Computer-Checked Proofs in the Logic of Homotopy Theory

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Homotopy theory

A branch of topology,
the study of spaces and continuous deformations

[image from wikipedia]

Homotopy

Deformation of one path into another

a

p

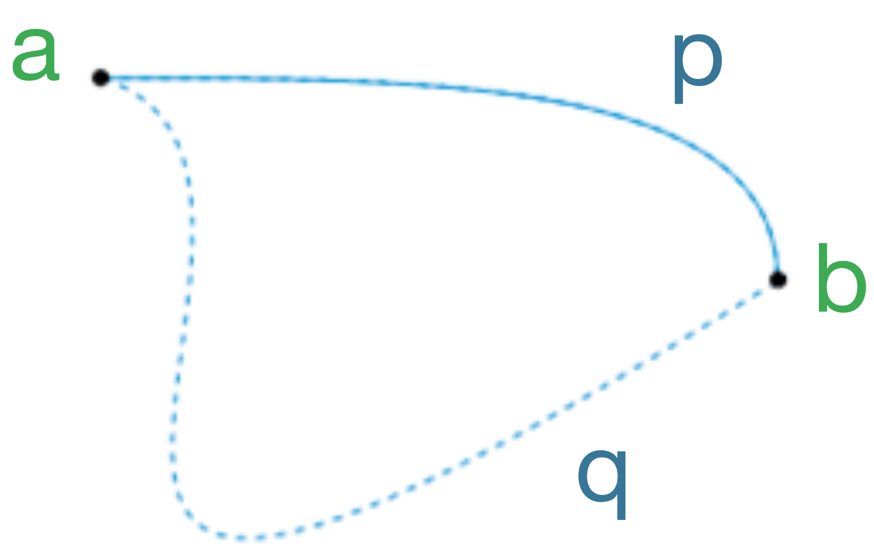
b

q

[image from wikipedia]

Homotopy

Deformation of one path into another



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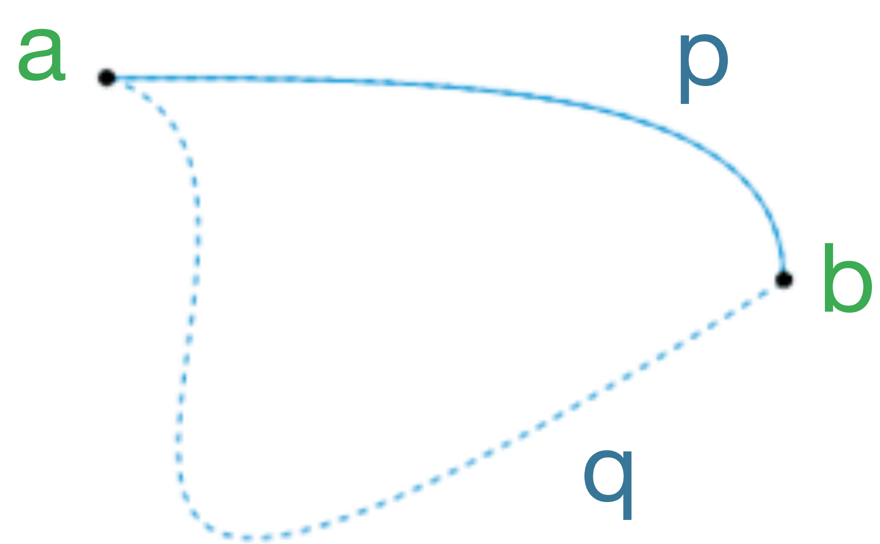

= 2-dimensional *path between paths*

[image from wikipedia]

Homotopy

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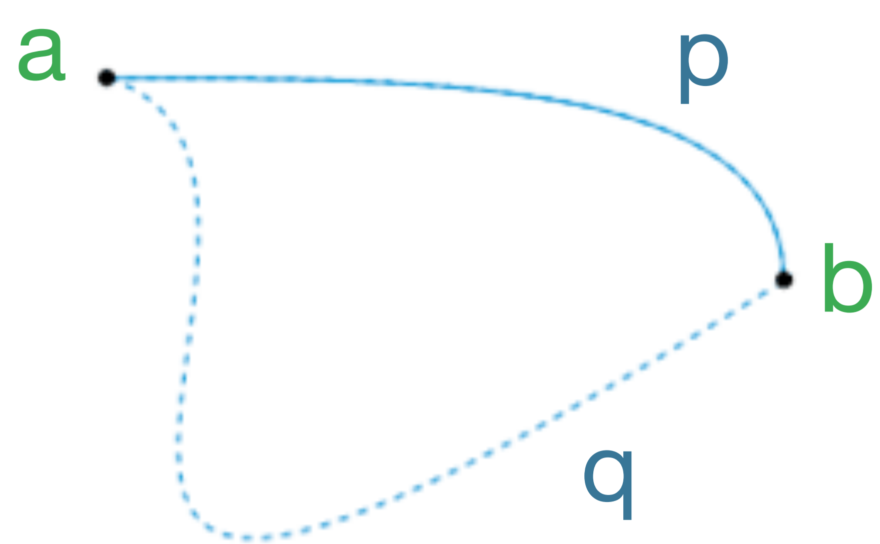


= 2-dimensional *path between paths*

Homotopy theory is the study of spaces by way of their paths, homotopies, homotopies between homotopies, ….

[image from wikipedia]

Synthetic vs Analytic

Synthetic geometry (Euclid)

POSTULATES.

I.

LET it be granted that a straight line may be drawn from any one point to any other point.

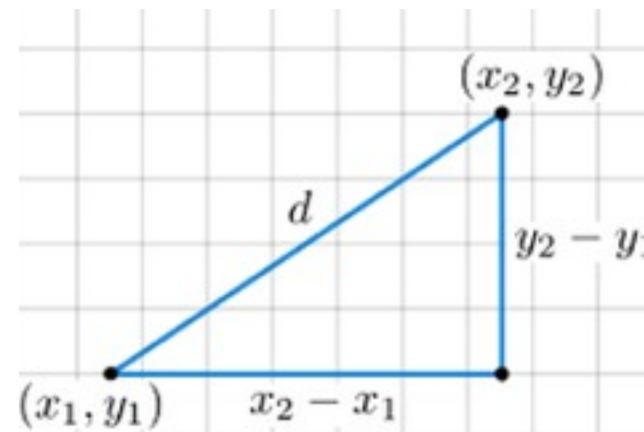
II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any centre, at any distance from that centre.

Analytic geometry (Descartes)



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Synthetic vs Analytic

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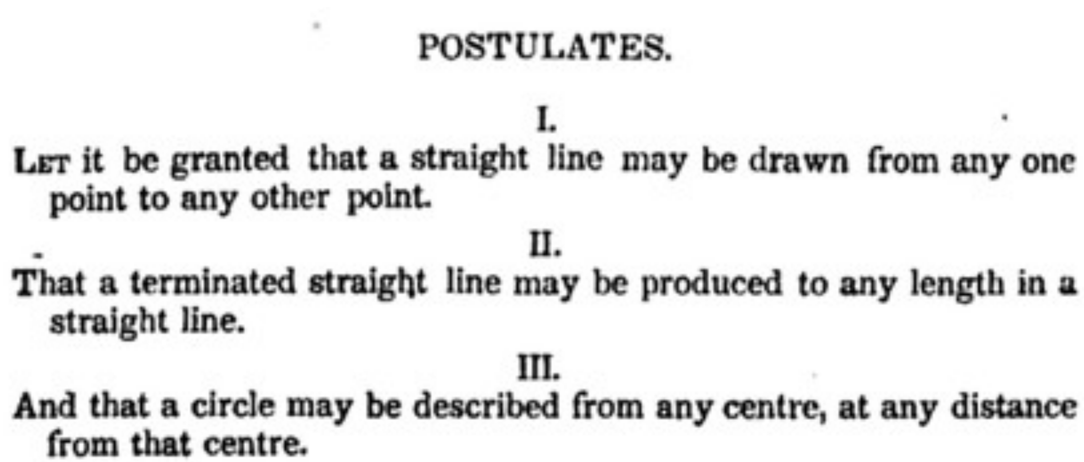
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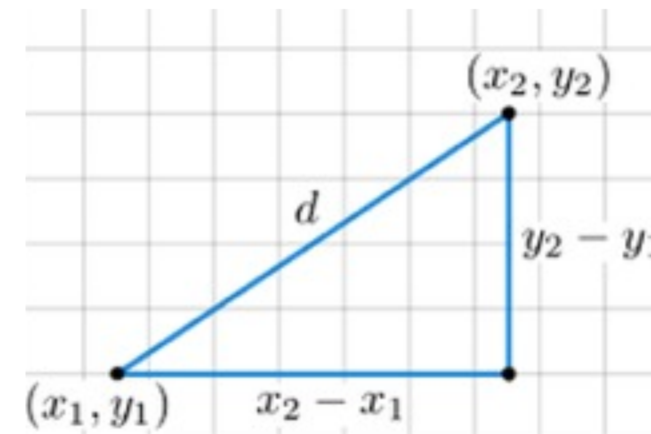

