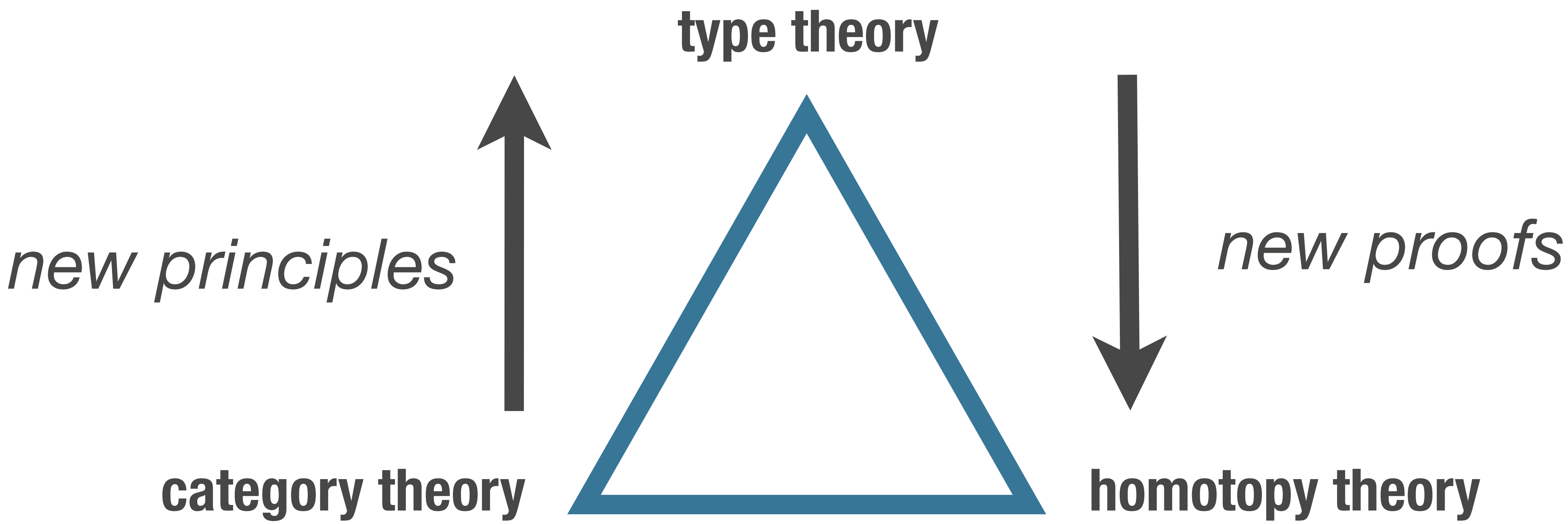
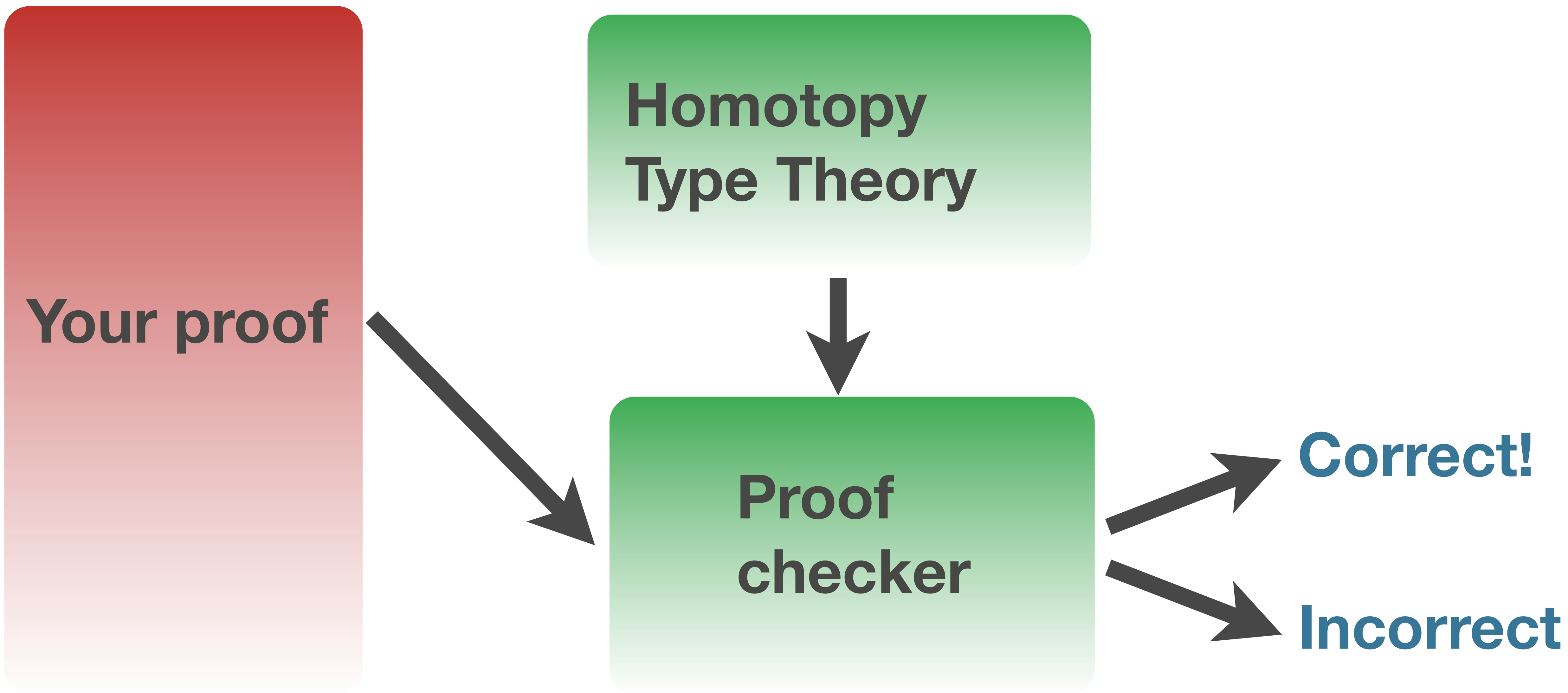
Homotopy Type Theory



Computer-checked proofs



Synthetic vs Analytic

Synthetic geometry (Euclid)

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I.

LET it be granted that a straight line may be drawn from any one point to any other point.

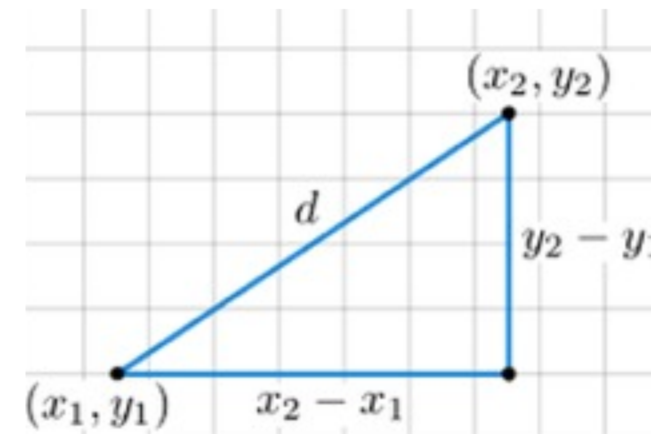
II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any centre, at any distance from that centre.

Analytic geometry (Descartes)



[image from wikipedia]

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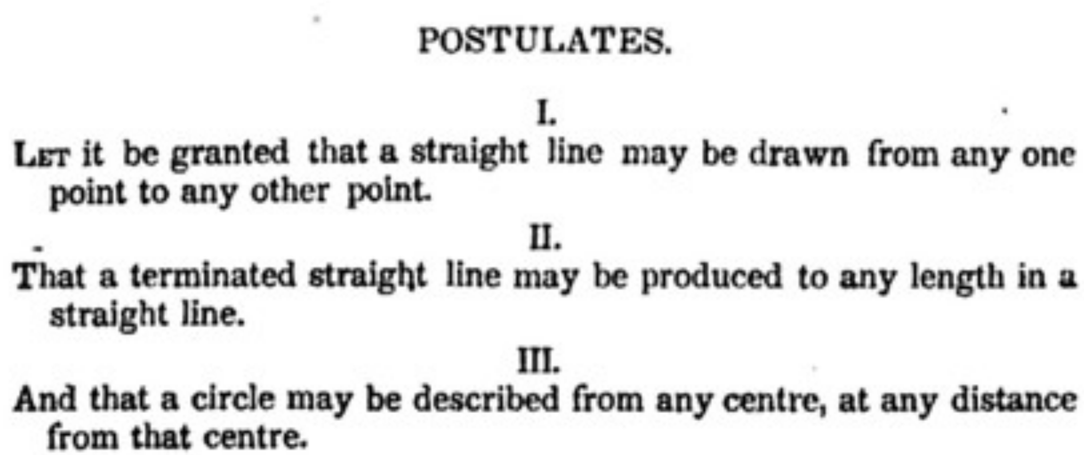
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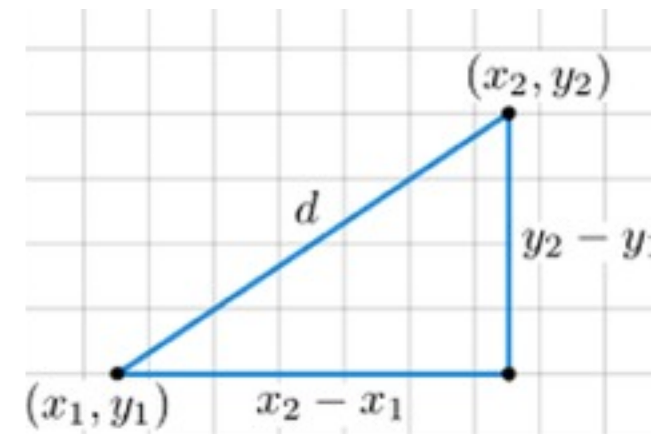

Classical homotopy theory is analytic:

- a space is a set of points equipped with a topology
- a path is a map [0,1] → X

[image from wikipedia]

Synthetic homotopy theory

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space	<type>
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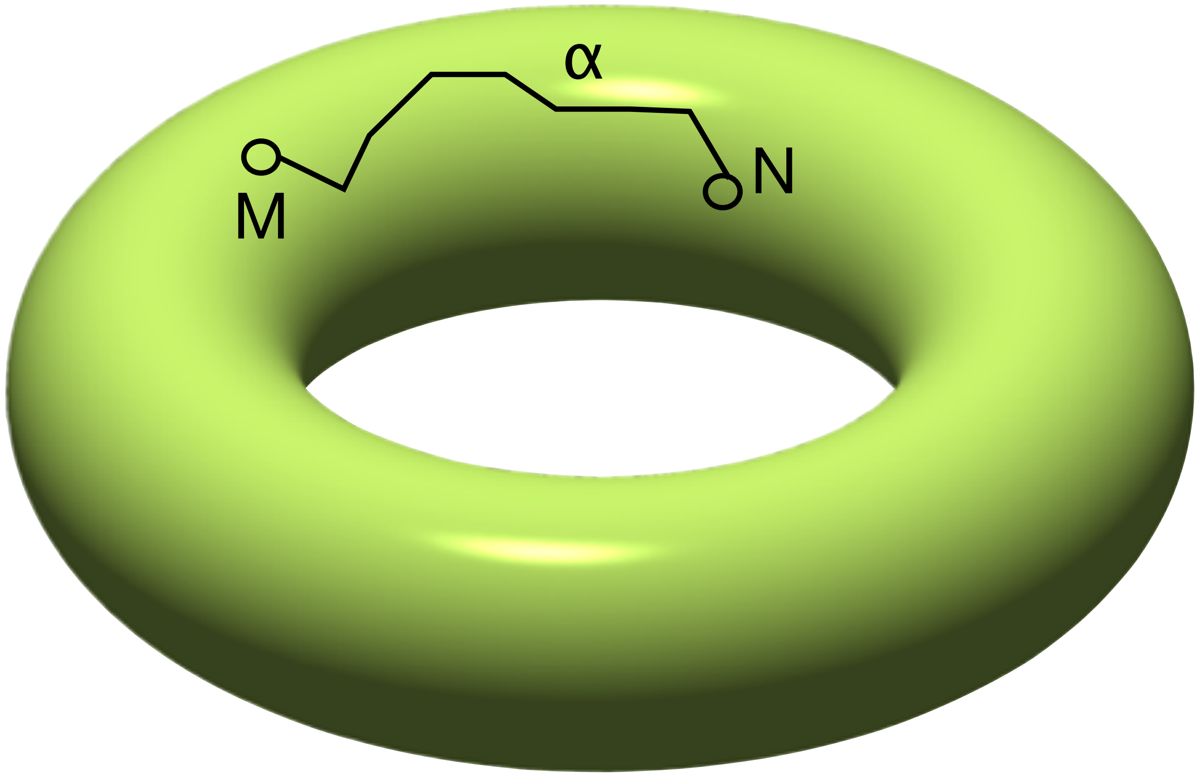
⋮	⋮

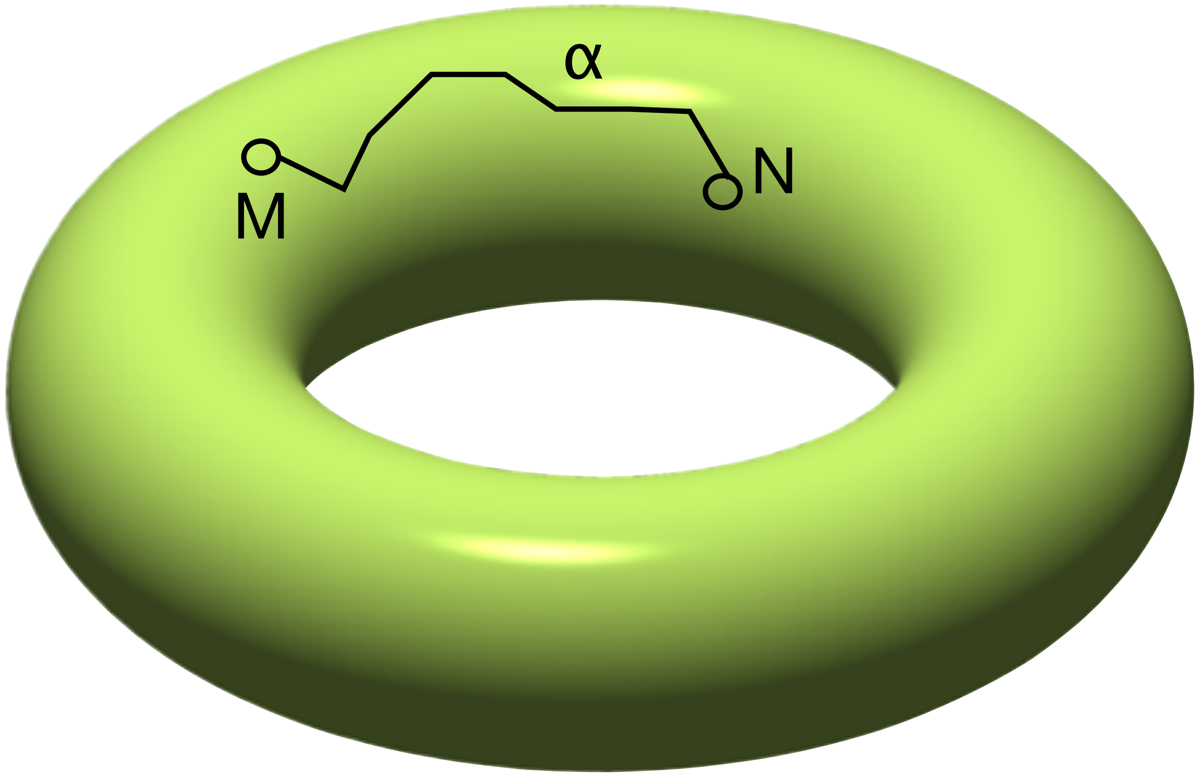
*A path is **not** a map [0,1]→X; it is a basic notion*

Spaces as types



Spaces as types

a space is a type A



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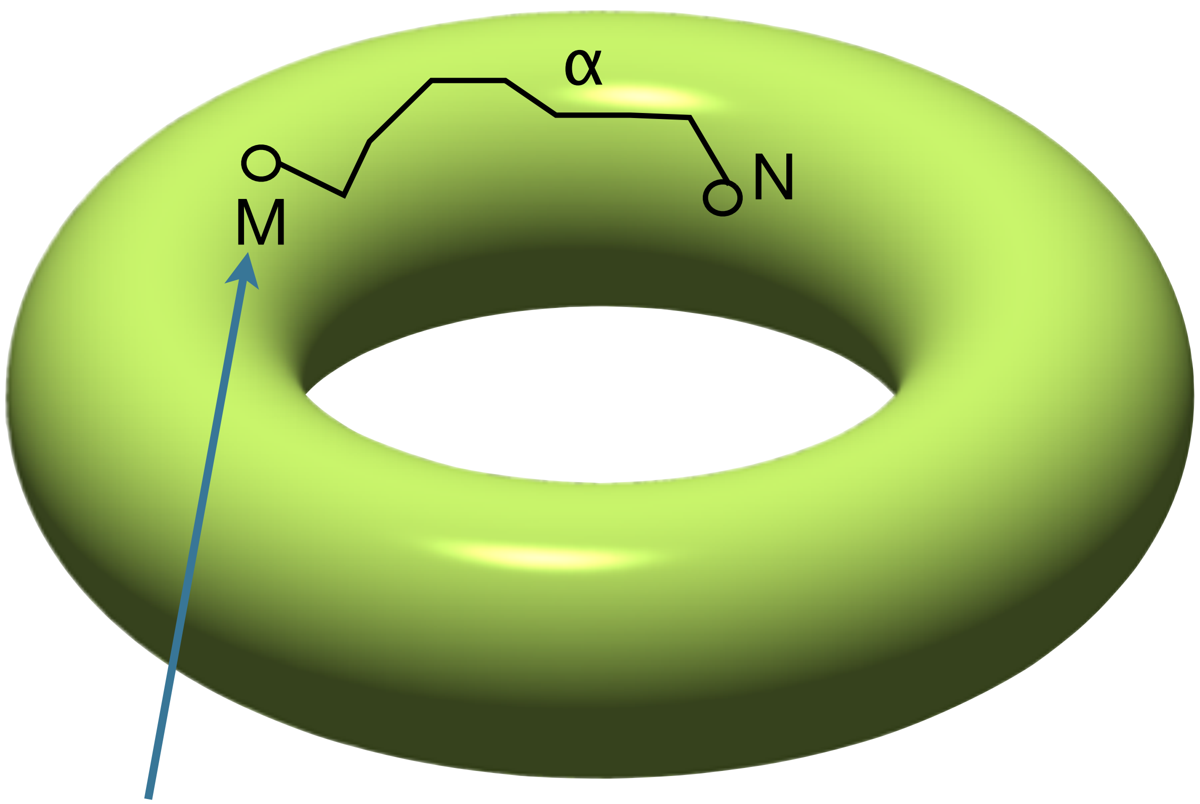

points are elements
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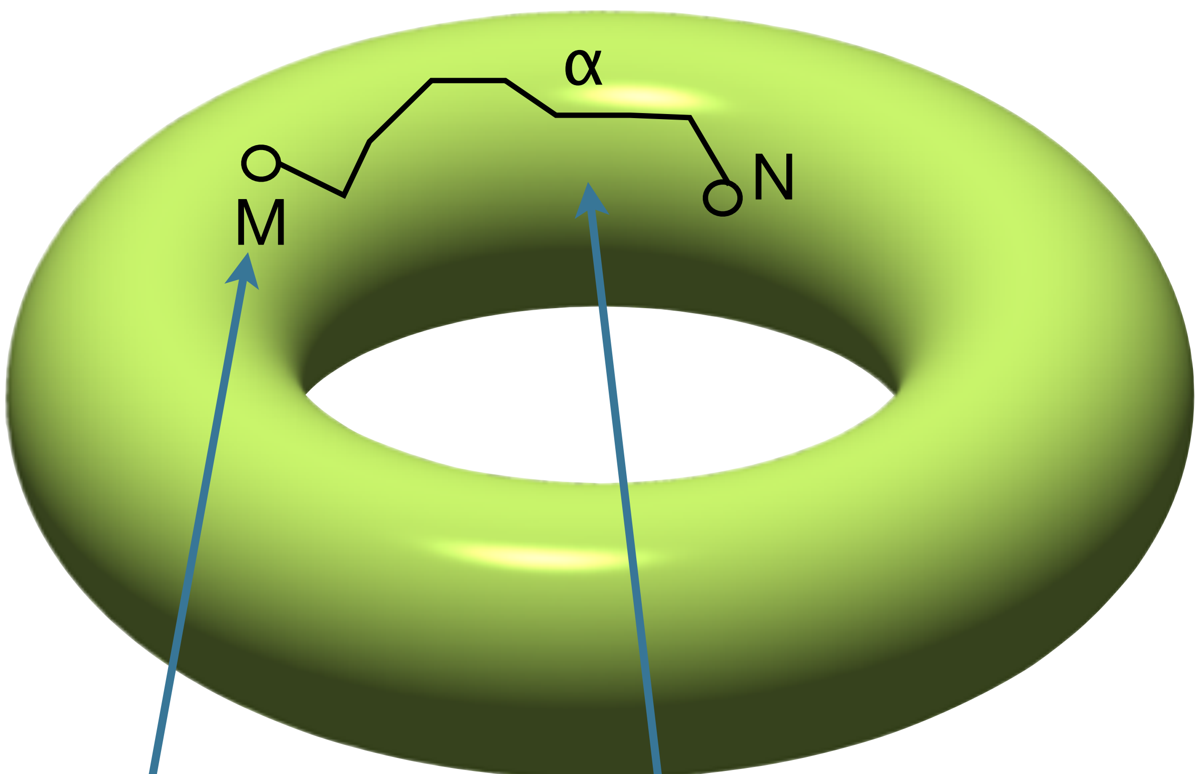

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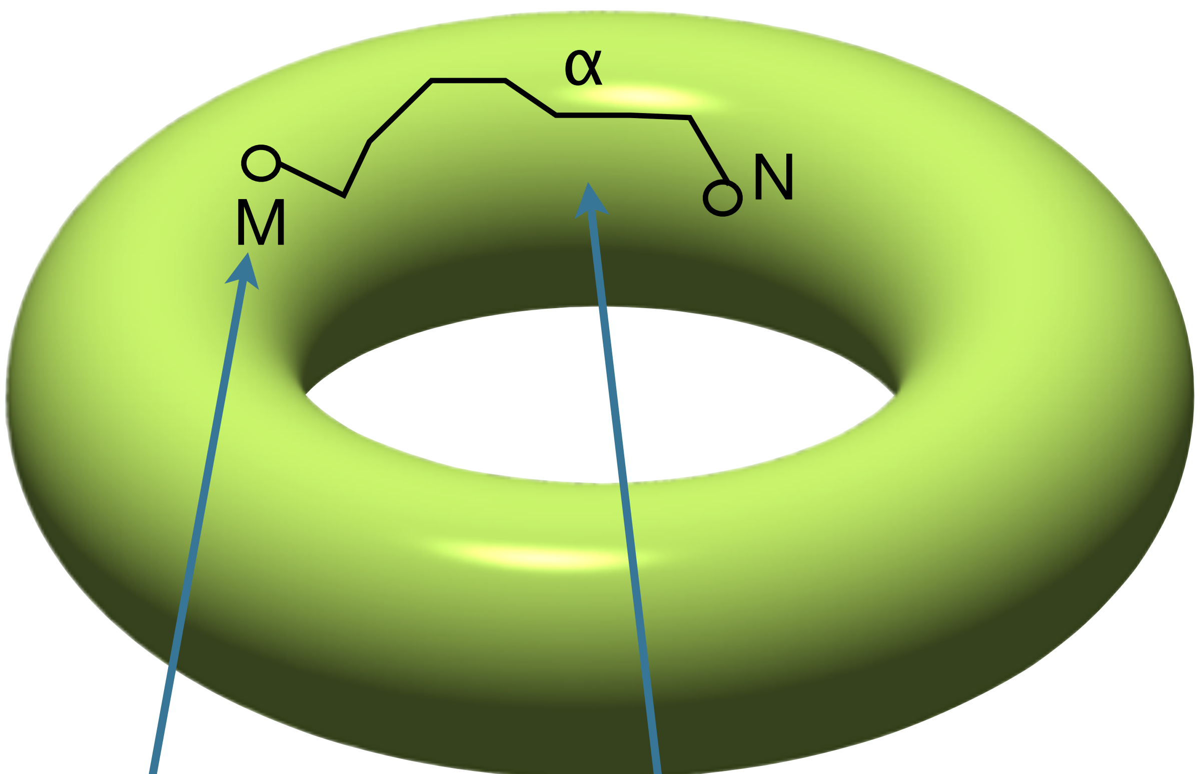

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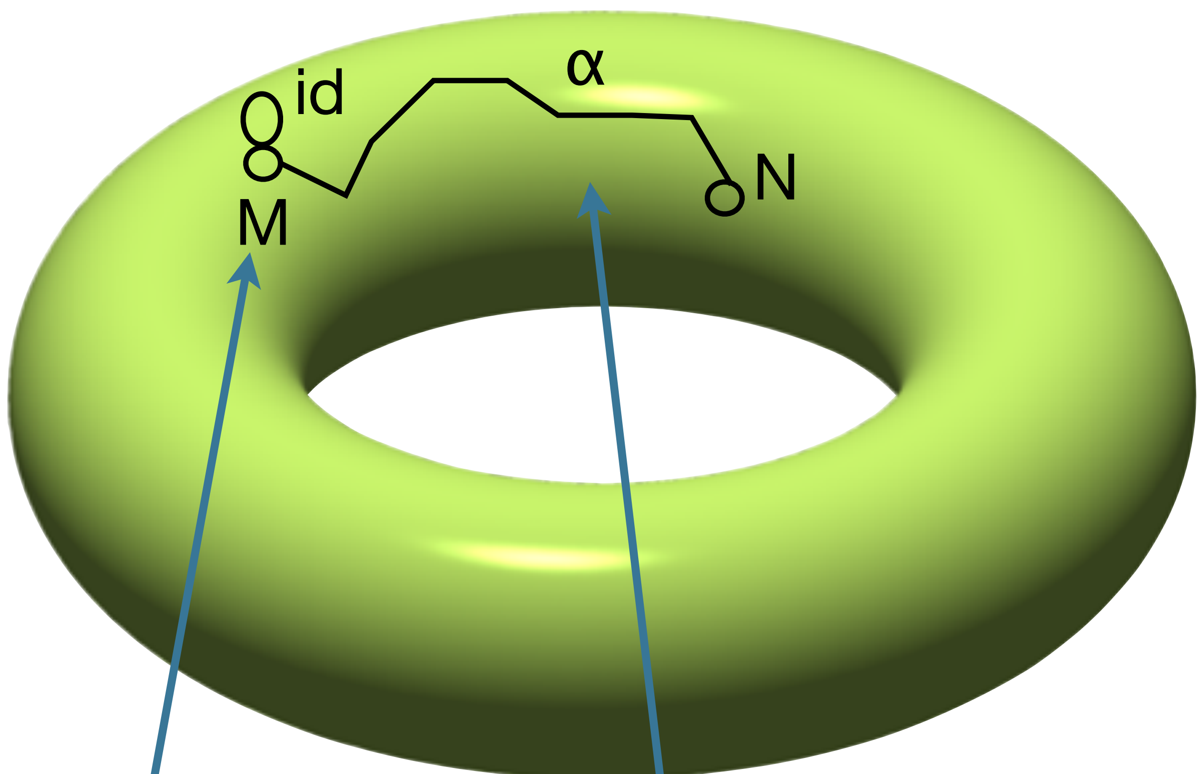

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`id      : M = M (refl)`

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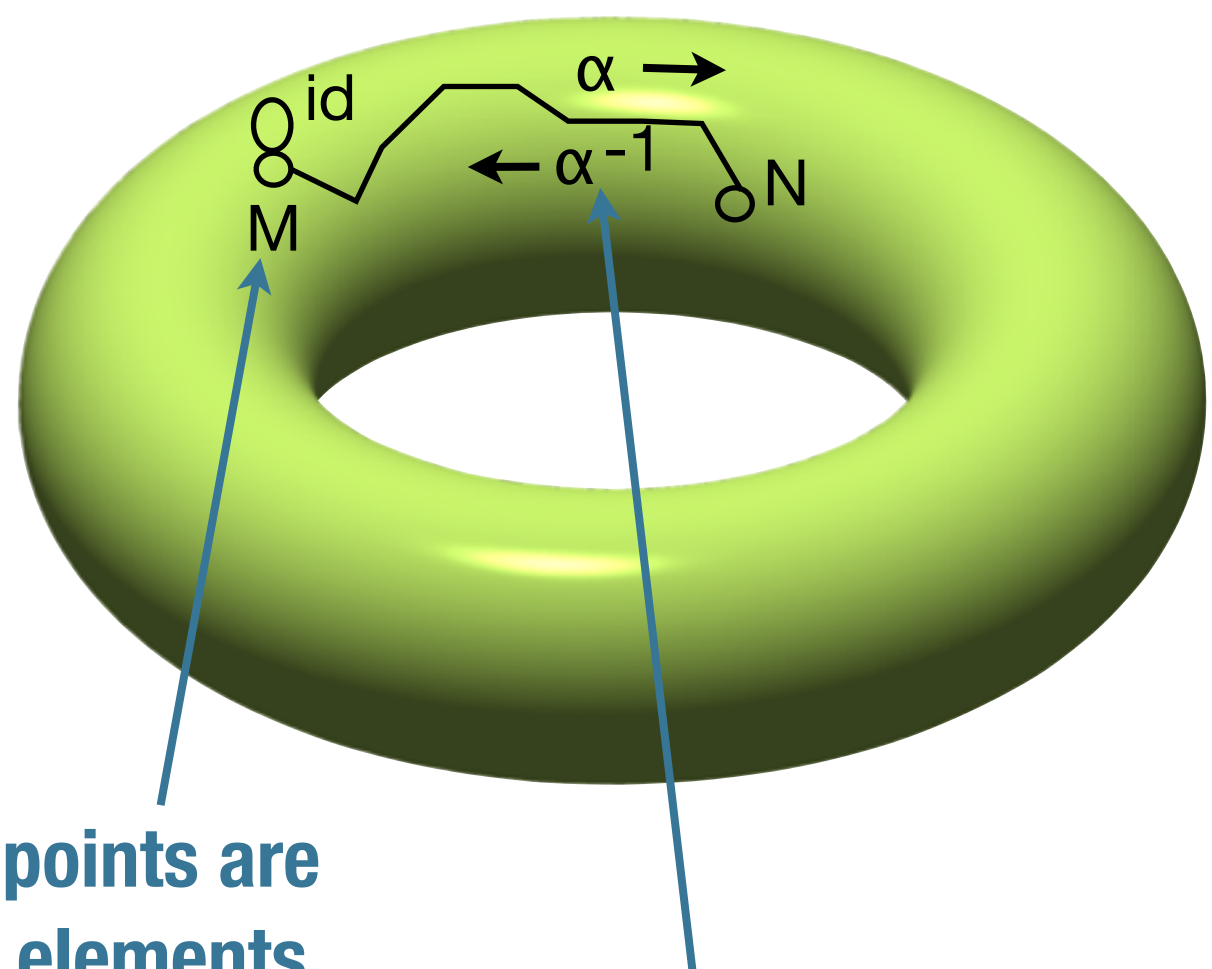

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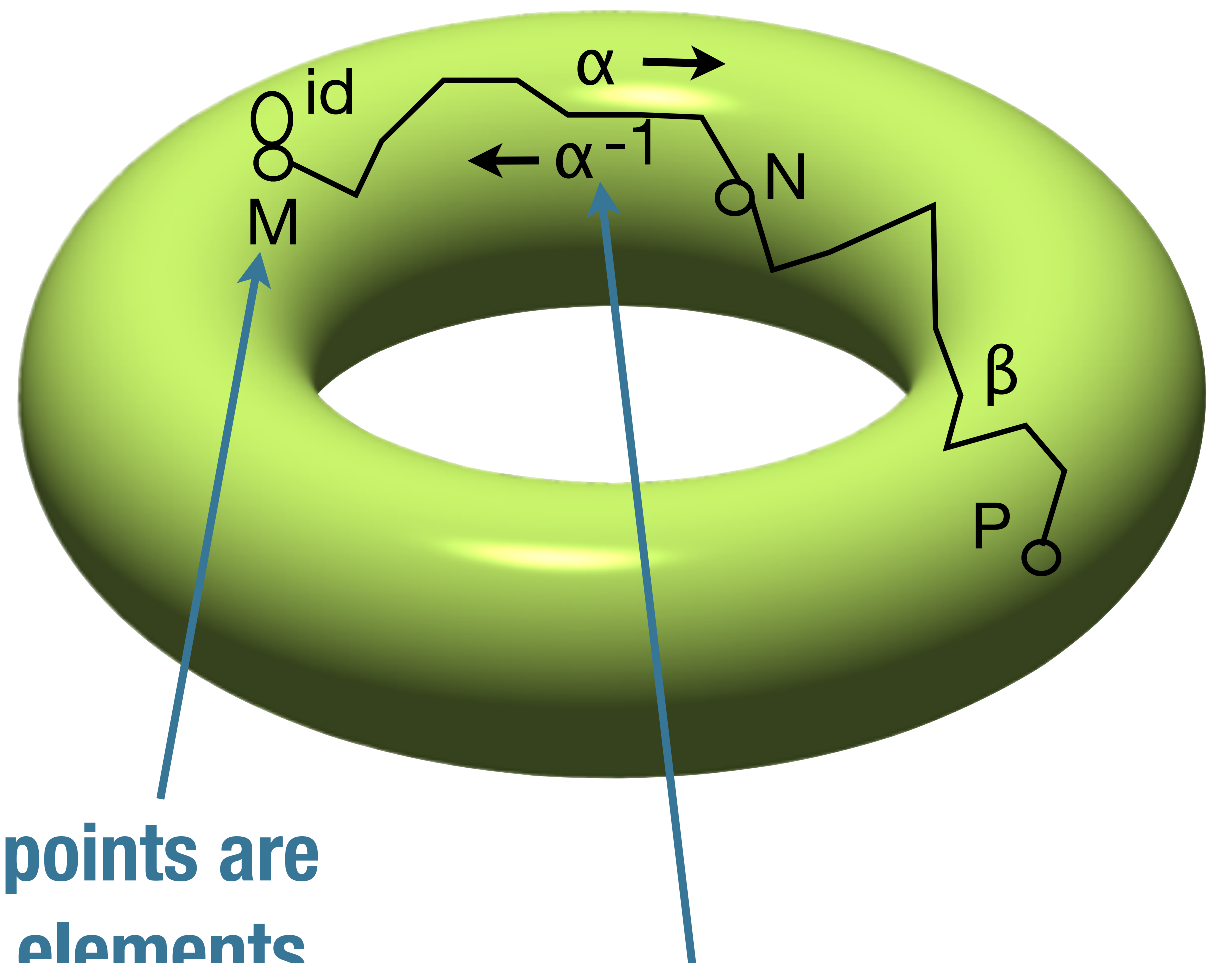