Classical homotopy theory is analytic:

- a space is a set of points equipped with a topology
- a path is a continuous map [0,1] → X

[image from wikipedia]

Synthetic homotopy theory

homotopy theory	**type theory**
space	<type>
points	<element> : <type>
paths	<proof> : <elem$_1$> = <elem$_2$>
homotopies	<2-proof> : <proof$_1$> = <proof$_2$>
⋮	⋮

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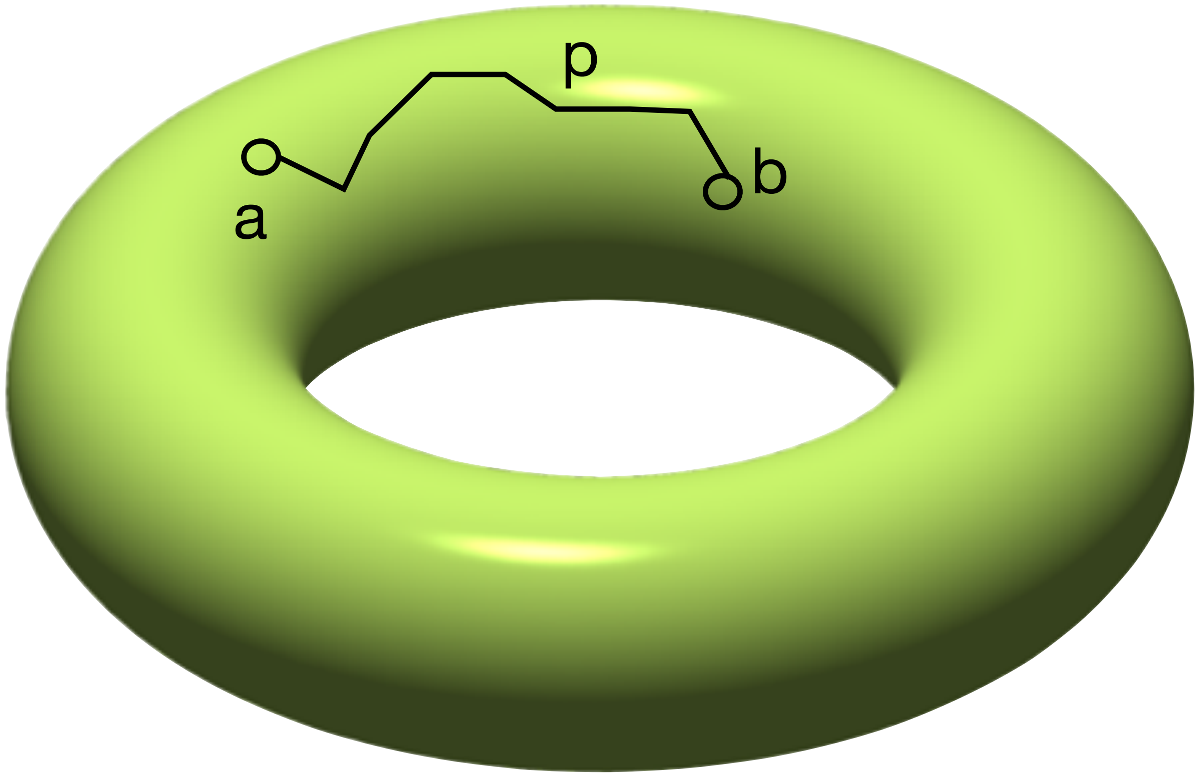
*A path is **not** a map [0,1]→X; it is a basic notion*

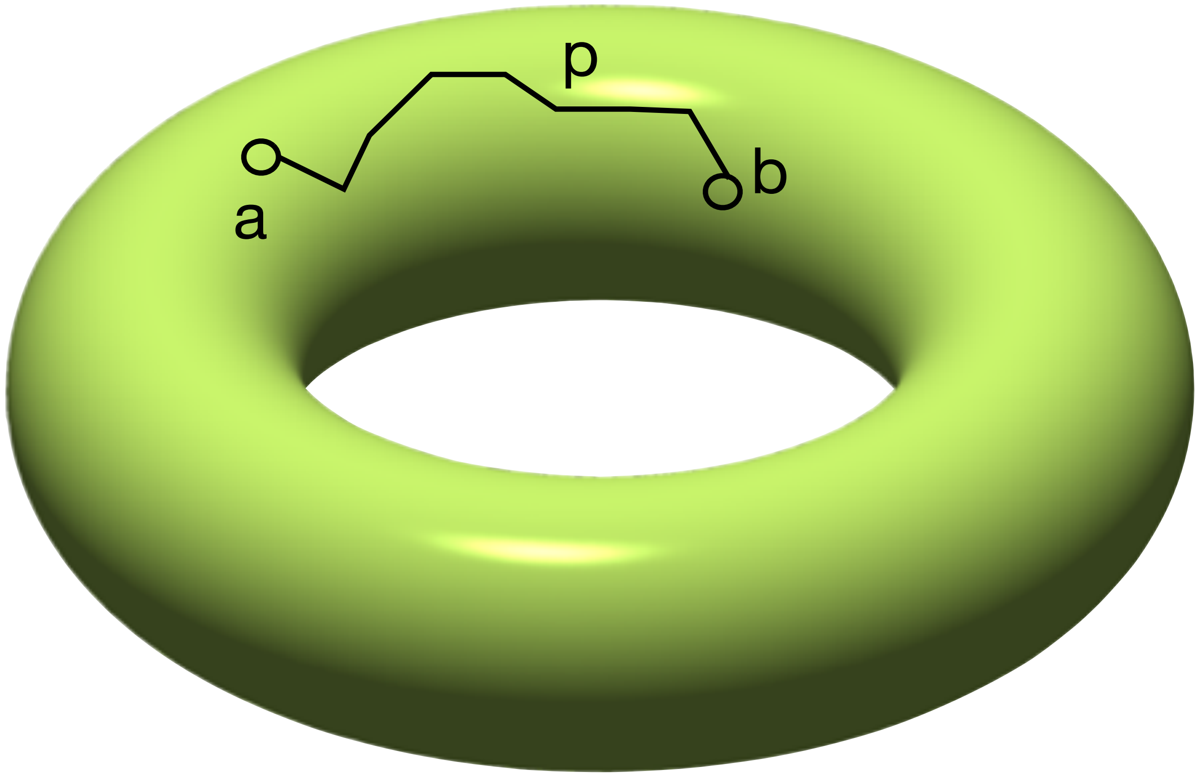
Id(<elem1>,<elem2>)