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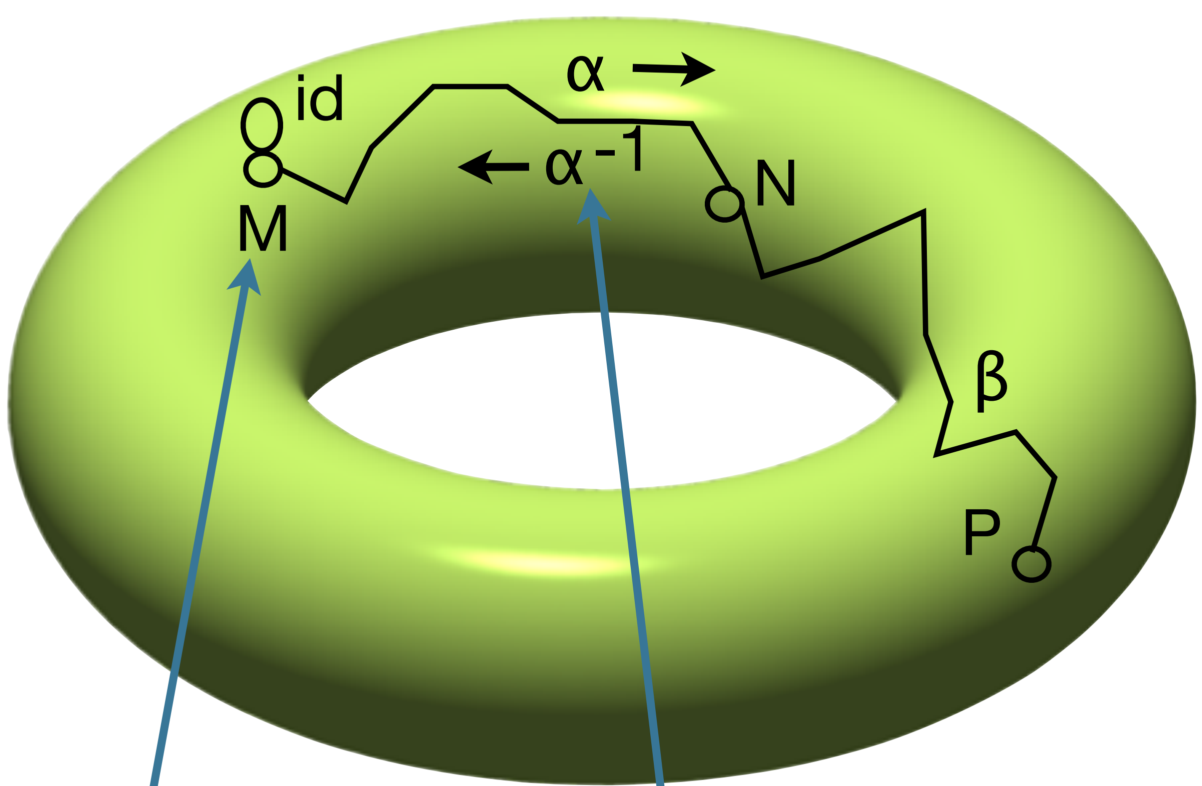

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homotopies

$id \circ \alpha = \alpha$

$\alpha^{-1} \circ \alpha = id$

$\gamma \circ (\beta \circ \alpha)$
$= (\gamma \circ \beta) \circ \alpha$

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$\mathrm{id} \circ \alpha = \alpha$
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*work in progress

Some results

Homotopy Theoretic **Type Theoretic**

Some results

Homotopy Theoretic | **Type Theoretic**

$\pi_1(S^1)$

Some results

Homotopy Theoretic | **Type Theoretic**

$\pi_1(S^1) \rightarrow \pi_1(S^1)$

Some results

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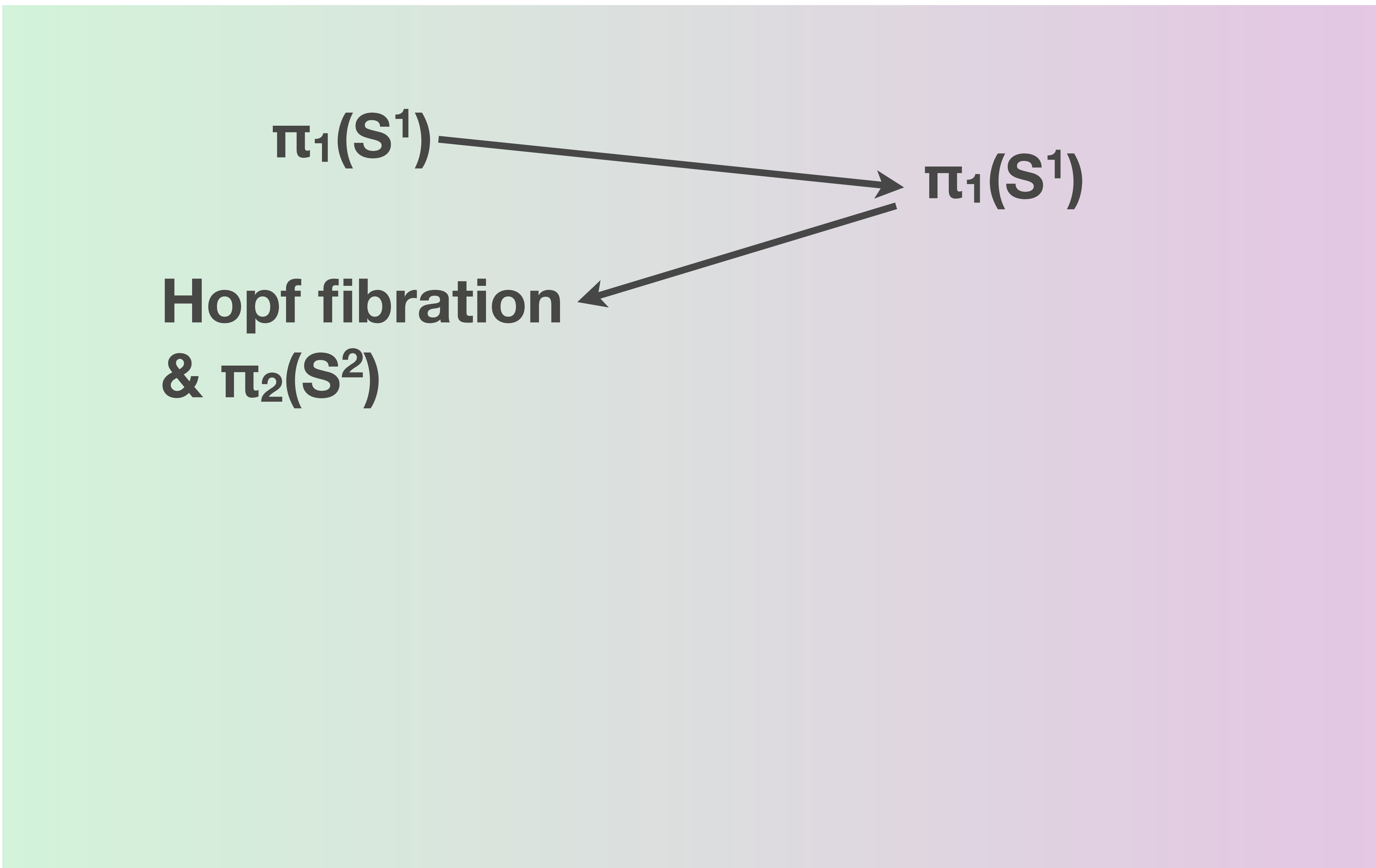

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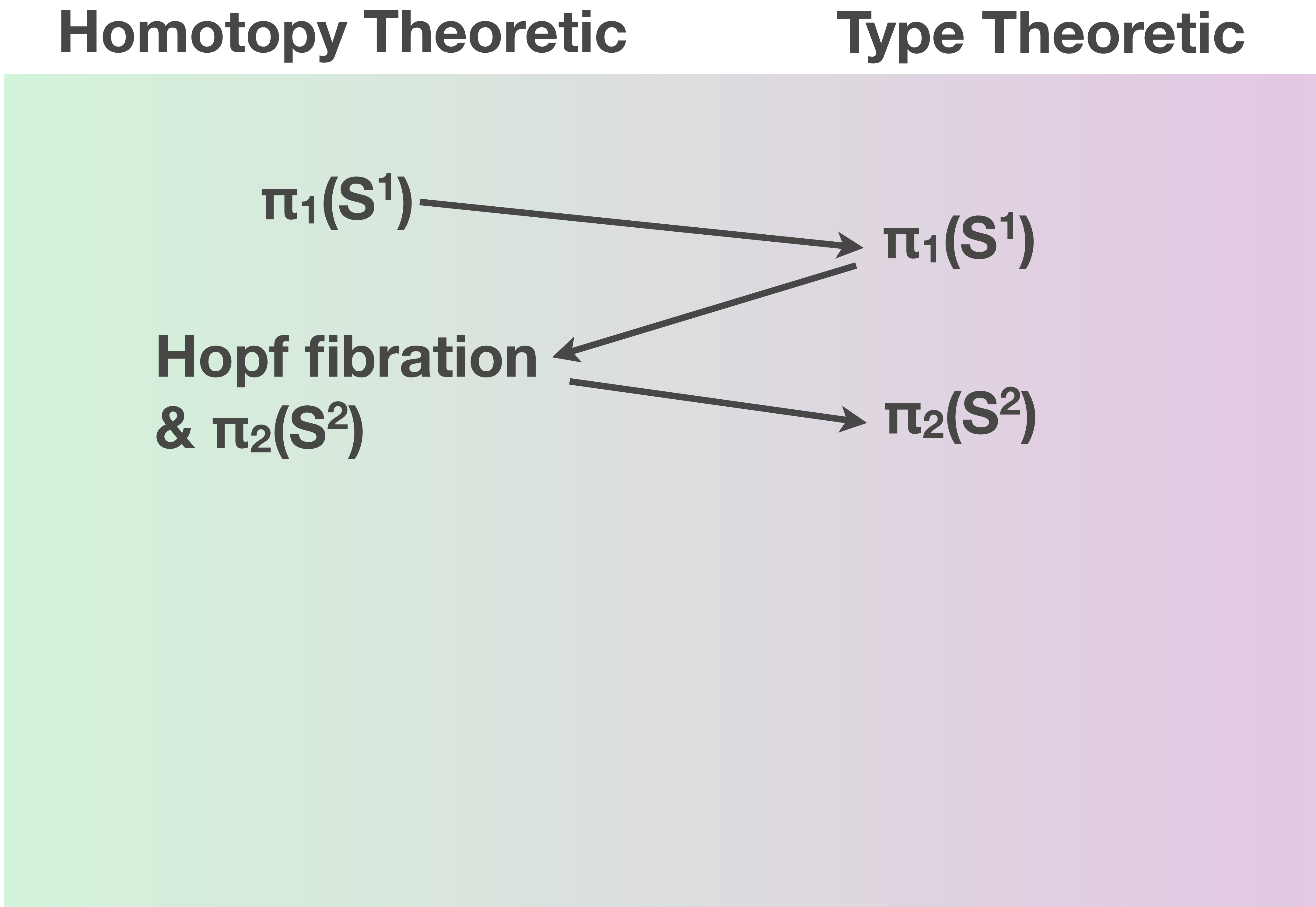

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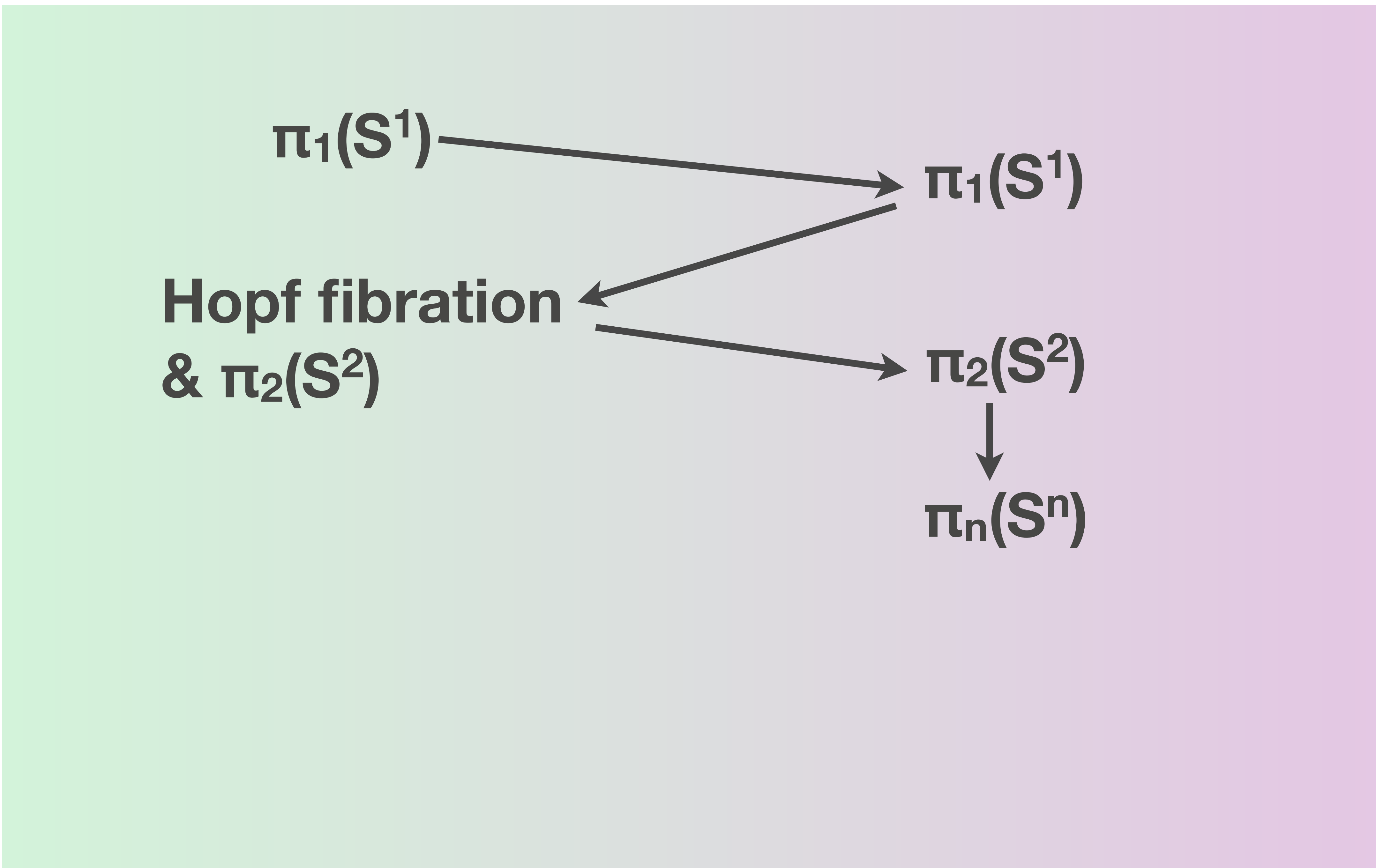

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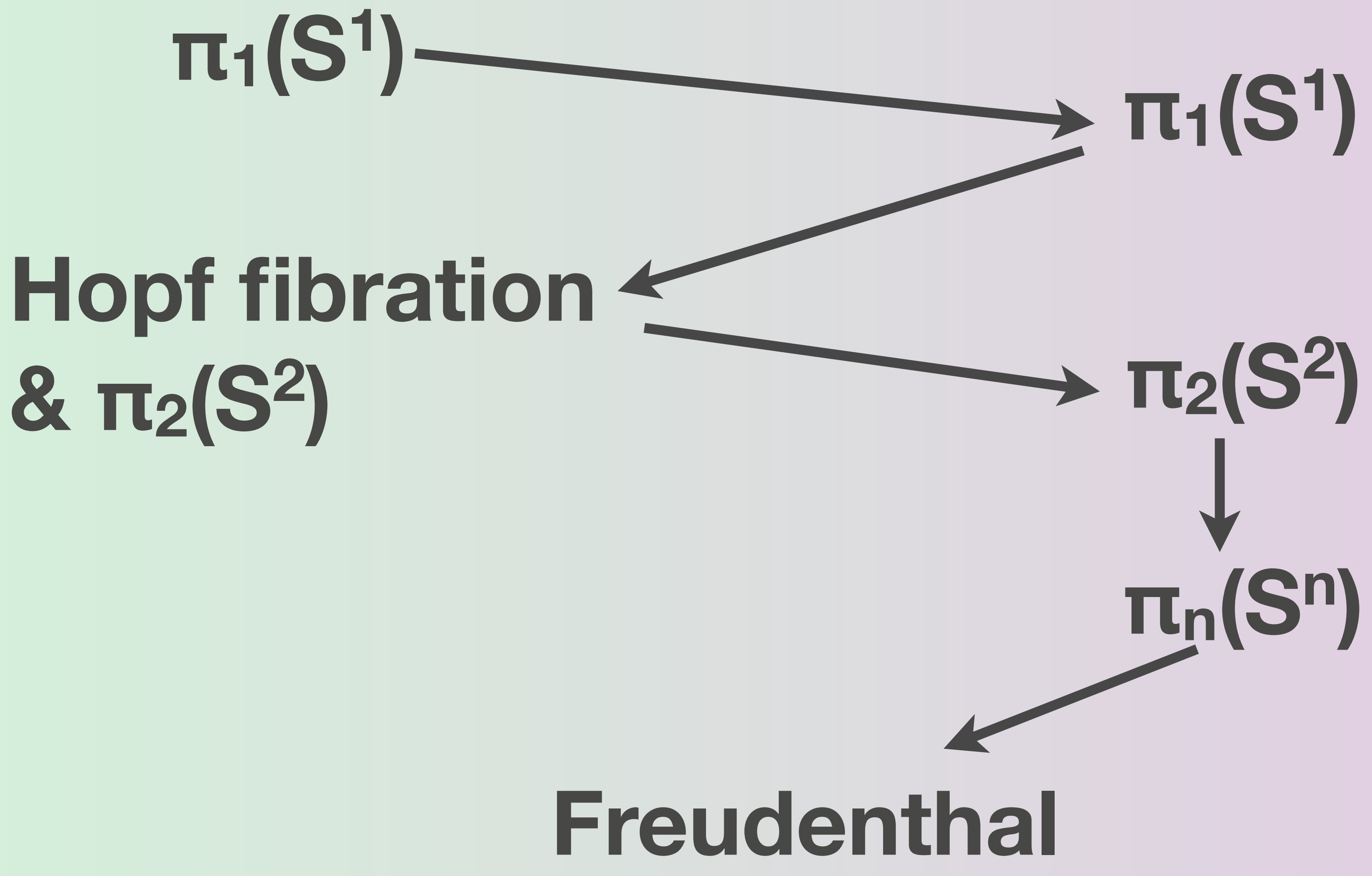