Spaces as types



Spaces as types

a space is a type A



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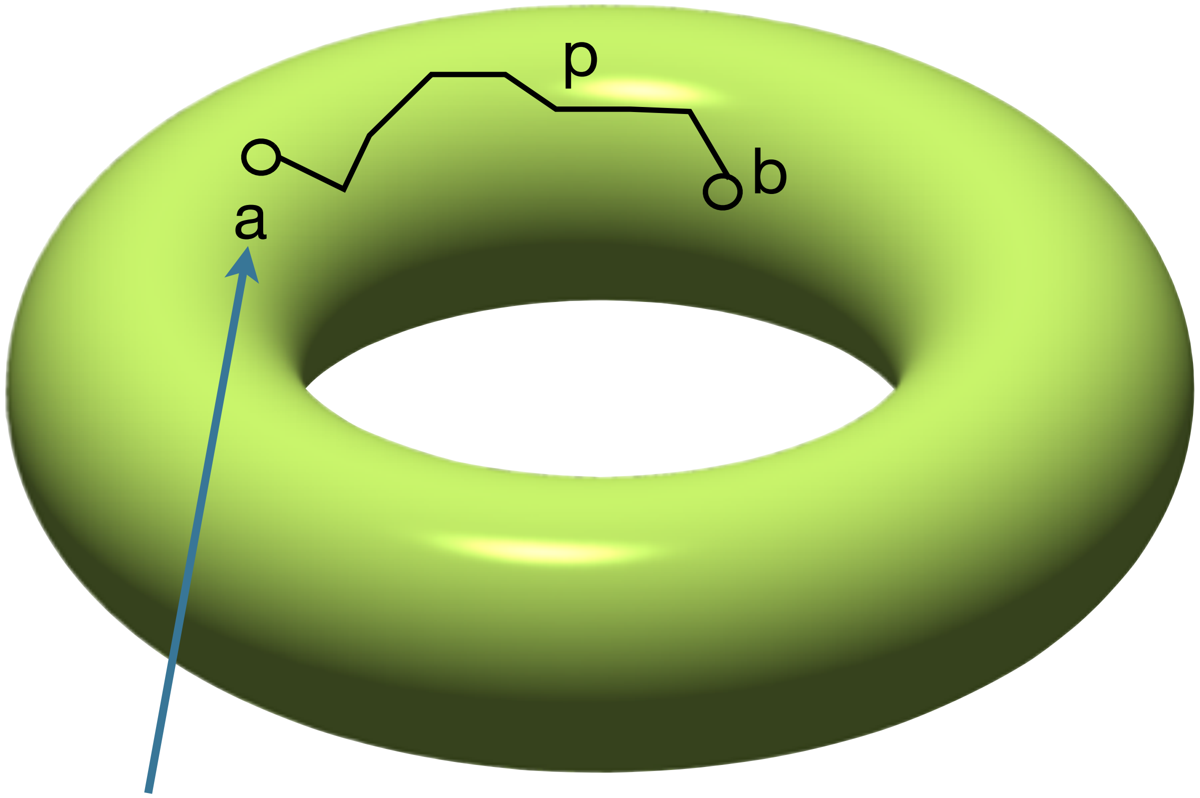
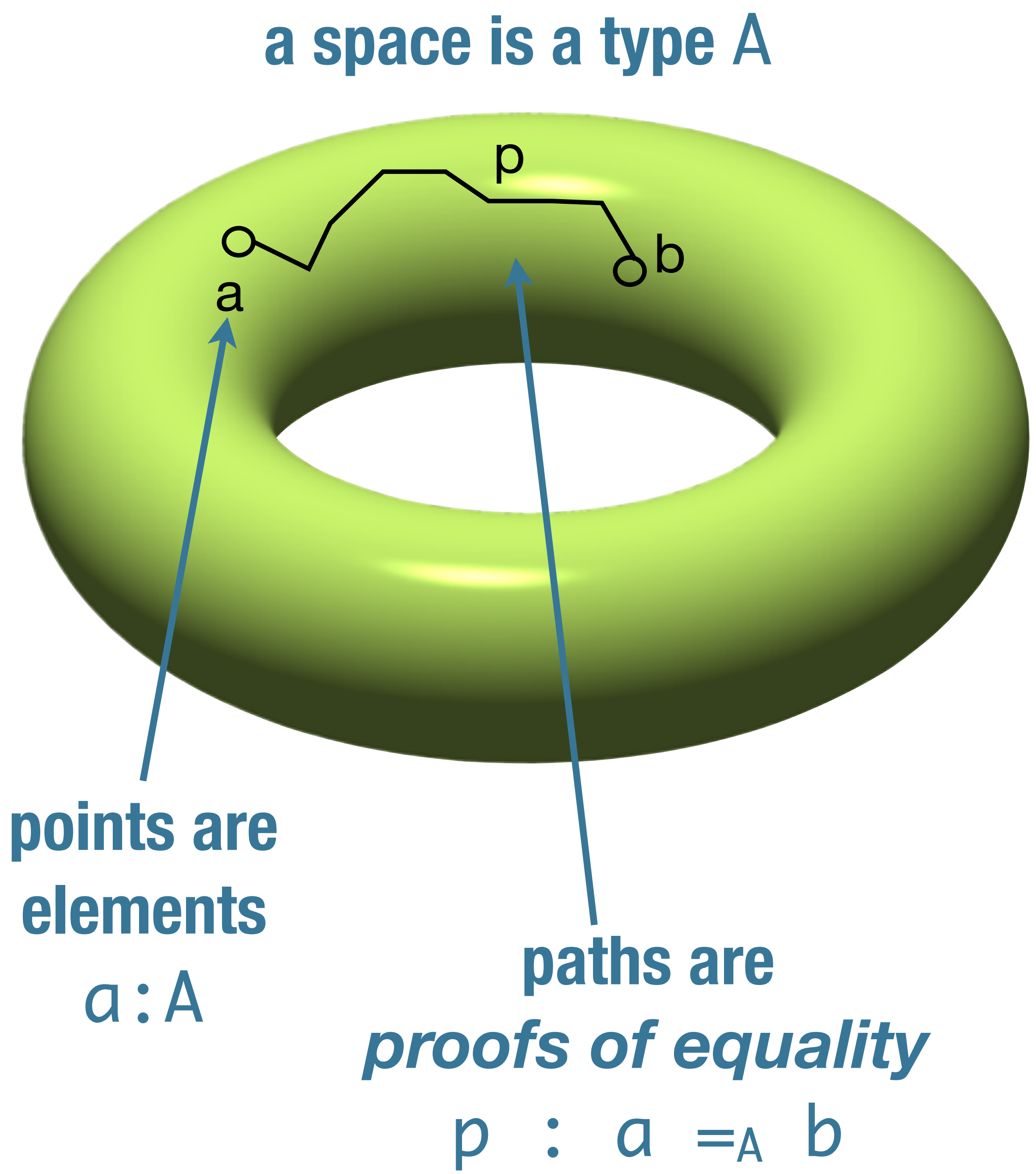


points are elements

$a : A$

Spaces as types



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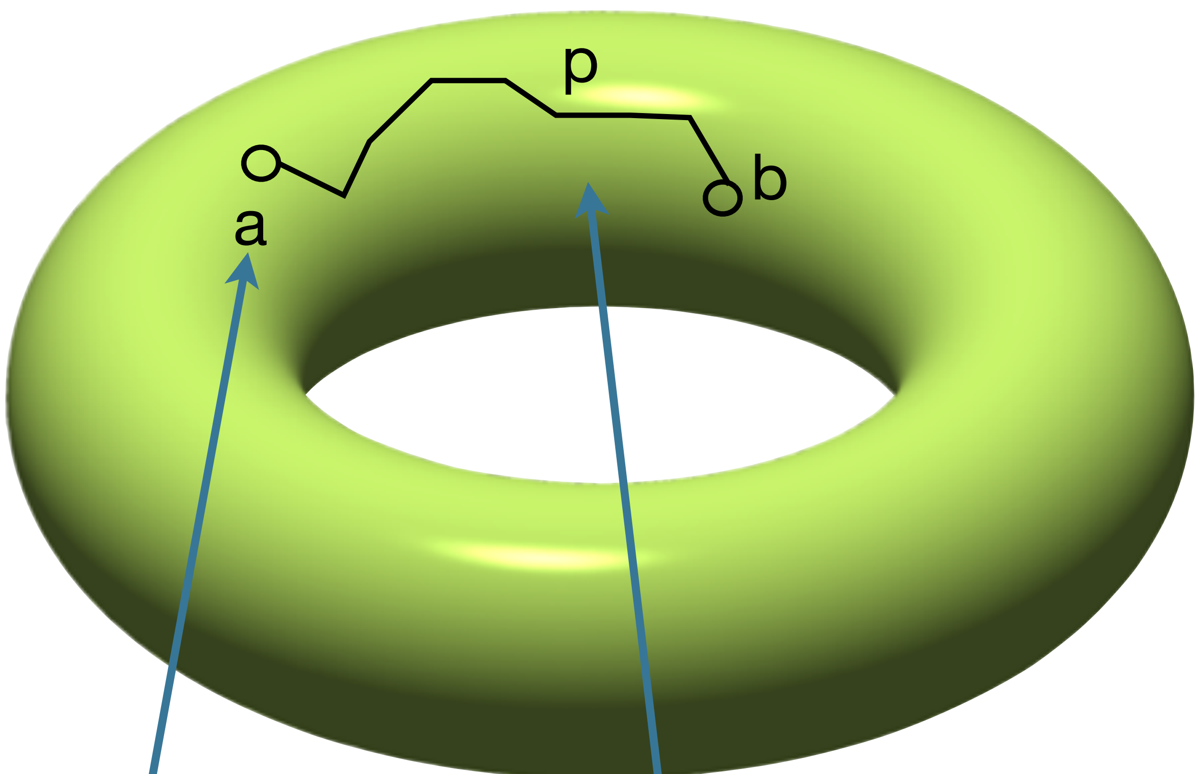