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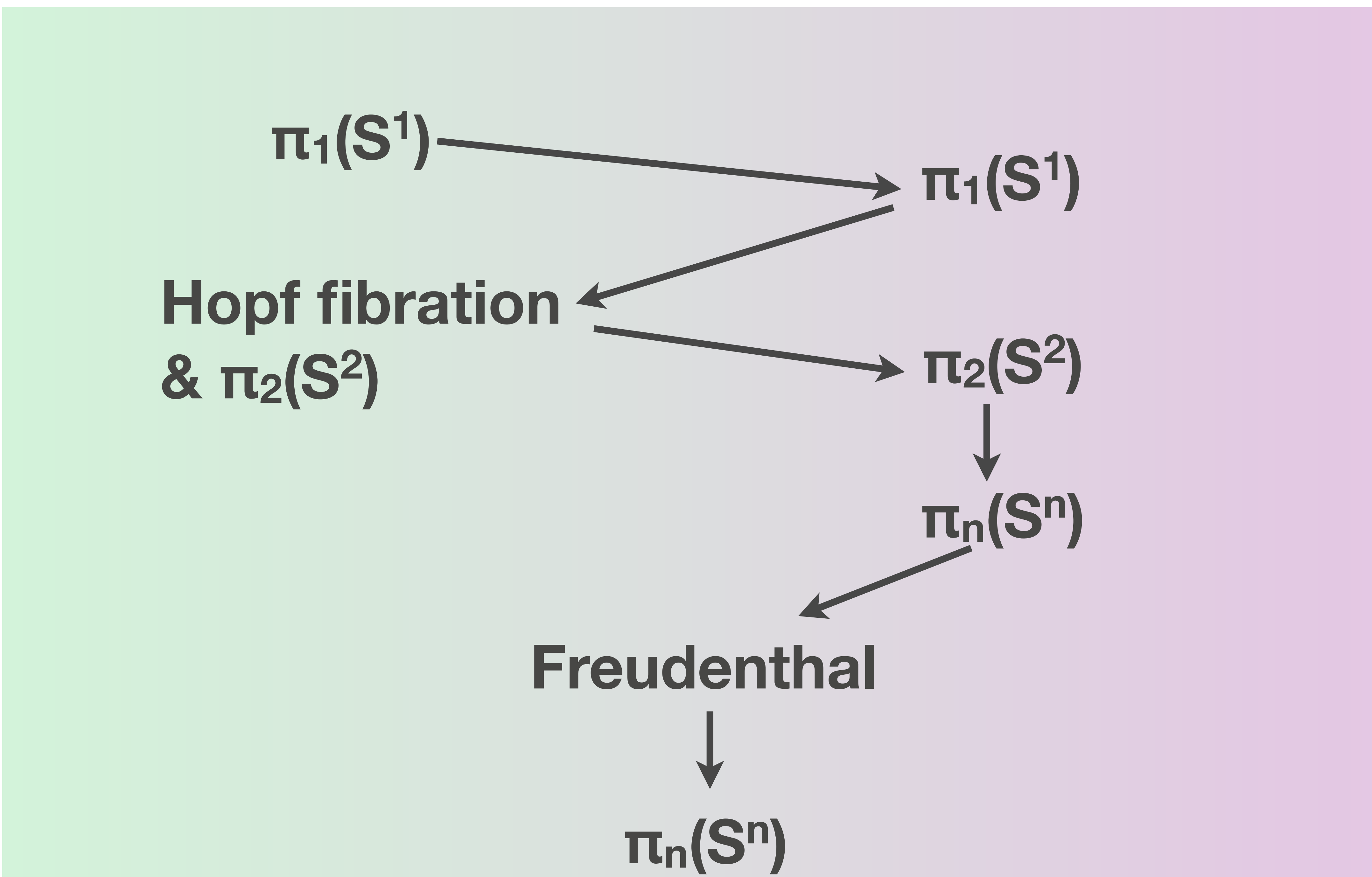

Outline

1. $\pi_1(S^1) = \mathbb{Z}$
2. The Hopf fibration
3. Connectedness and Freudenthal Suspension

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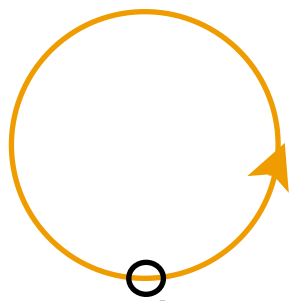
Higher inductive types

Circle is ***inductively generated*** by



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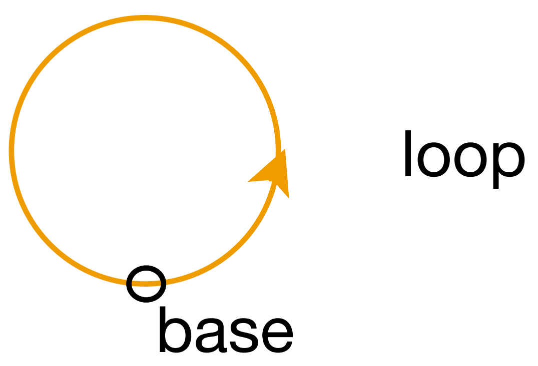
Circle is ***inductively generated*** by

```
base : Circle
loop : base = base
```



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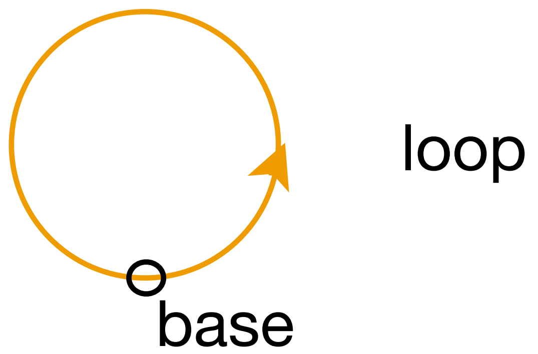

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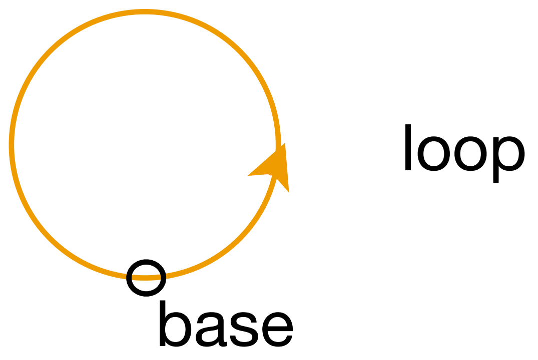

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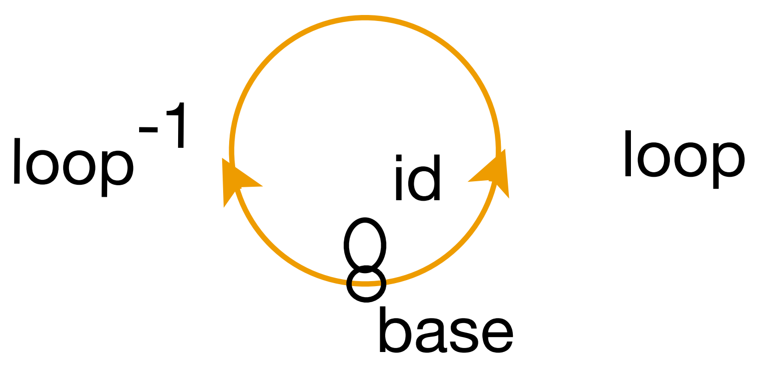

Free ∞-groupoid with these generators

```
id              inv : loop o loop^-1 = id
loop^-1         ...
loop o loop
```

Higher inductive types

Circle recursion:

function `Circle → X` determined by

```
base' : X
loop' : base' = base'
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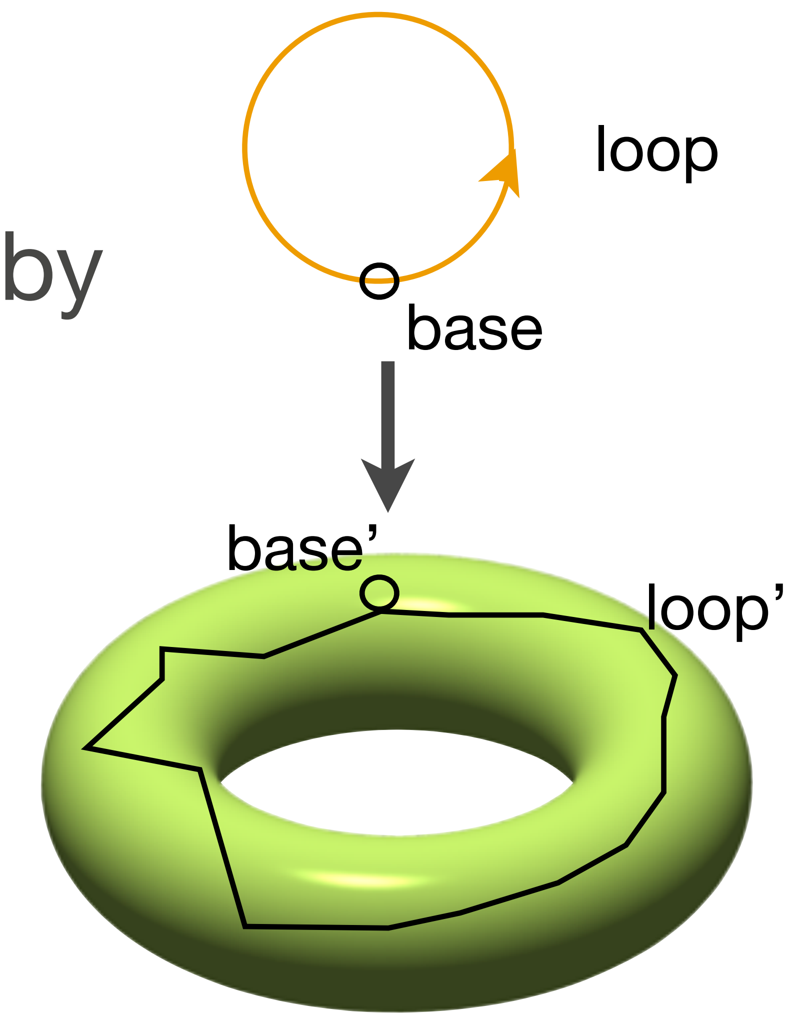

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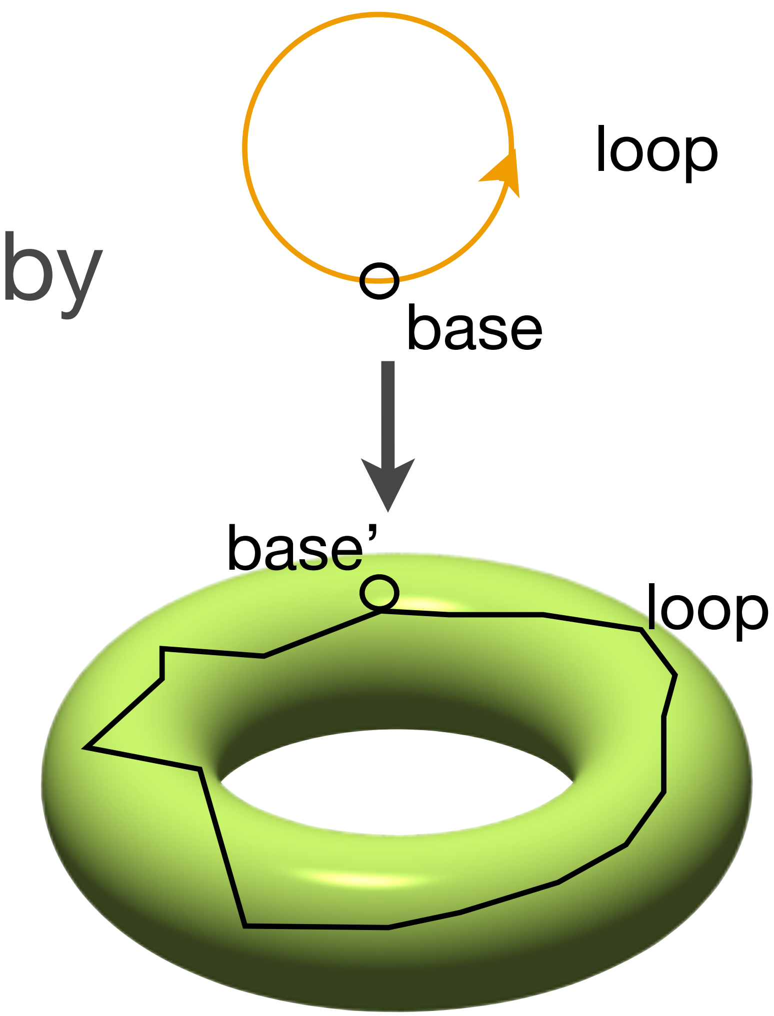

Circle induction: To prove a predicate P for all points on the circle, suffices to prove P(`base`), continuously in the loop

Fundamental group of circle

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i.e. the type `(base = base)`

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Corollary: $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$
$\pi_k(S^1)$ trivial otherwise

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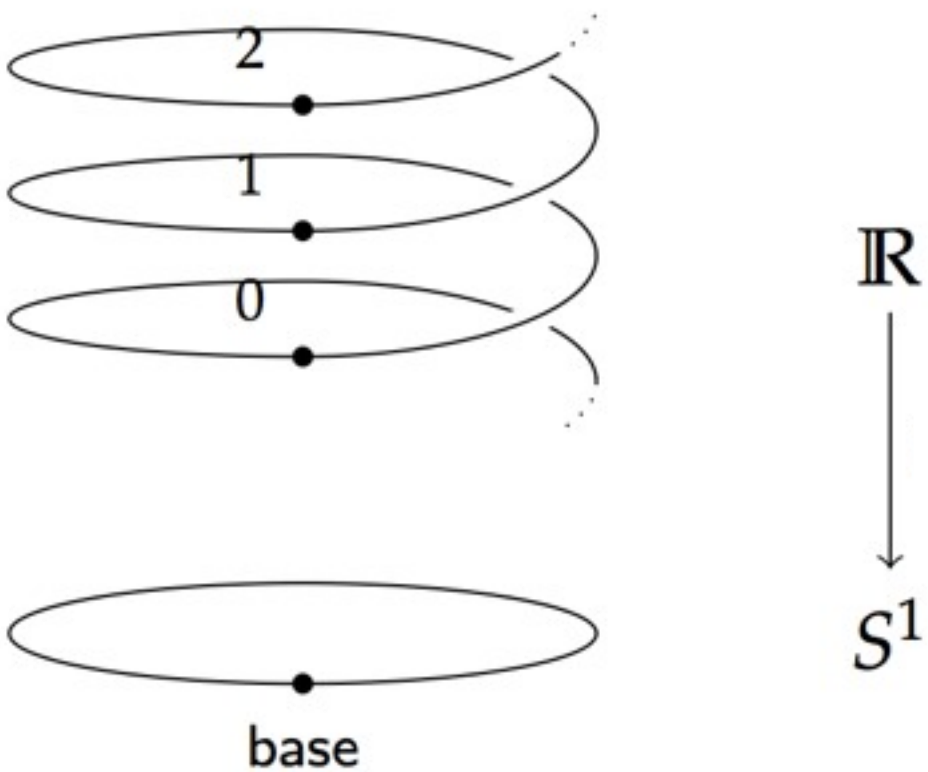
Universal Cover



`w : Ω(S¹) → ℤ`

defined by **lifting** a loop to the cover, and giving the other endpoint of 0

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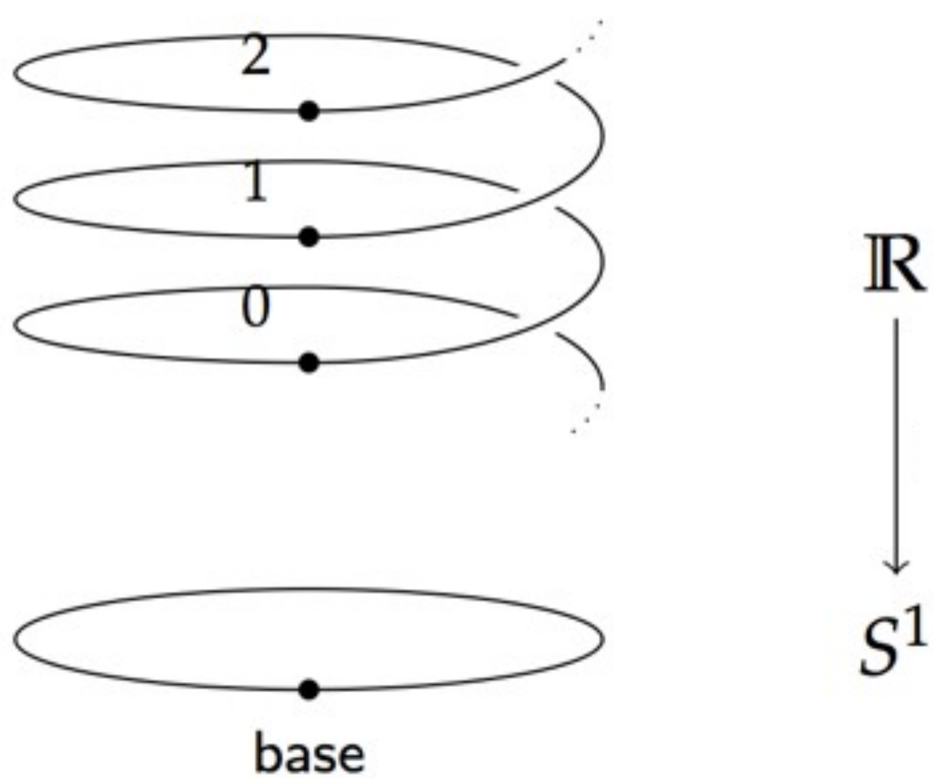

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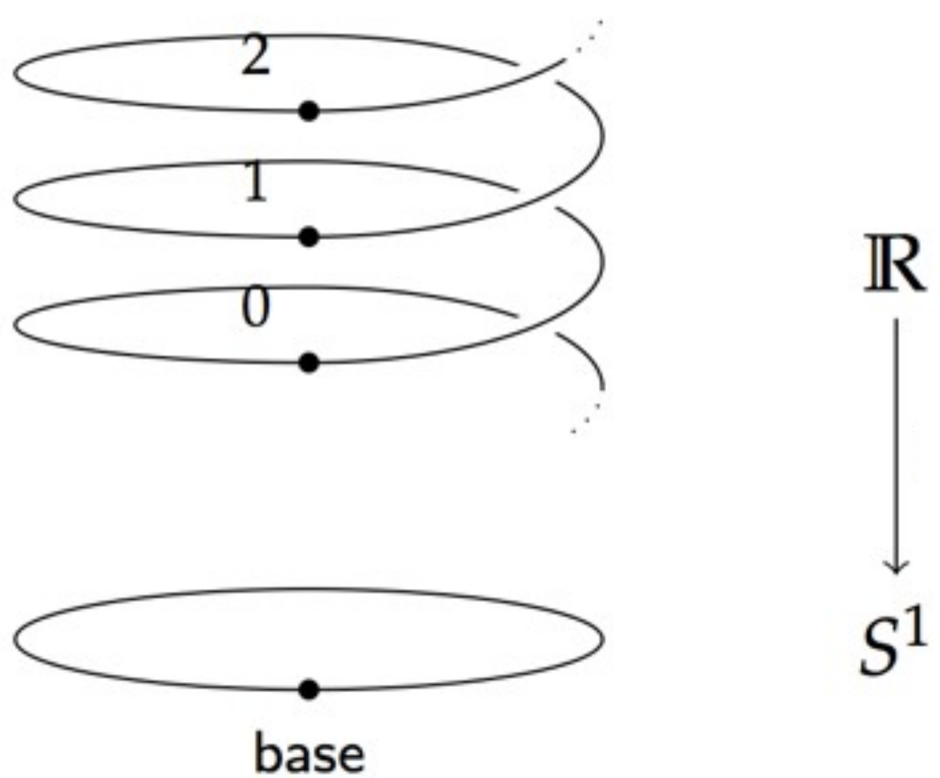

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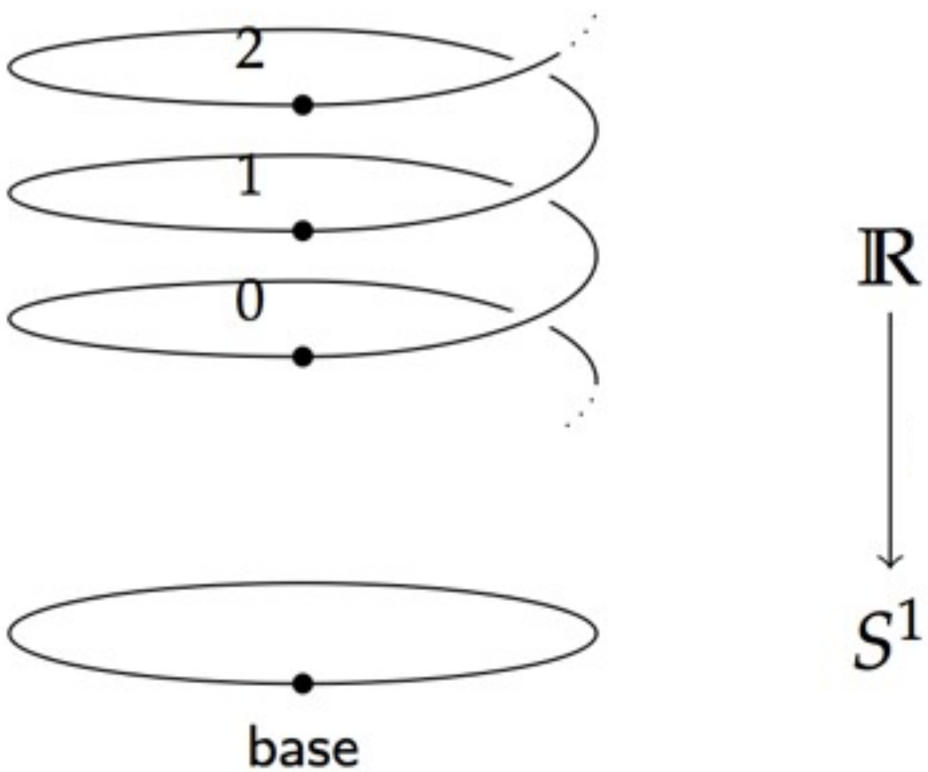
lifting `loop` adds 1

Universal Cover



$w : \Omega(S^1) \to \mathbb{Z}$

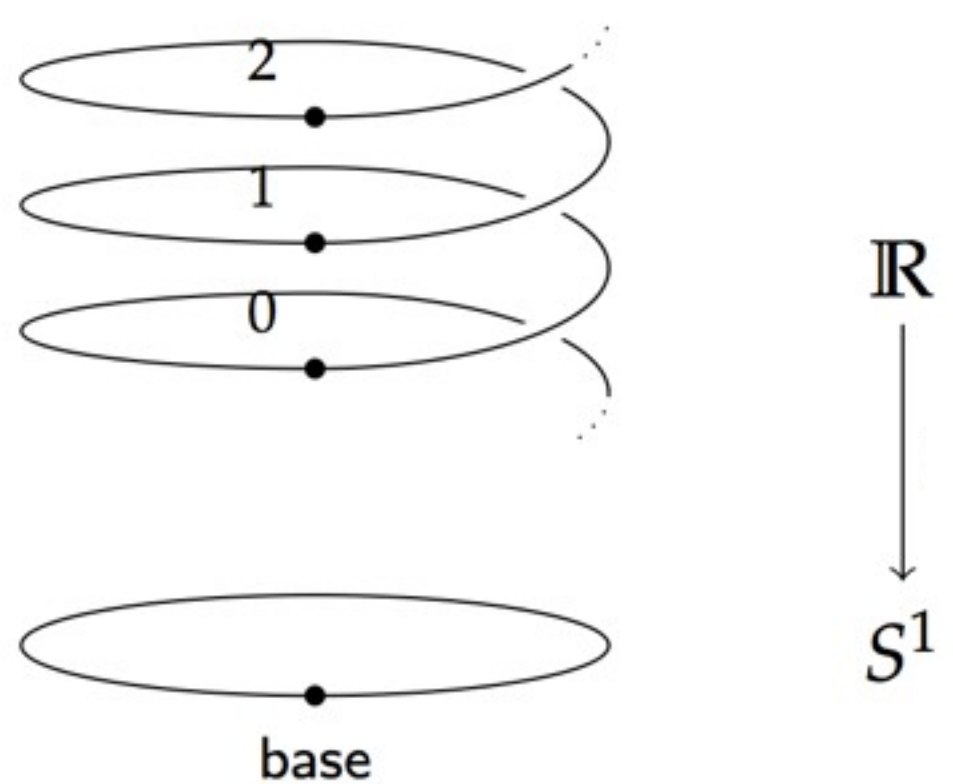
defined by **lifting** a loop to the cover, and giving the other endpoint of 0

lifting is functorial

lifting `loop` adds 1

lifting $\texttt{loop}^{-1}$ subtracts 1

Universal Cover



`w : Ω(S¹) → ℤ`

defined by **lifting** a loop to the cover, and giving the other endpoint of 0

lifting is functorial

lifting `loop` adds 1

lifting `loop`$^{-1}$ subtracts 1

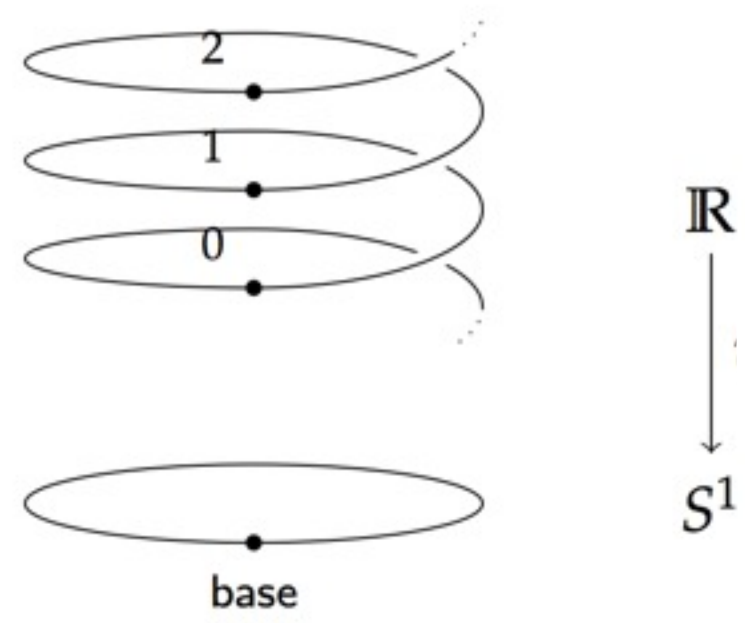
Example:

```
  w(loop o loop⁻¹)
= 0 + 1 - 1
= 0
```

Fibration = Family of types

Fibration (classically):

map p : E → B such that any path from p(e) to y lifts to a path in E from e to some point in $p^{-1}(y)$



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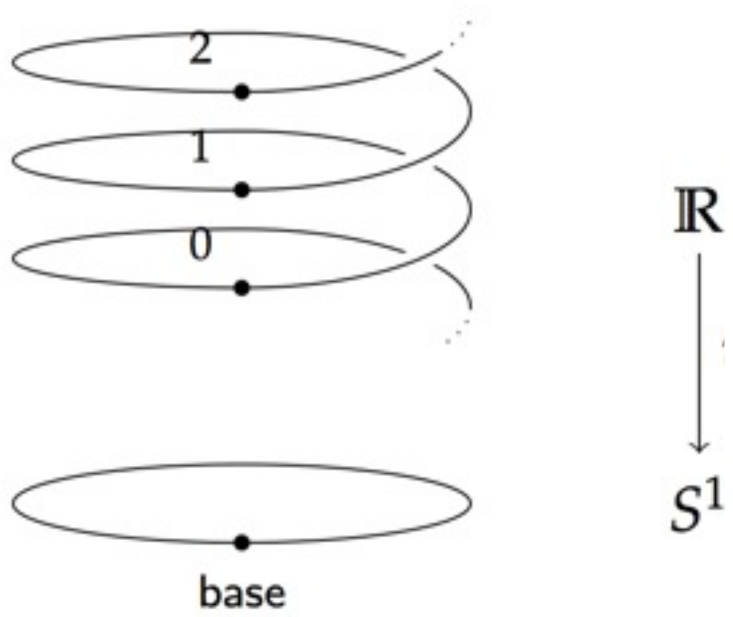

Family of types $(E(x))_{x:B}$

- *Fibers:* $E(b)$ is a type for all $b:B$
- *transport:* equivalence $E(b_1) \simeq E(b_2)$ for all $p : b_1 =_B b_2$

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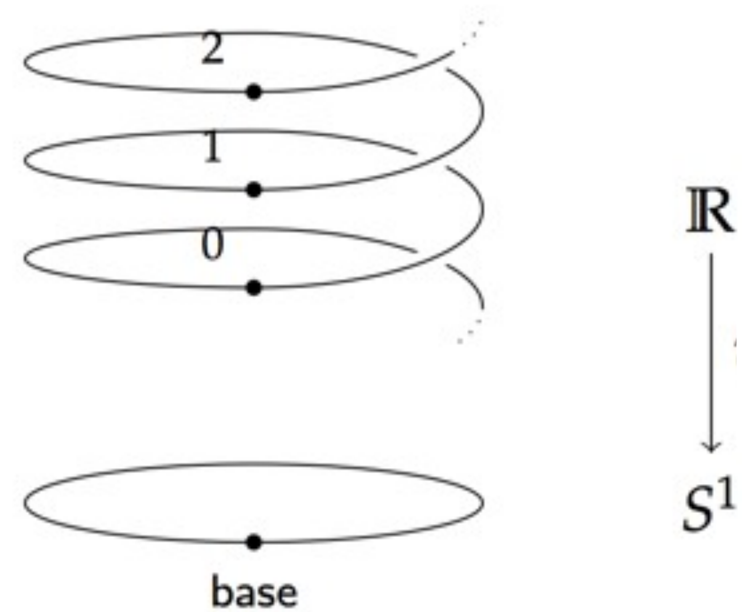

Family of types $(E(x))_{x:B}$

$p^{-1}(b)$

- *Fibers:* $E(b)$ is a type for all $b:B$
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Fibration = Family of types

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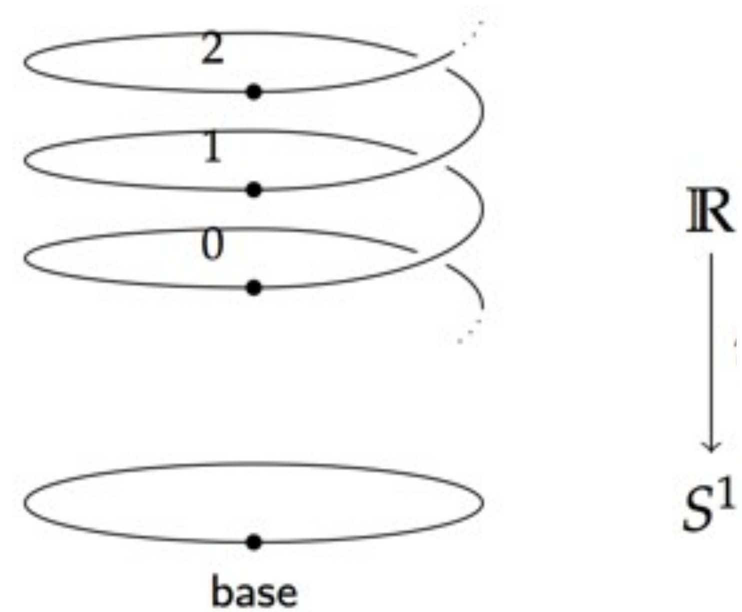
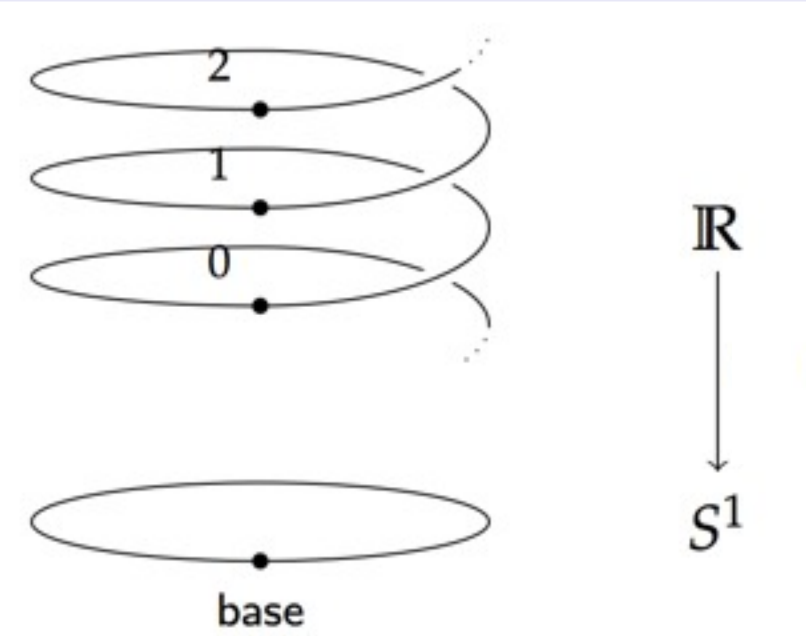


Family of types $(E(x))_{x:B}$

- *Fibers:* $E(b)$ is a type for all $b:B$ ← **$p^{-1}(b)$**
- *transport:* equivalence $E(b_1) \simeq E(b_2)$ for all $p : b_1 =_B b_2$

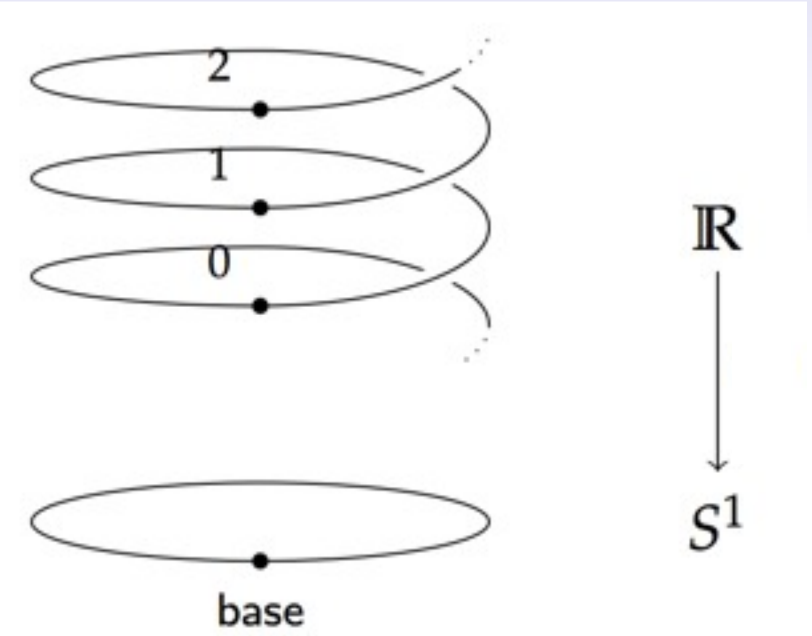
sends $e \in E(x)$ to other endpoint of lifting of p

Universal Cover



family of types $(\mathtt{Cover}(\mathtt{x}))_{\mathtt{x:S1}}$

Universal Cover

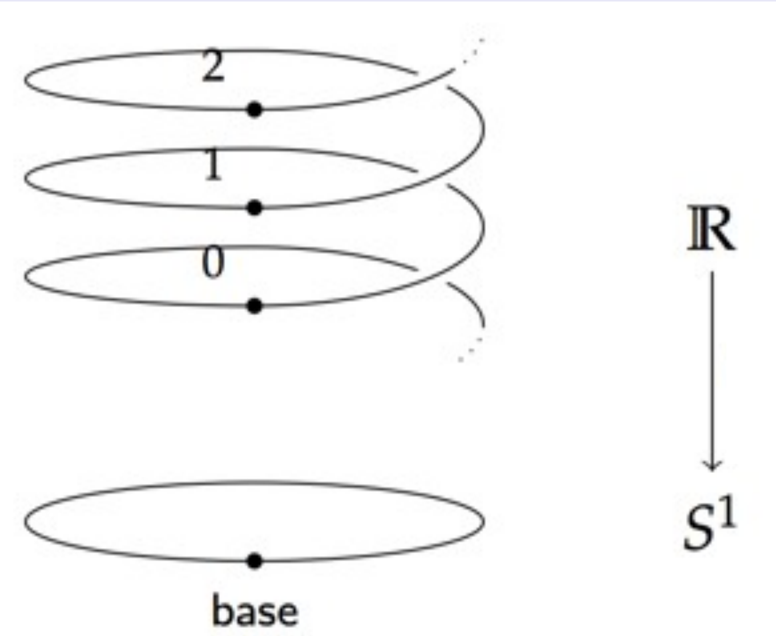


family of types (`Cover(x)`)$_{x:S1}$

By circle recursion, it suffices to give

- Fiber over `base`: $\mathbb{Z}$
- Equivalence $\mathbb{Z} \simeq \mathbb{Z}$ as lifting of `loop`: `successor`

Universal Cover

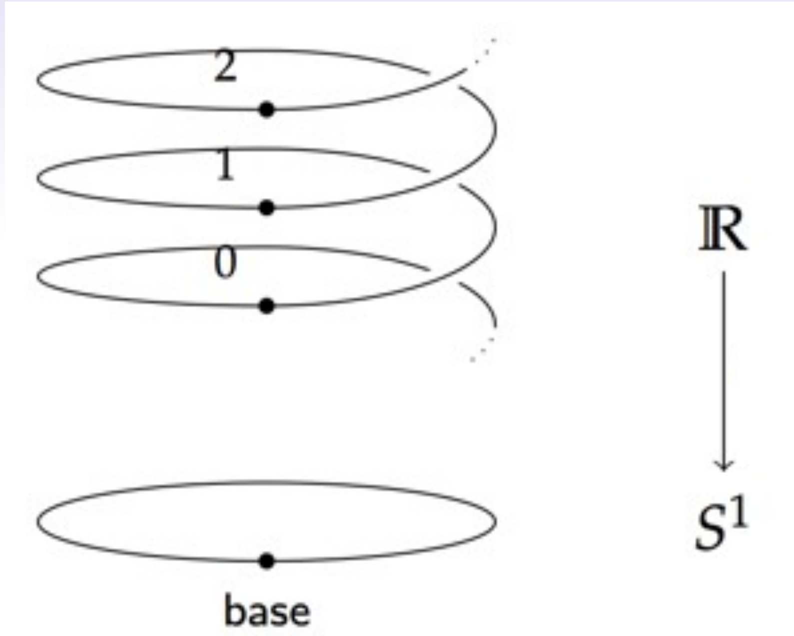


family of types $(\texttt{Cover(x)})_{\texttt{x:S1}}$

By circle recursion, it suffices to give

- Fiber over `base`: $\mathbb{Z}$
- Equivalence $\mathbb{Z} \simeq \mathbb{Z}$ as lifting of `loop`: `successor` ***uses univalence***

Universal Cover



family of types $(\mathtt{Cover(x)})_{\mathtt{x:S1}}$

By circle recursion, it suffices to give

- Fiber over `base`: $\mathbb{Z}$
- Equivalence $\mathbb{Z} \simeq \mathbb{Z}$ as lifting of `loop`: `successor`

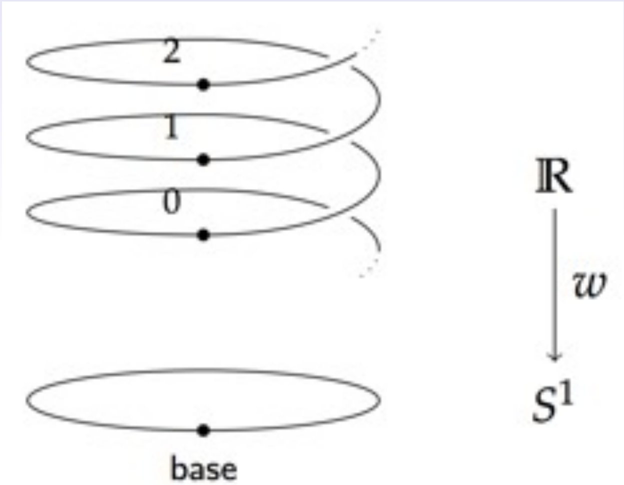
uses univalence

Defining equations:

`Cover(base) :=` $\mathbb{Z}$

$\mathtt{transport}_{\mathtt{Cover}}\mathtt{(loop)}$ `:= successor`

Winding number

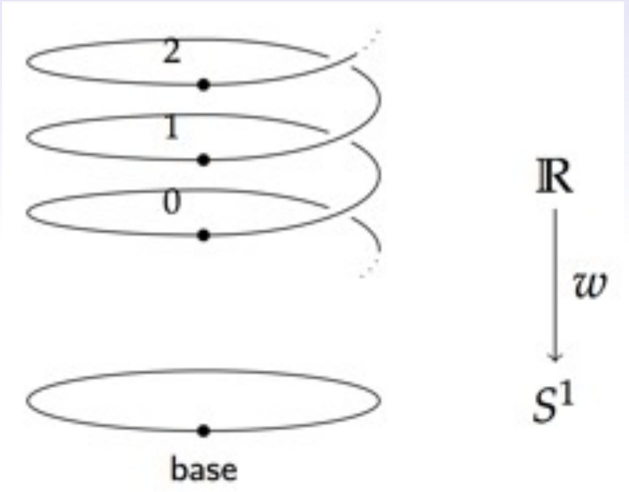


$$w : \Omega(S^1) \to \mathbb{Z}$$

$$w(p) = \mathrm{transport}_{\mathrm{Cover}}(p, 0)$$

lift p to cover, starting at 0

Winding number



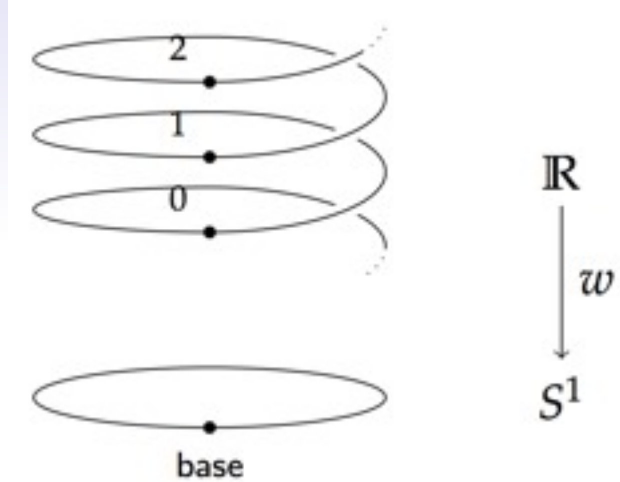
$w : \Omega(S^1) \to \mathbb{Z}$

$w(p) = \mathrm{transport}_{\mathrm{Cover}}(p, 0)$

lift p to cover, starting at 0

w(loop^{-1} o loop)

Winding number



$w : \Omega(S^1) \to \mathbb{Z}$

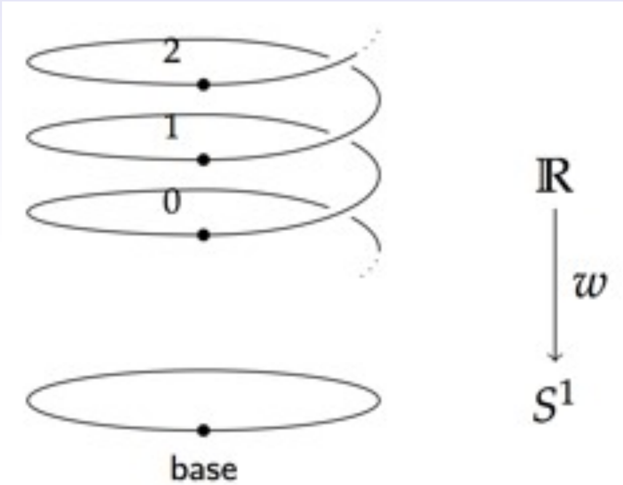
$w(p) = transport_{Cover}(p, 0)$

lift p to cover, starting at 0

$w(loop^{-1} \circ loop)$

$= transport_{Cover}(loop^{-1} \circ loop, 0)$

Winding number



lift p to cover, starting at 0

$w : \Omega(S^1) \to \mathbb{Z}$

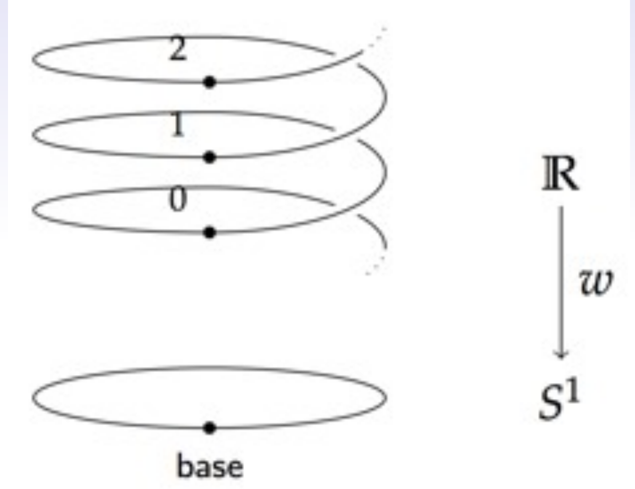
$w(p) = \text{transport}_{\text{Cover}}(p, 0)$

$w(\text{loop}^{-1} \circ \text{loop})$

$= \text{transport}_{\text{Cover}}(\text{loop}^{-1} \circ \text{loop}, 0)$

$= \text{transport}_{\text{Cover}}(\text{loop}^{-1}, \text{transport}_{\text{Cover}}(\text{loop}, 0))$

Winding number



$w : \Omega(S^1) \to \mathbb{Z}$

$w(p) = \text{transport}_{\text{Cover}}(p, 0)$

lift p to cover, starting at 0

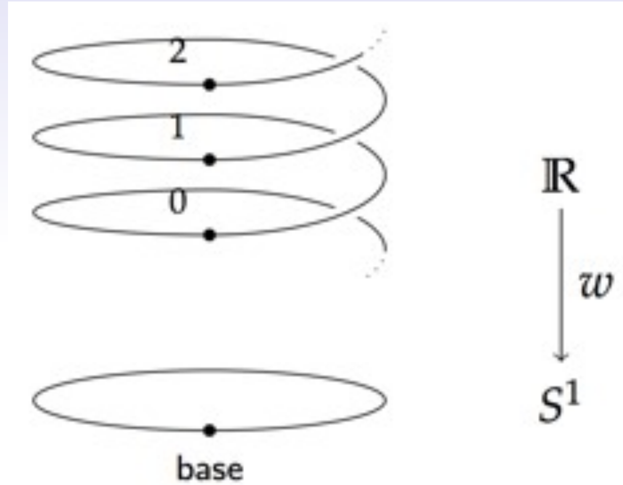
$w(\text{loop}^{-1} \circ \text{loop})$

$= \text{transport}_{\text{Cover}}(\text{loop}^{-1} \circ \text{loop}, 0)$

$= \text{transport}_{\text{Cover}}(\text{loop}^{-1}, \text{transport}_{\text{Cover}}(\text{loop}, 0))$

$= \text{transport}_{\text{Cover}}(\text{loop}^{-1}, 1)$

Winding number



$w : \Omega(S^1) \to \mathbb{Z}$

$w(p) = \mathrm{transport}_{\mathrm{Cover}}(p, 0)$

lift p to cover, starting at 0

$w(\mathrm{loop}^{-1} \circ \mathrm{loop})$

$= \mathrm{transport}_{\mathrm{Cover}}(\mathrm{loop}^{-1} \circ \mathrm{loop}, 0)$

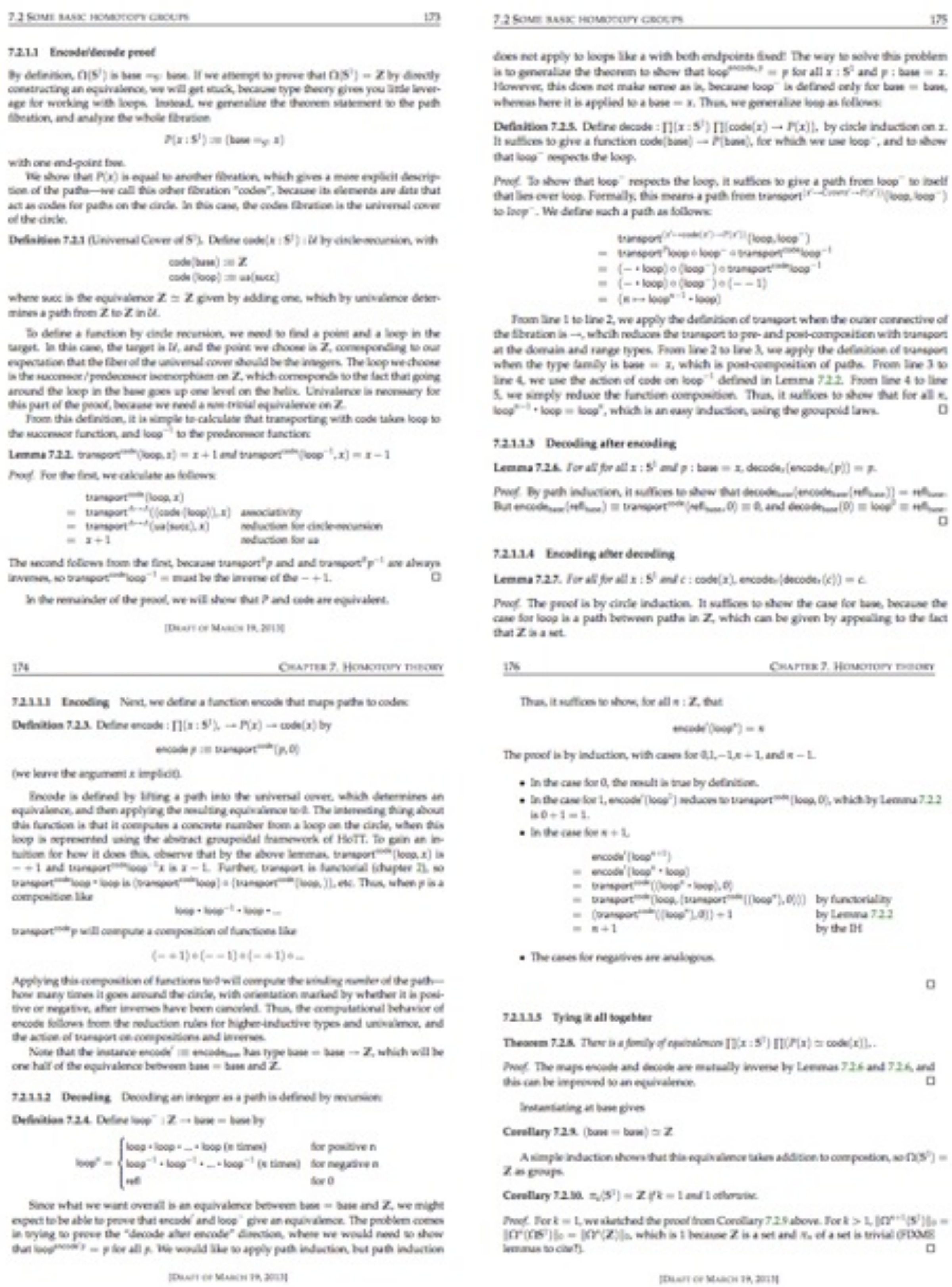
$= \mathrm{transport}_{\mathrm{Cover}}(\mathrm{loop}^{-1}, \mathrm{transport}_{\mathrm{Cover}}(\mathrm{loop}, 0))$

$= \mathrm{transport}_{\mathrm{Cover}}(\mathrm{loop}^{-1}, 1)$

$= 0$

Fundamental group of the circle

The book



Computer-checked

Outline

1. $\pi_1(S^1) = \mathbb{Z}$

2. **The Hopf fibration**

3. Connectedness and Freudenthal Suspension

The Hopf fibration

The Hopf fibration is a fibration with

- base $\mathbb{S}^2$
- fiber $\mathbb{S}^1$
- total space $\mathbb{S}^3$

$$\begin{array}{c} \mathbb{S}^1 \\ \big\downarrow \\ \mathbb{S}^3 \\ \big\downarrow \\ \mathbb{S}^2 \end{array}$$

The Hopf fibration

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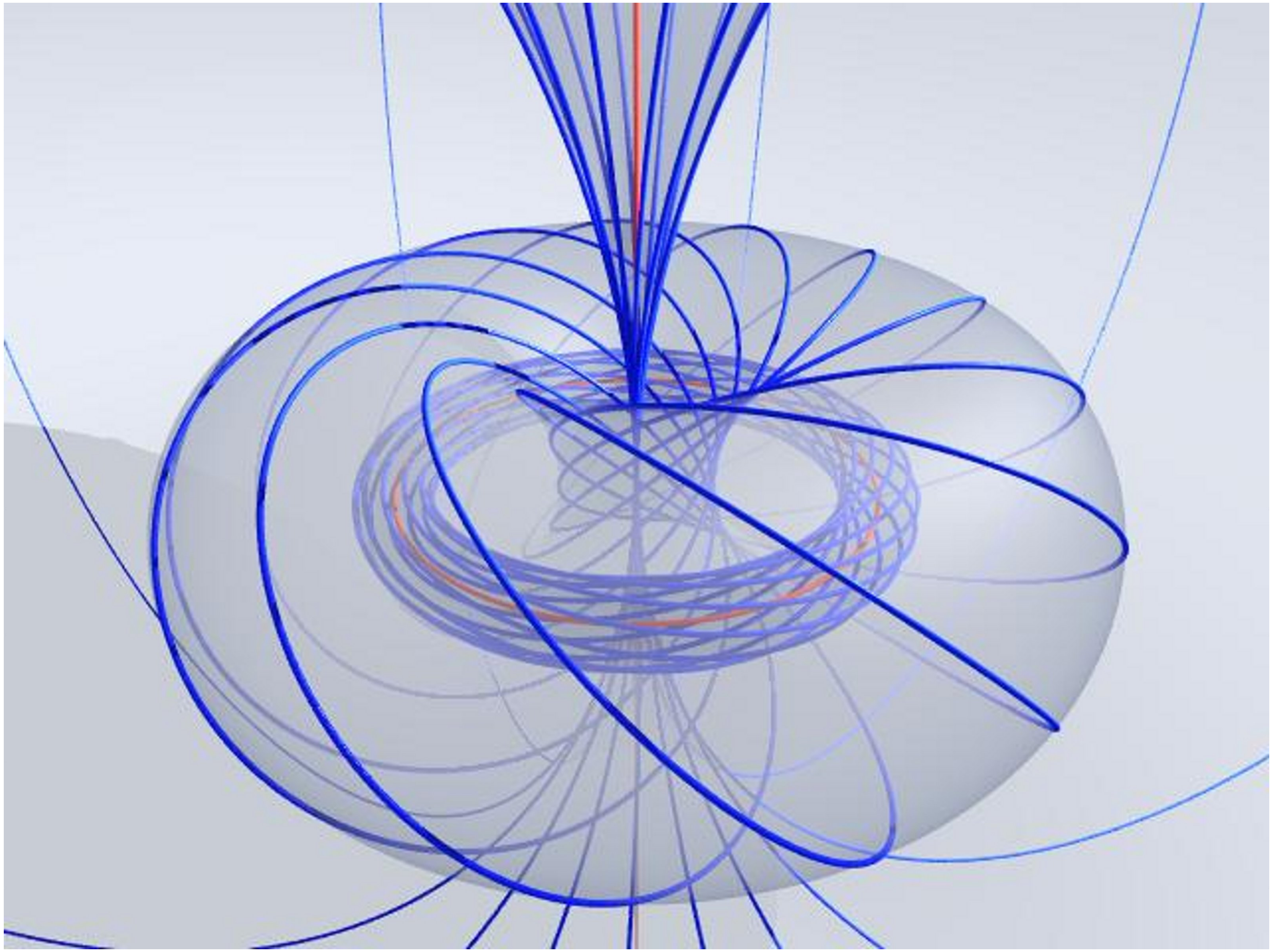
$$\begin{array}{c} \mathbb{S}^1 \\ \big\downarrow \\ \mathbb{S}^3 \\ \big\downarrow \\ \mathbb{S}^2 \end{array}$$

The Hopf fibration is a family of circles, parametrized by $\mathbb{S}^2$ and whose "union" is $\mathbb{S}^3$.

Picture



The spheres

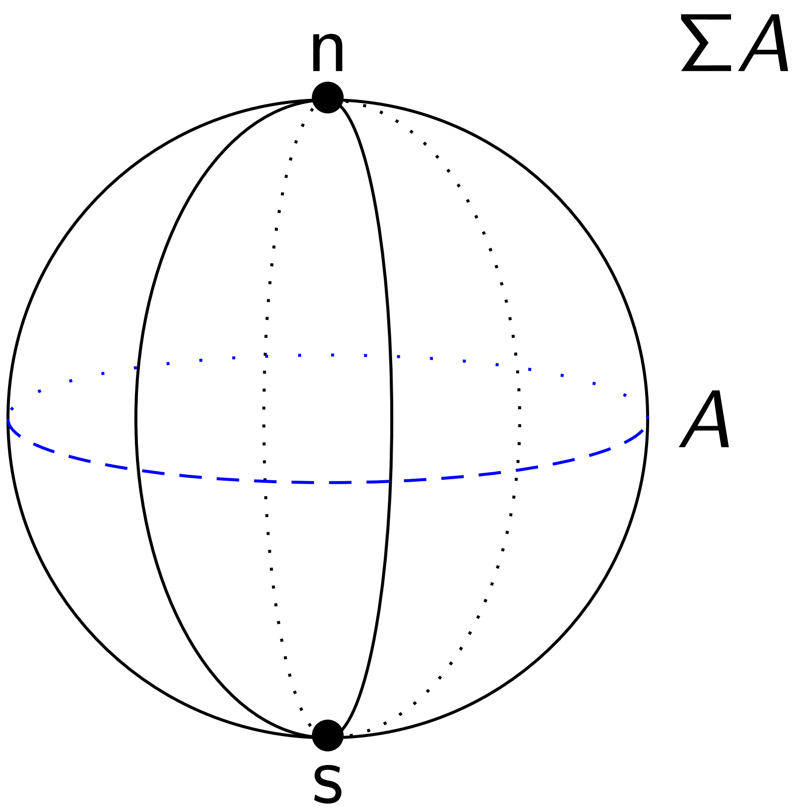
Definition

The suspension of a space A (denoted ΣA) is generated by

- Two points $\mathsf{n}, \mathsf{s} : \Sigma A$
- For every $a : A$, a path $\mathsf{m}(a) : \mathsf{n} =_{\Sigma A} \mathsf{s}$

Definition

$$\mathbb{S}^{n+1} := \Sigma \mathbb{S}^n$$



Fibrations over $\mathbb{S}^2$

A fibration over $\mathbb{S}^2$ is given by

- a space A (over n)

Fibrations over $\mathbb{S}^2$

A fibration over $\mathbb{S}^2$ is given by

- a space A (over n)
- a space B (over s)

Fibrations over $\mathbb{S}^2$

A fibration over $\mathbb{S}^2$ is given by

- a space A (over n)
- a space B (over s)
- a "circle of equivalences" between A and B (over m)

  $\iff$ a function $e : \mathbb{S}^1 \to (A \simeq B)$

  $\iff$ for every $x : \mathbb{S}^1$, an equivalence $e_x : A \simeq B$

The Hopf fibration in HoTT