points are elements

$a : A$

paths are

proofs of equality

$p \ : \ a =_A \ b$

path operations

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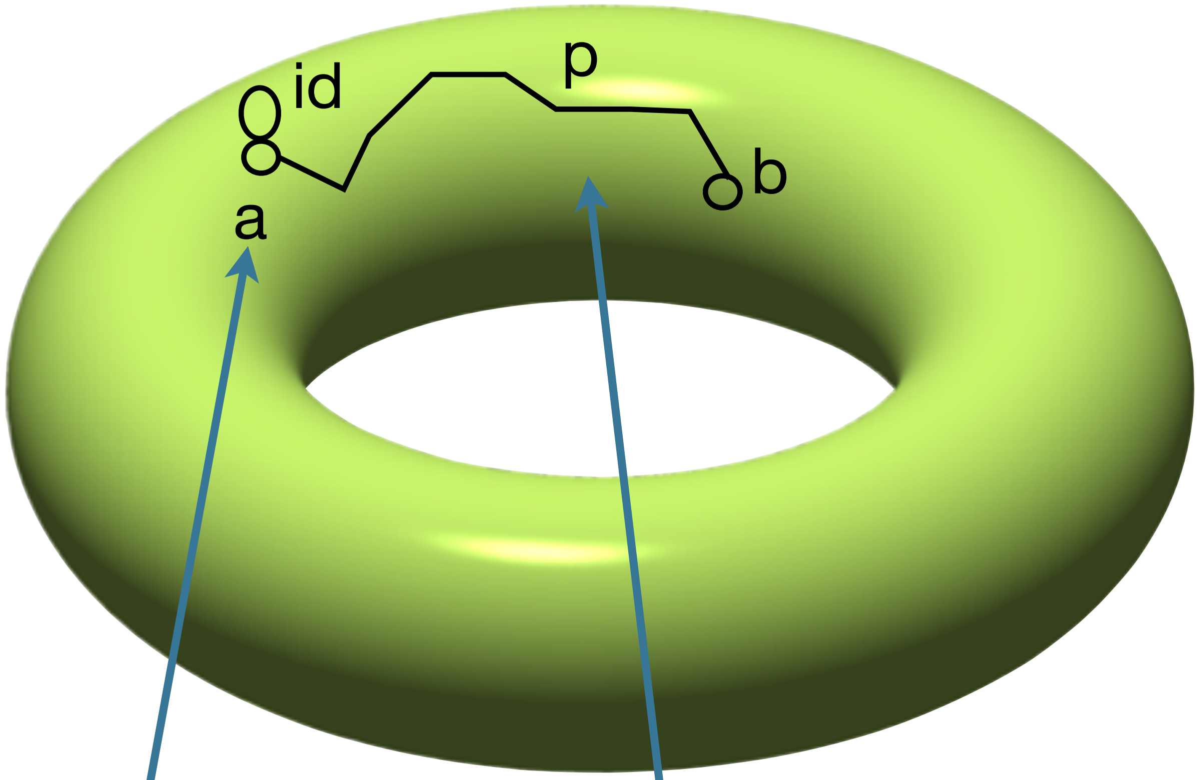

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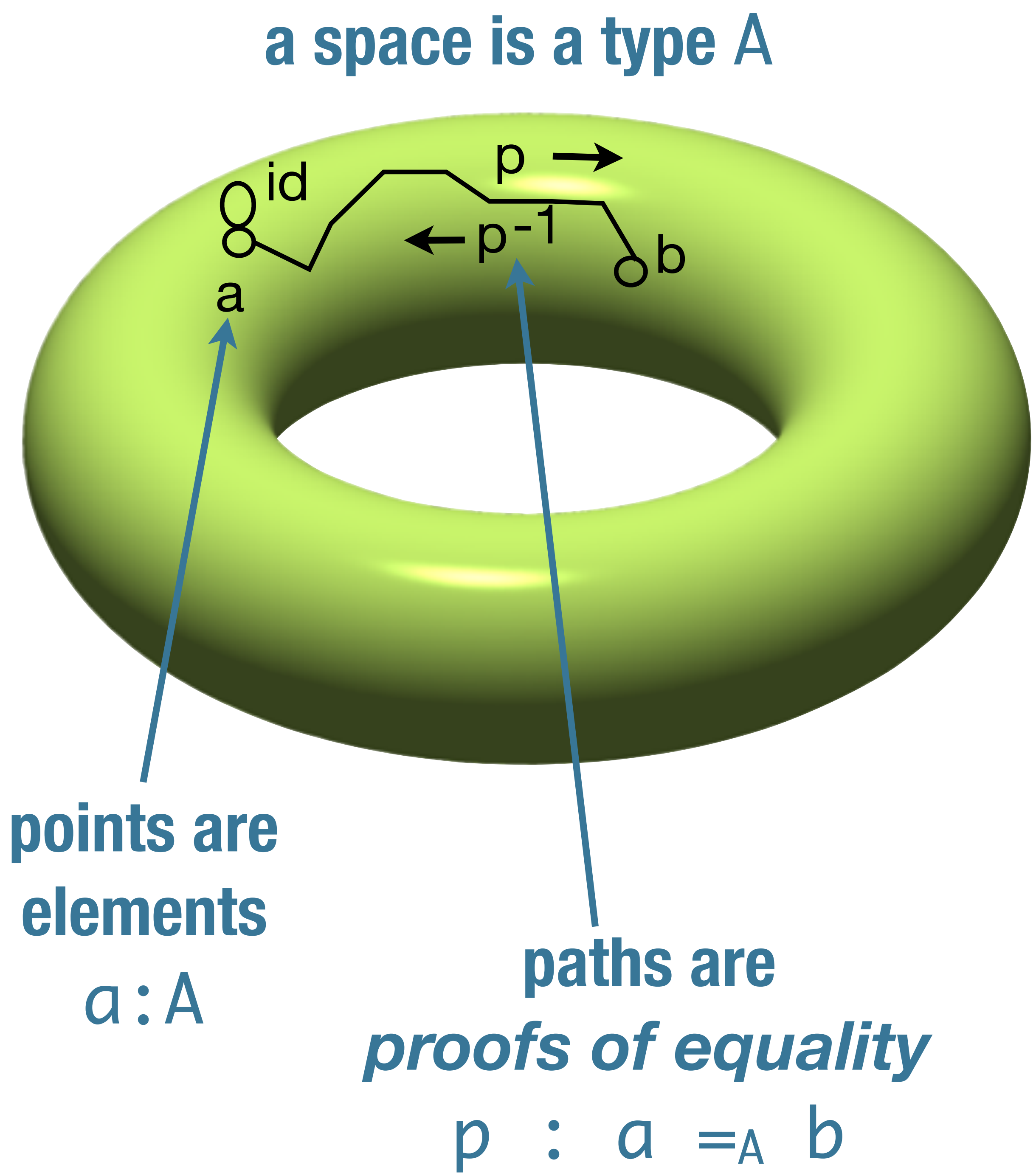
path operations

`id     : a = a (refl)`

Spaces as types



path operations

$$\mathtt{id} : a = a \ (\mathrm{refl})$$
$$p^{-1} : b = a \ (\mathrm{sym})$$

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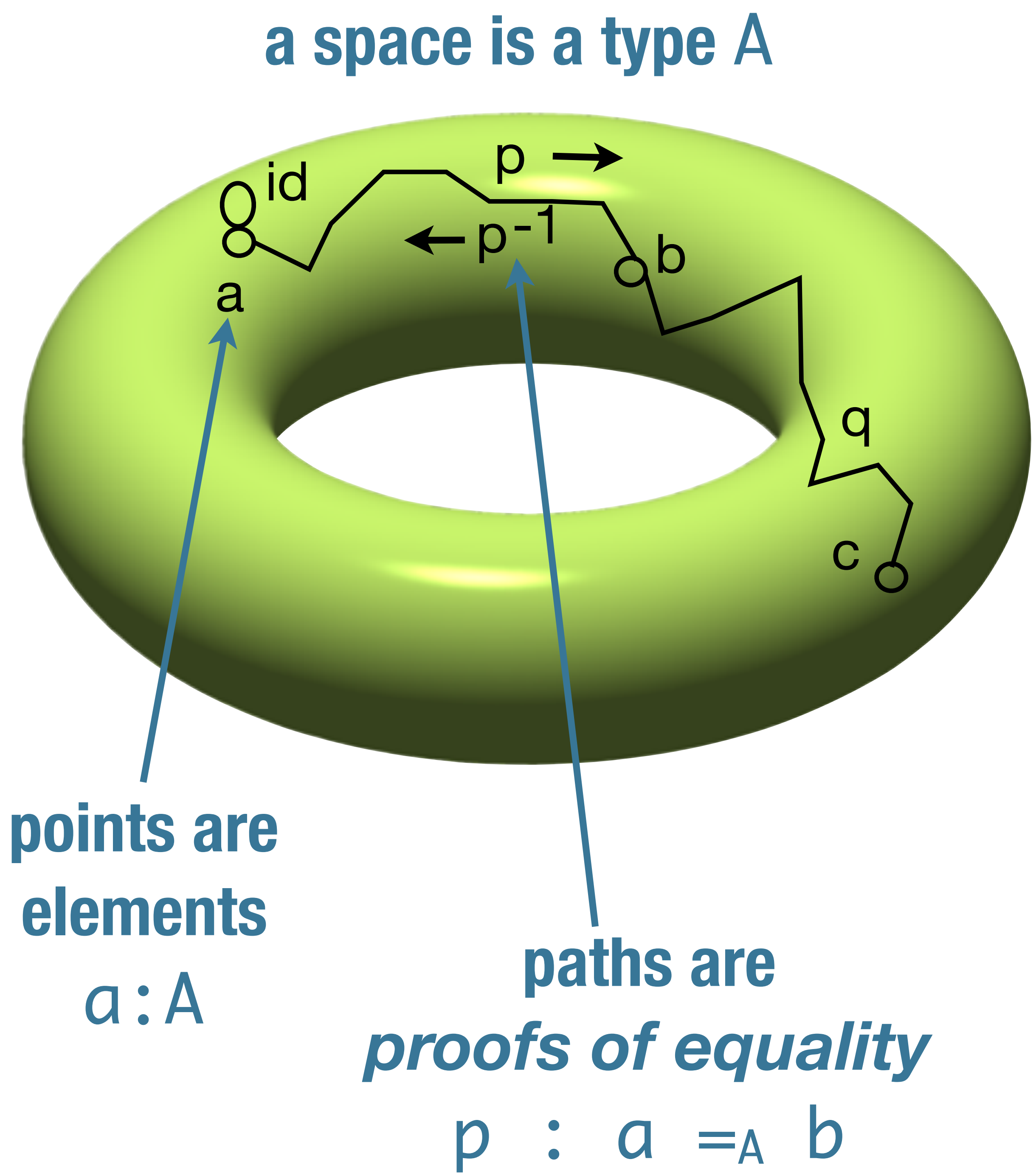

path operations

id : a = a (refl)

p^{-1} : b = a (sym)

q o p : a = c (trans)

Spaces as types

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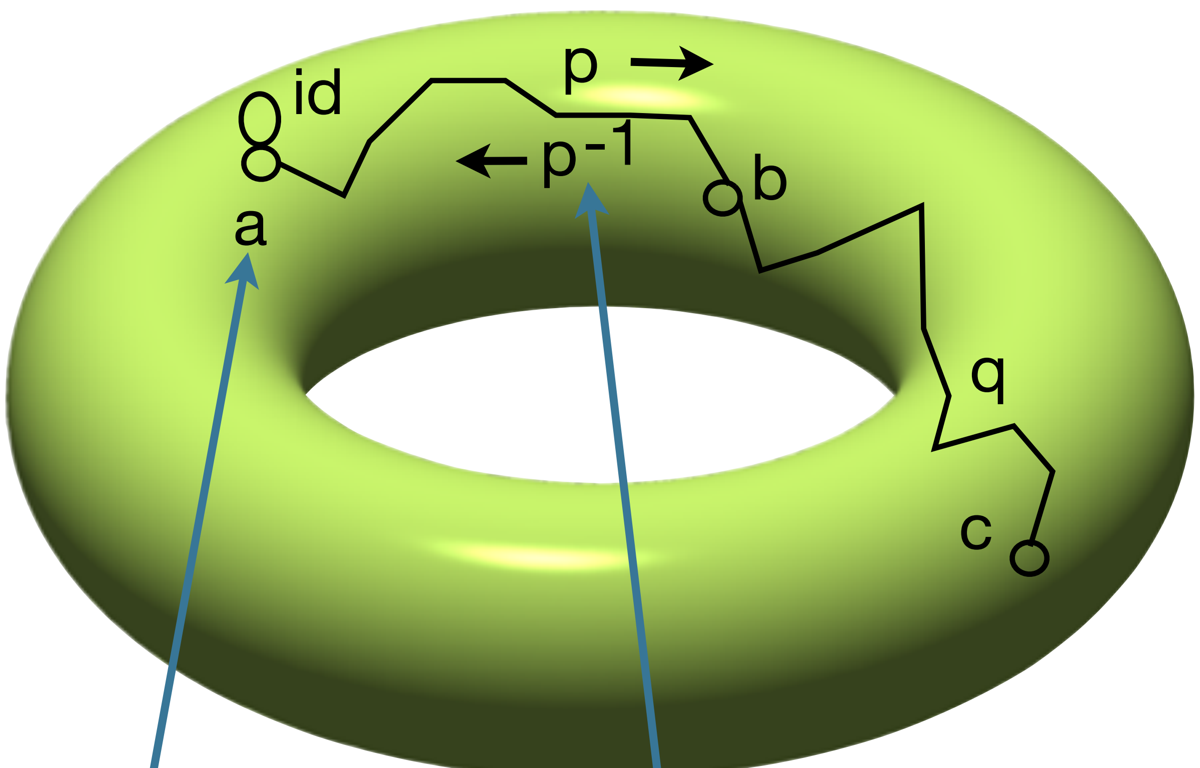

points are elements

$a : A$

paths are ***proofs of equality***

$p : a =_A b$

path operations

$id : a = a$ (refl)

$p^{-1} : b = a$ (sym)

$q \circ p : a = c$ (trans)

homotopies

$id \circ p = p$

$p^{-1} \circ p = id$

$r \circ (q \circ p) = (r \circ q) \circ p$

Spaces as types

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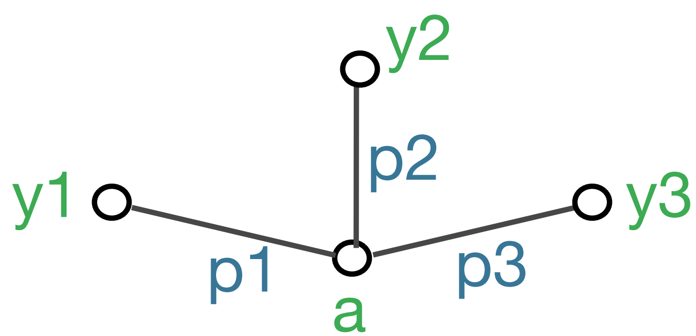
Path induction

Type of paths
from a to somewhere



is inductively
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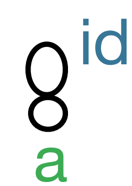

Path induction

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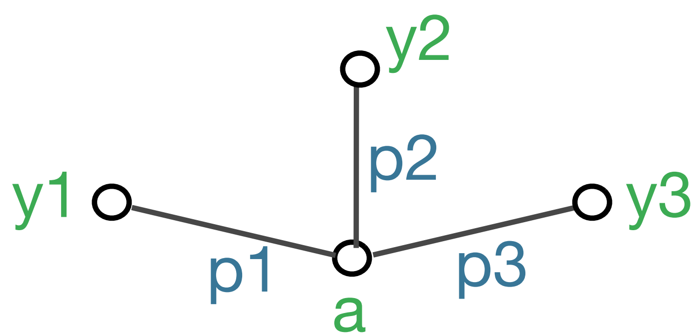


is inductively generated by

Fix a type A with element $a:A$.