A fibration over $\mathbb{S}^2$ with fiber $\mathbb{S}^1$ and total space $\mathbb{S}^3$?

The Hopf fibration in HoTT

A fibration over $\mathbb{S}^2$ with fiber $\mathbb{S}^1$ and total space $\mathbb{S}^3$?

- $\mathbb{S}^1$ over n
- $\mathbb{S}^1$ over s
- for $x : \mathbb{S}^1$, the equivalence $e_x : \mathbb{S}^1 \simeq \mathbb{S}^1$ is the "rotation of angle" x

The Hopf fibration in HoTT

A fibration over $\mathbb{S}^2$ with fiber $\mathbb{S}^1$ and total space $\mathbb{S}^3$?

- $\mathbb{S}^1$ over n
- $\mathbb{S}^1$ over s
- for $x : \mathbb{S}^1$, the equivalence $e_x : \mathbb{S}^1 \simeq \mathbb{S}^1$ is the "rotation of angle" x

Left to do:

- Define the rotation of angle x
- Prove that the total space is $\mathbb{S}^3$

Rotations of $\mathbb{S}^1$

We want

$$e : \mathbb{S}^1 \to (\mathbb{S}^1 \simeq \mathbb{S}^1)$$

Rotations of $\mathbb{S}^1$

We want

$$e : \mathbb{S}^1 \to (\mathbb{S}^1 \simeq \mathbb{S}^1)$$

By definition of $\mathbb{S}^1$, we need

- an equivalence $e_{\mathsf{base}} : \mathbb{S}^1 \simeq \mathbb{S}^1$
- a homotopy $e(\mathsf{loop}) : e_{\mathsf{base}} = e_{\mathsf{base}}$

Rotations of $\mathbb{S}^1$

We want

$$e : \mathbb{S}^1 \to (\mathbb{S}^1 \simeq \mathbb{S}^1)$$

By definition of $\mathbb{S}^1$, we need

- an equivalence $\mathrm{id}_{\mathbb{S}^1} : \mathbb{S}^1 \simeq \mathbb{S}^1$
- a homotopy $e(\mathsf{loop}) : \mathrm{id}_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1}$

Rotations of $\mathbb{S}^1$

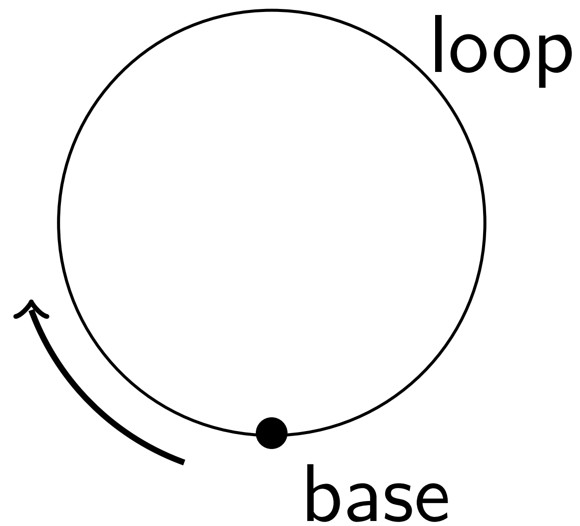
We want

$$e : \mathbb{S}^1 \to (\mathbb{S}^1 \simeq \mathbb{S}^1)$$

By definition of $\mathbb{S}^1$, we need

- an equivalence $\mathrm{id}_{\mathbb{S}^1} : \mathbb{S}^1 \simeq \mathbb{S}^1$
- a homotopy $e(\text{loop}) : \mathrm{id}_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1}$

e(loop) is the homotopy "turning once around the circle".



Homotopy turning once around the circle

A homotopy $\mathrm{id}_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1} \iff$ for every $x : \mathbb{S}^1$, a path $x = x$

Homotopy turning once around the circle

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We need:

- a path

$$p : \mathsf{base} = \mathsf{base}$$

- a (2-dimensional) path

$$q : p \cdot \mathsf{loop} = \mathsf{loop} \cdot p$$

Homotopy turning once around the circle

A homotopy $\mathrm{id}_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1} \iff$ for every $x : \mathbb{S}^1$, a path $x = x$

We need:

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$$\mathsf{loop} : \mathsf{base} = \mathsf{base}$$

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$$q : \mathsf{loop} \cdot \mathsf{loop} = \mathsf{loop} \cdot \mathsf{loop}$$

Homotopy turning once around the circle

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We need:

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$$\mathsf{loop} : \mathsf{base} = \mathsf{base}$$

- a (2-dimensional) path

$$\mathsf{refl}_{\mathsf{loop} \cdot \mathsf{loop}} : \mathsf{loop} \cdot \mathsf{loop} = \mathsf{loop} \cdot \mathsf{loop}$$

Total space

We just constructed a fibration with

- base $\mathbb{S}^2$
- fiber $\mathbb{S}^1$

What is the total space?

Homotopy pushouts

Given a span

$$Y \xleftarrow{f} X \xrightarrow{g} Z$$

Definition

The homotopy pushout $Y \sqcup^X Z$ is the space generated by

- For all $y : Y$, a point $\mathsf{l}(y) : Y \sqcup^X Z$
- For all $z : Z$, a point $\mathsf{r}(z) : Y \sqcup^X Z$
- For all $x : X$, a path $\mathsf{g}(x) : \mathsf{l}(f(x)) = \mathsf{r}(g(x))$

The suspension of A is the homotopy pushout of

$$1 \longleftarrow A \longrightarrow 1$$

Total space

By gluing/descent/flattening, the total space is the homotopy pushout of:

$$\mathbb{S}^1 \xleftarrow{e} \mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow{p_2} \mathbb{S}^1$$

Total space

By gluing/descent/flattening, the total space is the homotopy pushout of:

$$\mathbb{S}^1 \xleftarrow{e} \mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow{p_2} \mathbb{S}^1$$

This span is equivalent to the following:

$$\mathbb{S}^1 \xleftarrow{p_1} \mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow{p_2} \mathbb{S}^1$$

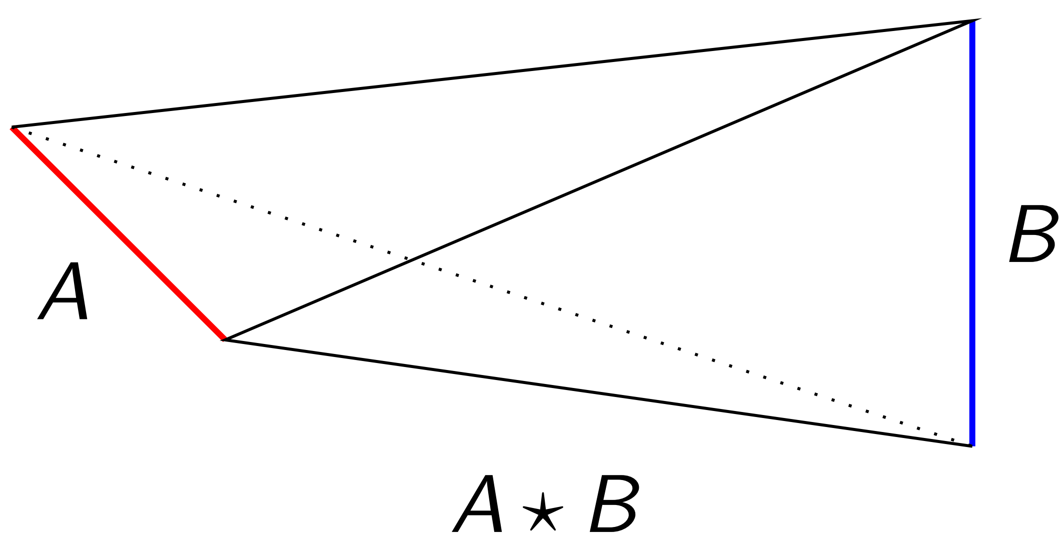
whose total space is $\mathbb{S}^1 \star \mathbb{S}^1$

Join

Definition

The join of A and B is the homotopy pushout of

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$$



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Definition

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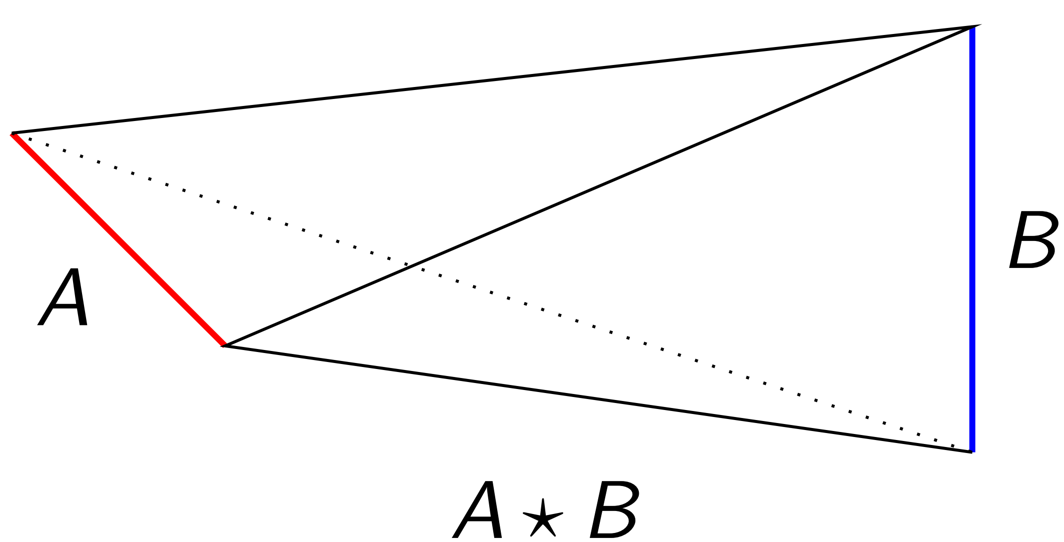


We have

$$\mathbb{S}^0 \star A = \Sigma A$$

$$(A \star B) \star C = A \star (B \star C)$$

Total space

$$\begin{aligned}
\mathbb{S}^1 \star \mathbb{S}^1 &= (\Sigma\mathbb{S}^0) \star \mathbb{S}^1 \\
&= (\mathbb{S}^0 \star \mathbb{S}^0) \star \mathbb{S}^1 \\
&= \mathbb{S}^0 \star (\mathbb{S}^0 \star \mathbb{S}^1) \\
&= \Sigma(\Sigma\mathbb{S}^1) \\
&= \mathbb{S}^3
\end{aligned}$$

Total space

$$
\begin{aligned}
\mathbb{S}^1 \star \mathbb{S}^1 &= (\Sigma\mathbb{S}^0) \star \mathbb{S}^1 \\
&= (\mathbb{S}^0 \star \mathbb{S}^0) \star \mathbb{S}^1 \\
&= \mathbb{S}^0 \star (\mathbb{S}^0 \star \mathbb{S}^1) \\
&= \Sigma(\Sigma\mathbb{S}^1) \\
&= \mathbb{S}^3
\end{aligned}
$$

We have the Hopf fibration in homotopy type theory.

Long exact sequence

Long exact sequence of homotopy groups of the Hopf fibration:

$$\vdots$$

$$\pi_4(\mathbb{S}^1) \longrightarrow \pi_4(\mathbb{S}^3) \longrightarrow \pi_4(\mathbb{S}^2)$$

$$\pi_3(\mathbb{S}^1) \longrightarrow \pi_3(\mathbb{S}^3) \longrightarrow \pi_3(\mathbb{S}^2)$$

$$\pi_2(\mathbb{S}^1) \longrightarrow \pi_2(\mathbb{S}^3) \longrightarrow \pi_2(\mathbb{S}^2)$$

$$\pi_1(\mathbb{S}^1) \longrightarrow \pi_1(\mathbb{S}^3) \longrightarrow \pi_1(\mathbb{S}^2)$$

Long exact sequence

Long exact sequence of homotopy groups of the Hopf fibration:

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & & \swarrow & \\
0 & \longrightarrow & \pi_4(\mathbb{S}^3) & \longrightarrow & \pi_4(\mathbb{S}^2) \\
 & & & \swarrow & \\
0 & \longrightarrow & \pi_3(\mathbb{S}^3) & \longrightarrow & \pi_3(\mathbb{S}^2) \\
 & & & \swarrow & \\
0 & \longrightarrow & \pi_2(\mathbb{S}^3) & \longrightarrow & \pi_2(\mathbb{S}^2) \\
 & & & \swarrow & \\
\mathbb{Z} & \longrightarrow & \pi_1(\mathbb{S}^3) & \longrightarrow & \pi_1(\mathbb{S}^2)
\end{array}$$

Long exact sequence

Long exact sequence of homotopy groups of the Hopf fibration:

$$
\begin{array}{ccccc}
 & & \vdots & & \\
0 & \longrightarrow & \pi_4(\mathbb{S}^3) & \longrightarrow & \pi_4(\mathbb{S}^2) \\
0 & \longrightarrow & \pi_3(\mathbb{S}^3) & \longrightarrow & \pi_3(\mathbb{S}^2) \\
0 & \longrightarrow & {\color{red}0} & \longrightarrow & \pi_2(\mathbb{S}^2) \\
\mathbb{Z} & \longrightarrow & {\color{red}0} & \longrightarrow & {\color{red}0}
\end{array}
$$

Long exact sequence

Long exact sequence of homotopy groups of the Hopf fibration:

$$
\begin{array}{ccccc}
 & & \vdots & & \\
0 & \longrightarrow & \pi_4(\mathbb{S}^3) & \xrightarrow{\sim} & \pi_4(\mathbb{S}^2) \\
0 & \longrightarrow & \pi_3(\mathbb{S}^3) & \xrightarrow{\sim} & \pi_3(\mathbb{S}^2) \\
0 & \longrightarrow & 0 & \longrightarrow & \pi_2(\mathbb{S}^2) \\
 & & \sim & & \\
\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

Homotopy groups

Theorem

We have

$$\pi_2(\mathbb{S}^2) = \mathbb{Z}$$

$$\pi_k(\mathbb{S}^2) = \pi_k(\mathbb{S}^3) \textit{ for } k \geq 3$$

In particular

Theorem

Assuming $\pi_3(\mathbb{S}^3) = \mathbb{Z}$

$$\pi_3(\mathbb{S}^2) = \mathbb{Z}$$

$\pi_4(\mathbb{S}^3)$

Theorem

There exists a natural number n such that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}$.

$\pi_4(\mathbb{S}^3)$

Theorem

There exists a natural number n such that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}$.

- Classical mathematics: cannot compute n, unless the proof is nice enough

$\pi_4(\mathbb{S}^3)$

Theorem

There exists a natural number n such that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}$.

- Classical mathematics: cannot compute n, unless the proof is nice enough
- Constructive mathematics: disallow the axiom of choice and excluded middle $\implies$ every proof is nice enough

$\pi_4(\mathbb{S}^3)$

Theorem

There exists a natural number n such that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}$.

- Classical mathematics: cannot compute n, unless the proof is nice enough
- Constructive mathematics: disallow the axiom of choice and excluded middle $\implies$ every proof is nice enough

In this case we can compute the value of n and get 2*

*work in progress

Outline

1. $\pi_1(S^1) = \mathbb{Z}$
2. The Hopf fibration
3. **Connectedness and Freudenthal Suspension**

Part III: Freudenthal and friends

1. Truncatedness

2. Connectedness

3. Freudenthal Suspension Theorem

Truncatedness

Definition

A type X is n-truncated (or an n-type) if, by induction on $n \geq -2$:

- $n = -2$: if X is contractible, i.e. $X \simeq 1$;
- $n > -2$: if each path space $(x =_X x')$ of X is $(n-1)$-truncated.

Proposition

Suppose X is n-truncated, for $n \geq -1$. Then $\pi_k(X, x_0) \simeq 1$, for all $k > n$ and $x_0 : X$.

[In **Top** and **SSet**, the converse holds; but not in all classical settings, cf. Whitehead's theorem and hypercompleteness.]

Truncations

Definition

For any type X, and $n \geq -1$, the n-truncation $\tau_n X$ is the higher inductive type generated by:

- for $x : X$, an element $[x]_n : \tau_n X$;
- for $f : \mathbb{S}^{n+1} \to \tau_n X$, and $t : \mathbb{S}^{n+1}$, a path $f(t) = f(0)$.

Proposition

$\tau_n X$ is the free n-truncated type on X: any $f : X \to Y$, with Y n-truncated, factors uniquely through $\tau_n X$.

[Classically: iteratively glue cells on to X to kill homotopy in dimensions $> n$.]

Connectedness (of types)

Definition

X is *n-connected* if $\tau_{n+1}X$ is contractible.

Proposition

TFAE:

- *X is n-connected;*
- *every map from X to an n-type is constant;*
- *(when $n \geq 0$) $\pi_k(X, x_0) \simeq 1$, for all $k \leq n$ and $x_0 : X$.*

Connectedness (trivial low homotopy groups) is dual to truncatedness (trivial high homotopy groups).

Connectedness (of maps)

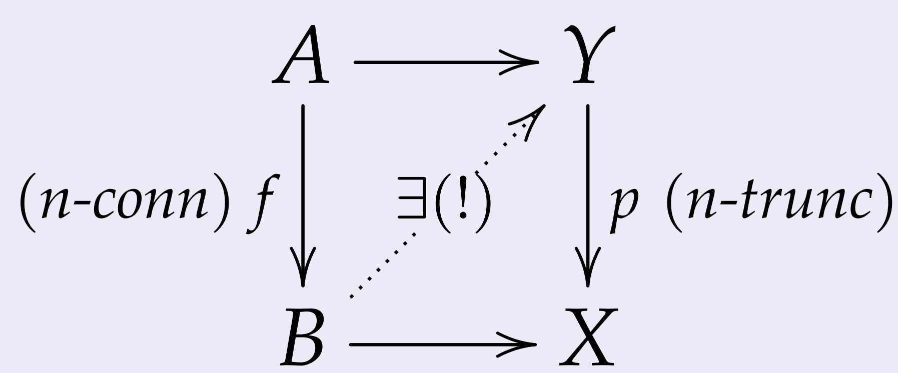
Definition

$f\colon A \to B$ is n-connected if each (homotopy) fiber $f^{-1}(b)$ is n-connected. (Warning: indexing conventions vary by ± 1.)

Proposition

TFAE:

- *f is n-connected;*
- *f is weakly (or strongly) orthogonal to maps with n-truncated fibers;*

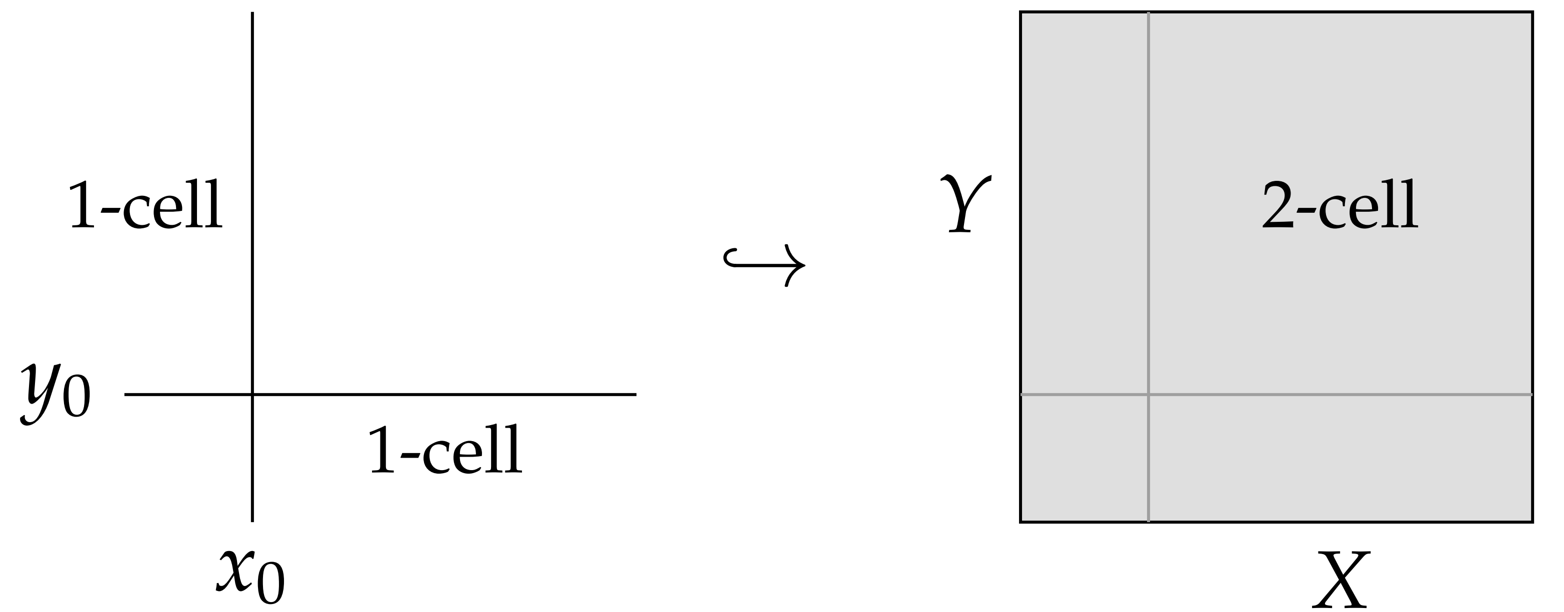


- *f is equivalent to the inclusion of A into some extension by cells of dimensions $> n$.*

Additivity of connectedness

Lemma (Wedge-product connectedness)

Suppose (X, x_0) *is i-connected,* (Y, y_0) *is j-connected. Then the inclusion* $X \sqcup_1 Y \hookrightarrow X \times Y$ *is* $(i + j)$*-connected.*



Type-theoretically: to define a function of two variables $f(x, y)$ into an $(i + j)$-type, enough to define in the cases $f(x_0, y)$ and $f(x, y_0)$, agreeing in the case $f(x_0, y_0)$.

Freudenthal

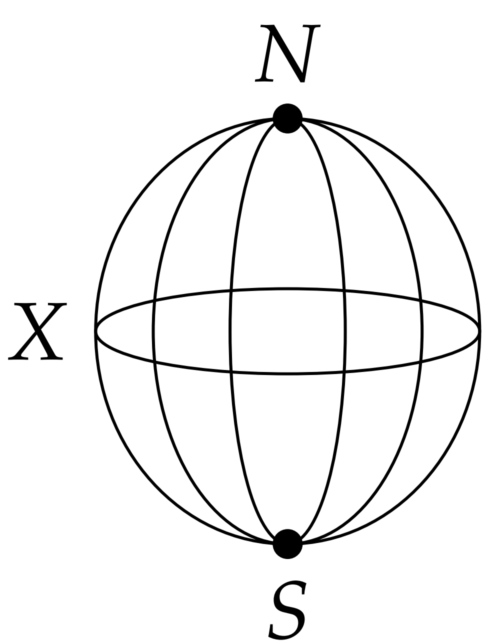
Definition

Recall: the suspension ΣX is generated by

- $N, S : \Sigma X$;
- for each $x : X$, a path $m(x) : N =_{\Sigma X} S$.

Theorem (Freudenthal Suspension Theorem)

Suppose (X, x_0) is n-connected. Then the canonical map $X \to \Omega(\Sigma X, N)$ is $2n$-connected.



Idea: want $X \to \Omega(\Sigma X, N)$ to be an equivalence. Generally (e.g. for $\Sigma \mathbb{S}^1 \simeq \mathbb{S}^2$) it isn't; but within a certain dimension range, it is.

Important application: stable homotopy groups of spheres.