For a family of types $C(y:A,\ p:a=y)$,

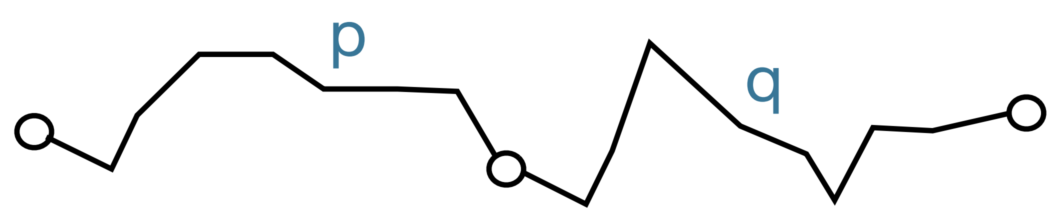
to give an element of

$$C(y,p) \text{ for all } y \text{ and } p:a=y,$$

suffices to give an element of

$$C(a,id)$$

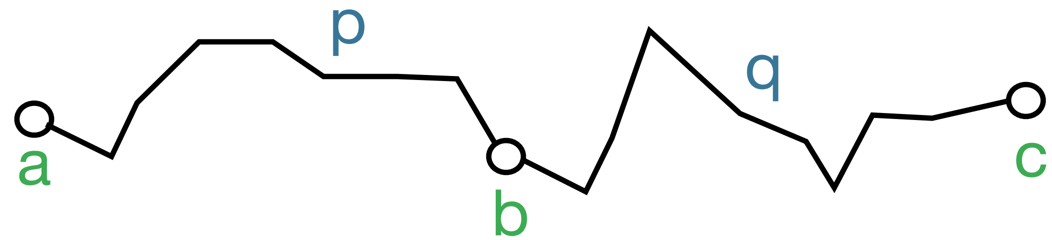
Composition, analytically



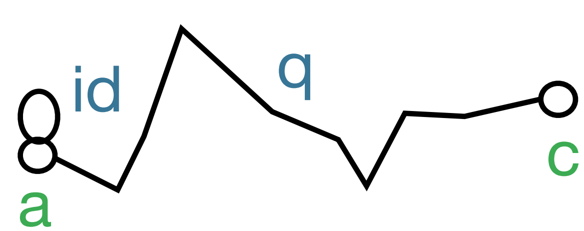
Given paths p and q `: [0,1] → X` where `p(1) = q(0)`
define **composition** by:

```
(q o p)(x) = p(2x)        if 0 ≤ x ≤ 1/2
           | q(2x - 1) if 1/2 ≤ x ≤ 1
```

Composition, synthetically



Given paths `p : a=b` and `q : b=c`
define **composition** `(q o p)` by path induction:



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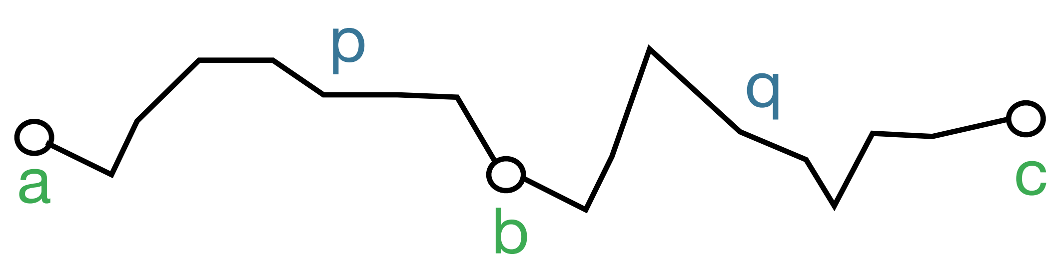
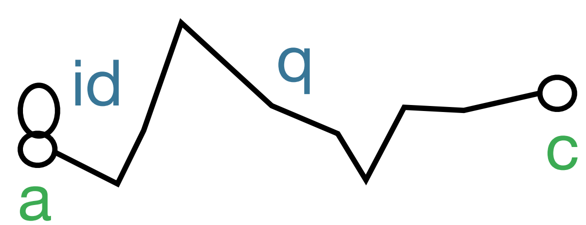


Given paths $\mathtt{p}:a=b$ and $\mathtt{q}:b=c$

define **composition** $(\mathtt{q} \circ \mathtt{p})$ by path induction:

- Suffices to consider case where b is a, and $\mathtt{p}$ is $\mathtt{id}$



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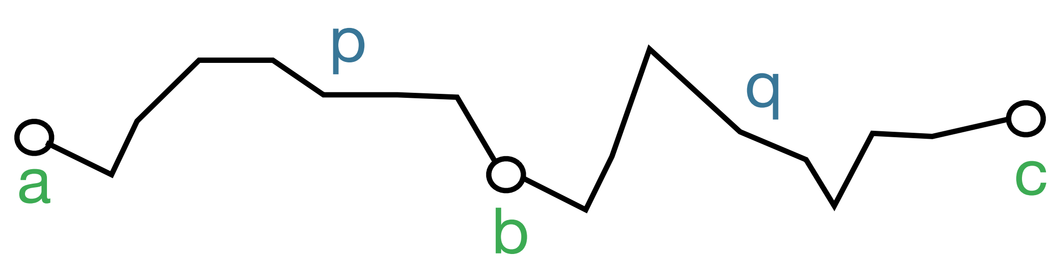
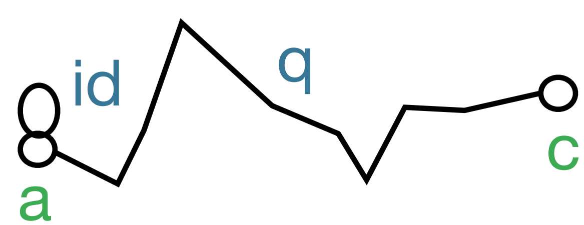


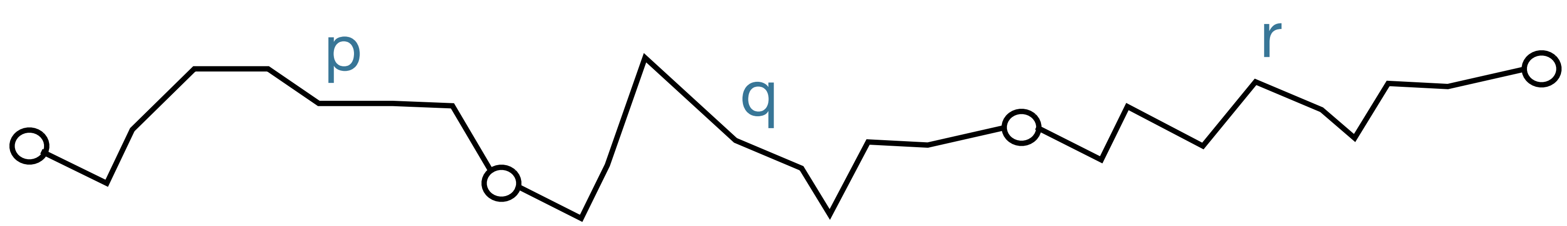
Given paths `p`:`a`=`b` and `q`:`b`=`c`
define **composition** `(q o p)` by path induction:

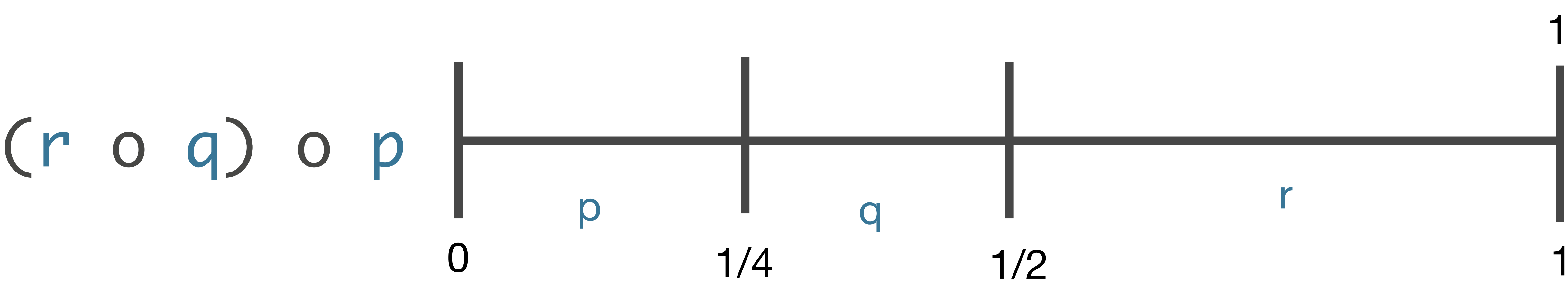
- Suffices to consider case where `b` is `a`, and `p` is `id`



- In this case the composite is `q`

Associativity, analytically





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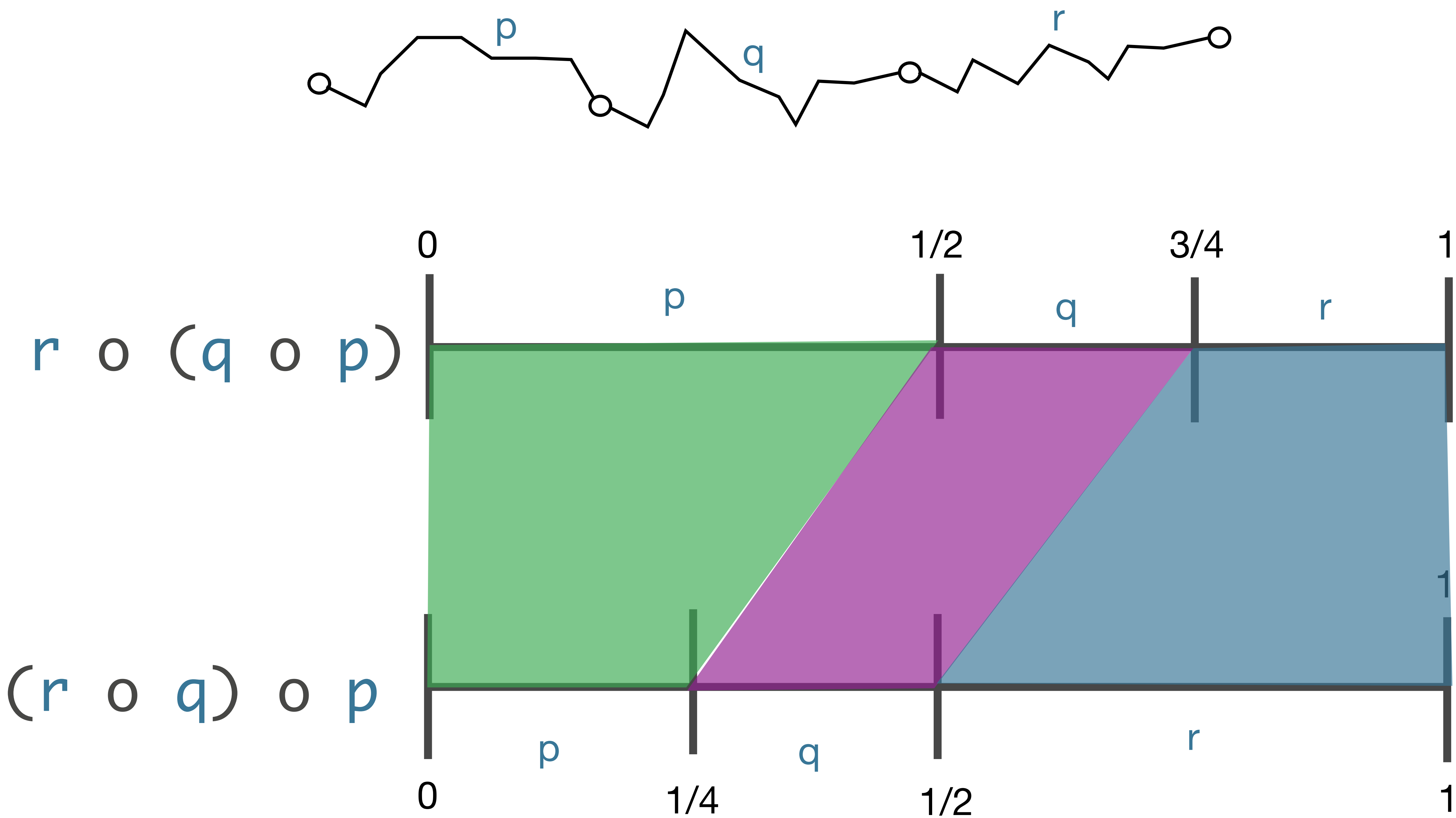

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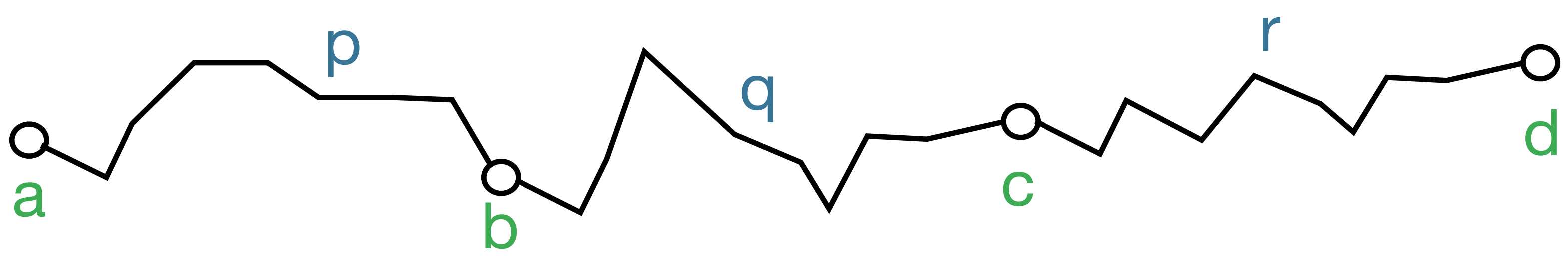
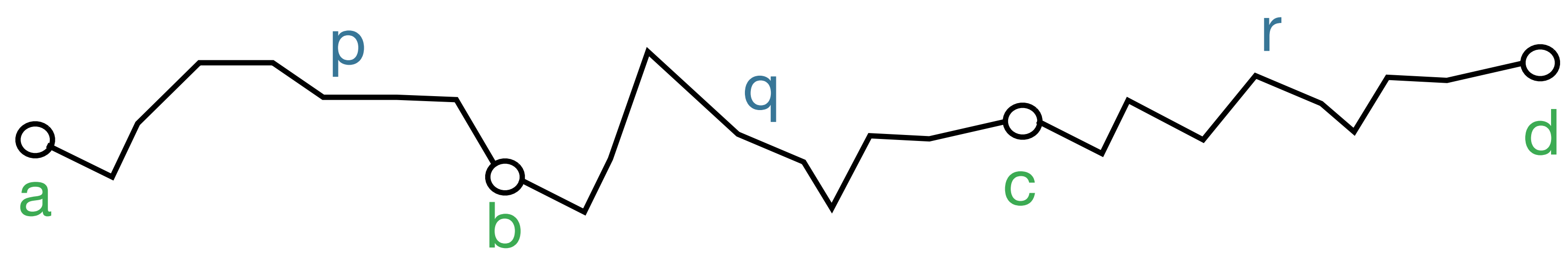


$$\forall\ a,b,c,d,\ p:a=b,\ q:b=c,\ r:c=d.$$
$$r \circ (q \circ p) = (r \circ q) \circ p$$

Associativity, synthetically



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- By path induction, suffices to consider case where all points are a and all paths are `id`:
  `id o (id o id) = (id o id) o id`