Proof: weak Freudenthal

For now, prove a weaker statement. (Same approach, with more work, yields full FST.)

Theorem (Weak Freudenthal)

Suppose (X, x_0) is n-connected. Then the canonical map $\tau_{2n}(X) \to \tau_{2n}\Omega(\Sigma X, N)$ is an equivalence.

Proof.

Heuristic: to prove a result of the form $X \approx \Omega(Y, y_0)$, generalise X to a dependent type $\bar{X}_y$ over $y : Y$, with $\bar{X}_{y_0} \simeq X$, and prove $\bar{X}_y \approx (y_0 =_Y y)$ for all $y : Y$.

So: define type $\bar{X}_y$ depending on $y : \Sigma X$, and maps $\bar{m}_y : \bar{X}_y \to \tau_{2n}(N = y)$, using universal property of ΣX.

Weak Freudenthal, cont'd

Proof.

To give $\bar{X}_y, \bar{m}_y$ for all $y : \Sigma X$, need:

- types and maps $\bar{m}_N : \bar{X}_N \to \tau_{2n}(N = N)$, and $\bar{m}_S : \bar{X}_S \to \tau_{2n}(N = S)$;
- transport equivalences $\mathsf{transport}_{\bar{X}} m(x_1) : \bar{X}_N \to \bar{X}_S$, for each $x_1 : X$, commuting with $\bar{m}_N, \bar{m}_S$.

over S: $\quad \bar{m}_S := \tau_{2n}(m) : \tau_{2n}(X) \to \tau_{2n}(N = S)$

over N: $\quad \bar{m}_N := \tau_{2n}(x \mapsto m(x) \circ m(x_0)^{-1}) : \tau_{2n}(X) \to \tau_{2n}(N = N)$

and over $m(x)$, need to define for each $x_1 : X$ the action $\mathsf{transport}_{\bar{X}}(m(x), -) : \bar{X}_N \to \bar{X}_S$.

Weak Freudenthal, cont'd

Proof.

…transport over $m(x_1)$: need to give, for each $x_1 : X$ and $z : \bar{X}_N = \tau_{2n}(X)$, some element of $\bar{X}_S = \tau_{2n}(X)$.

Since RHS is $2n$-truncated, may assume z is of form $[x_2]$, some $x_2 : X$. Also, by wedge-product connectedness lemma, enough to assume one of x_1, x_2 is x_0. So: when $x_1 = x_0$, return $[x_2]$. When $x_2 = x_0$, return $[x_1]$. (Check: when $x_1 = x_2 = x_0$, these agree)

(Roughly: defining a multiplication $X \times \tau_{2n}(X) \to \tau_{2n}(X)$, with x_0 as a two-sided unit.)

So: have $\bar{m}_y : \bar{X}_y \to (N = y)$, for all $y : \Sigma X$.

Define converse $\bar{n}_y : (N = y) \to \bar{X}_y$ by $n_y(p) := \mathsf{transport}_{\bar{X}}[x_0]$. Not hard to prove $\bar{m}, \bar{n}$ mutually inverse; so, each $\bar{m}_y$ is an equivalence, as desired. □

Consequences

From (weak) Freudenthal, immediately have:

Corollary (Homotopy groups of spheres stabilise)

$\pi_{n+k}(\mathbb{S}^n) \simeq \pi_{n+1+k}(\mathbb{S}^{n+1})$, *for* $n \geq k+2$.

In particular,

Corollary

$\pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$, *for all* $n \geq 1$.

Proof.

- $n = 1$: by universal cover.
- $n = 2$: by LES of Hopf fibration.
- $n \geq 2$: by stabilisation. □

$\pi_k(S^n)$ in HoTT

k^{th} homotopy group

n-dimensional sphere

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	**Z**	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	**Z**	**Z**	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12} \times Z_2$	$Z_{84} \times Z_2^2$	Z_2^2
S^3	0	0	**Z**	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12} \times Z_2$	$Z_{84} \times Z_2^2$	Z_2^2
S^4	0	0	0	**Z**	Z_2	Z_2									
S^5	0	0	0	0	**Z**	Z_2	Z_2	Z_{24}							
S^6	0	0	0	0	0	**Z**	Z_2	Z_2	Z_{24}	0					
S^7	0	0	0	0	0	0	**Z**	Z_2	Z_2	Z_{24}	0	0			
S^8	0	0	0	0	0	0	0	**Z**	Z_2	Z_2	Z_{24}	0	0	Z_2	

[image from wikipedia]

More results

James construction

Refinement of Freudenthal: describes $\Omega(\Sigma X)$ precisely, via a filtration.

Theorem

Suppose (X, x_0) is n-connected, for $n \geq 0$. There is a sequence

$$1 \longrightarrow X \longrightarrow J_2(X) \longrightarrow J_3(X) \longrightarrow J_4(X) \longrightarrow \cdots$$

with the maps having respective connectivities $(n-1)$, $2n$, $(3n+1)$, $\ldots$, and such that $J_\infty(X) := \varinjlim_n J_n(X) \simeq \Omega(\Sigma X)$.

Conceptually, $J_\infty(X)$ is the free monoid on X; as X is connected, this is the free group on X.

Blakers–Massey

Generalization of Freudenthal: describes path spaces in pushouts.

Theorem (Blakers–Massey theorem)

Suppose given maps f, g as below, with f i-connected, g j-connected.

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{inr}} \\ X & \xrightarrow{\mathsf{inl}} & X \sqcup_Z Y \end{array}$$

Then for all $x : X$, $y : Y$, the canonical map $Z_{x,y} \to (\mathsf{inl}\, x = \mathsf{inr}\, y)$ is $(i + j)$-connected.

van Kampen

Another tool for pushouts of types:

Theorem (van Kampen theorem)

For any pointed maps $f : Z \to X$ and $g : Z \to Y$, with Z 0-connected, the fundamental group of the pushout of f and g is the amalgamated free product (pushout of groups) of $\pi_1(X)$ and $\pi_1(Y)$ over $\pi_1(Z)$:

$$\pi_1(X \sqcup_Z Y) \simeq \pi_1(X) *_{\pi_1(Z)} \pi_1(Y).$$

Can also be generalised to non-connected Z.

Covering spaces

The (beautiful) classical theory of covering spaces transfers straightforwardly. In particular:

Definition

A covering space of a connected type X is a dependent family of 0-types over X.

Theorem

Covering spaces of X correspond to sets with an action of $\pi_1(X)$.

Eilenberg–Mac Lane spaces; cohomology

Eilenberg–Mac Lane spaces of Abelian groups can be constructed as HIT's:

Theorem

For any (n-truncated) Abelian group G and natural number $n > 0$, there is a type $K(G, n)$ such that $\pi_n(K(G, n)) \simeq G$, and $\pi_n(K(G, n)) \simeq 1$ for $k \neq n$.

These (and other spectra) can be used to define cohomology of types.

Conclusion

We can do
computer-checked proofs
in **synthetic** homotopy theory

January 14, 2013

$\pi_1(S^1) = \mathbb{Z}$

$\pi_{k<n}(S^n) = 0$

April 11, 2013

$\pi_1(S^1) = \mathbb{Z}$

$\pi_{k<n}(S^n) = 0$

Hopf fibration

$\pi_2(S^2) = \mathbb{Z}$

$\pi_3(S^2) = \mathbb{Z}$

James Construction

$\pi_4(S^3) = \mathbb{Z}_?$

Freudenthal

$\pi_n(S^n) = \mathbb{Z}$

K(G,n)

Cohomology axioms

Blakers-Massey

Van Kampen

Covering spaces

Whitehead for n-types