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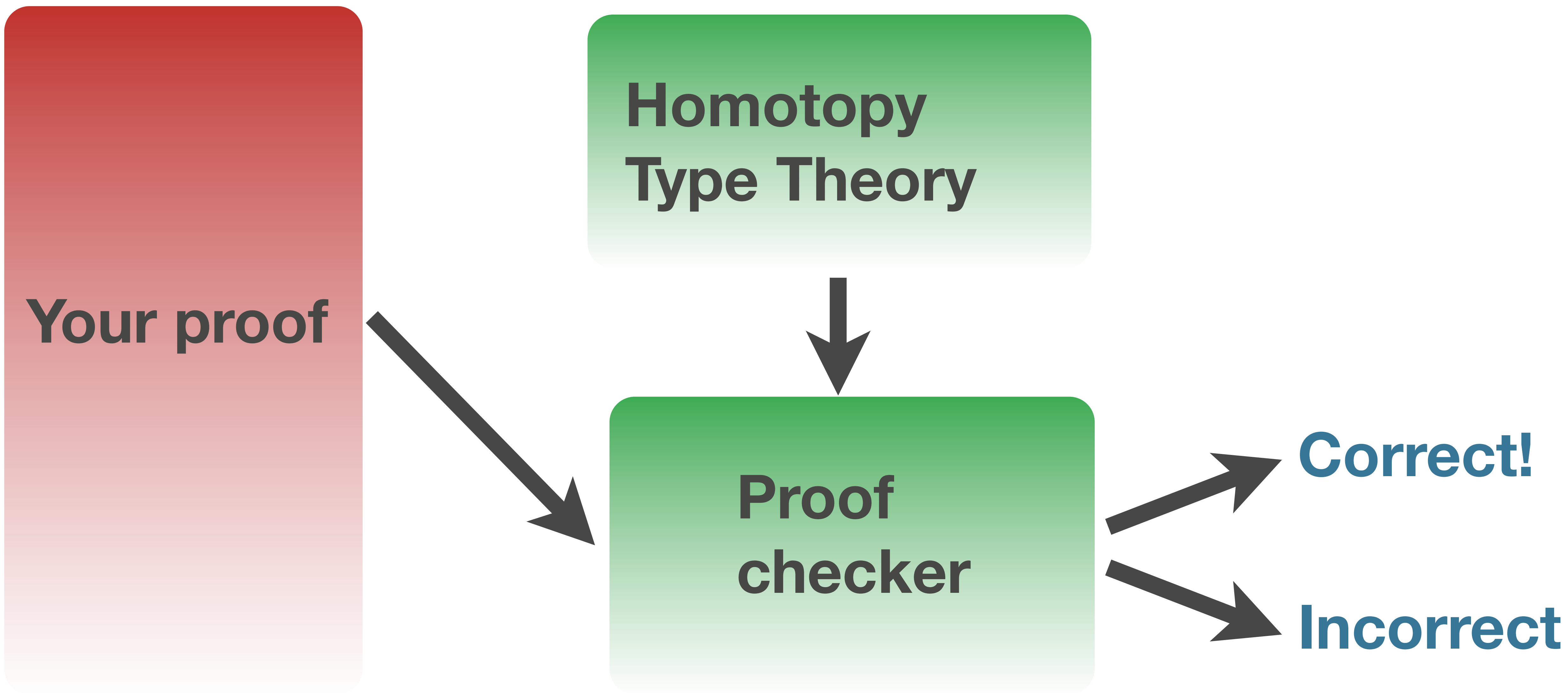
  $\mathrm{id} \circ (\mathrm{id} \circ \mathrm{id}) = (\mathrm{id} \circ \mathrm{id}) \circ \mathrm{id}$
- By definition of o, both sides equal `id`

Type theory is
a **logic** of
homotopy theory

Computer-checked proofs



Computer-checked proofs

```
_∘_  : {A : Type} {a b c : A}
     → Path b c → Path a b → Path a c
q ∘ id = q


∘-assoc  : {A : Type} {a b c d : A}
    (p : Path a b) (q : Path b c) (r : Path c d)
  → Path (r ∘ (q ∘ p)) ((r ∘ q) ∘ p)
∘-assoc id id id = id
```

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*work in progress

Homotopy in HoTT

$\pi_1(S^1) = \mathbb{Z}$

$\pi_{k<n}(S^n) = 0$

Hopf fibration

$\pi_2(S^2) = \mathbb{Z}$

$\pi_3(S^2) = \mathbb{Z}$

James Construction

$\pi_4(S^3) = \mathbb{Z}_?$

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K(G,n)

Cohomology axioms

Blakers-Massey

Van Kampen

Covering spaces

Whitehead for n-types

[Brunerie, Finster, Hou, Licata, Lumsdaine, Shulman]

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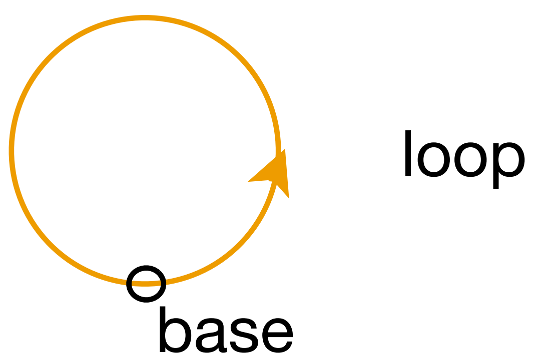
Example:

The Fundamental Group of the Circle

The Circle

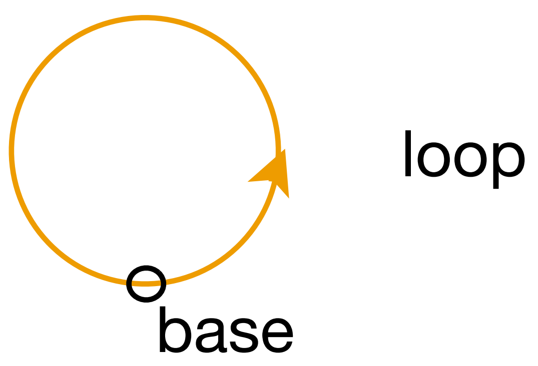
Circle S^1 is ***inductively generated*** by



The Circle

Circle S^1 is ***inductively generated*** by

```
base : S¹
loop : base = base
```



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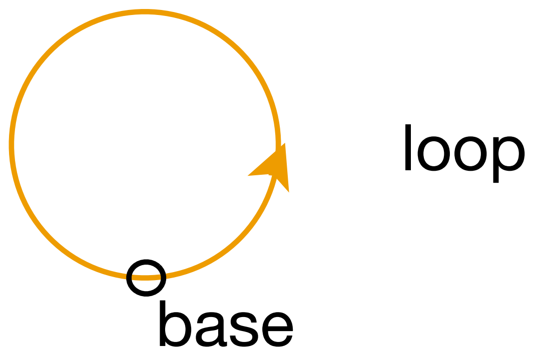

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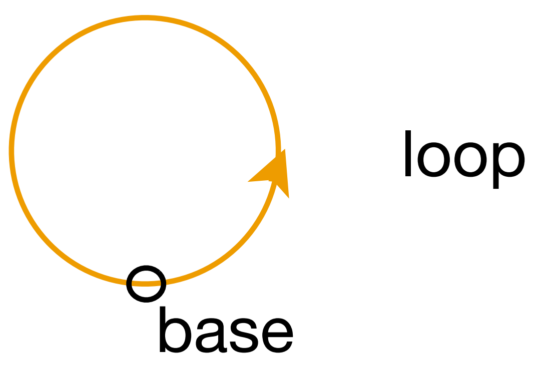

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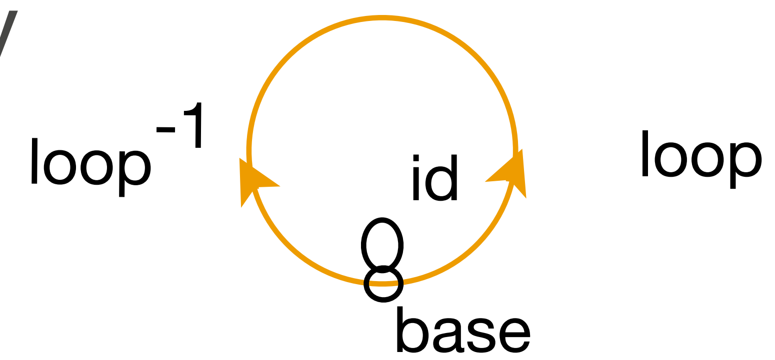


Free type: equipped with structure by path induction

`id` `inv : loop o loop⁻¹ = id`

`loop⁻¹` `...`

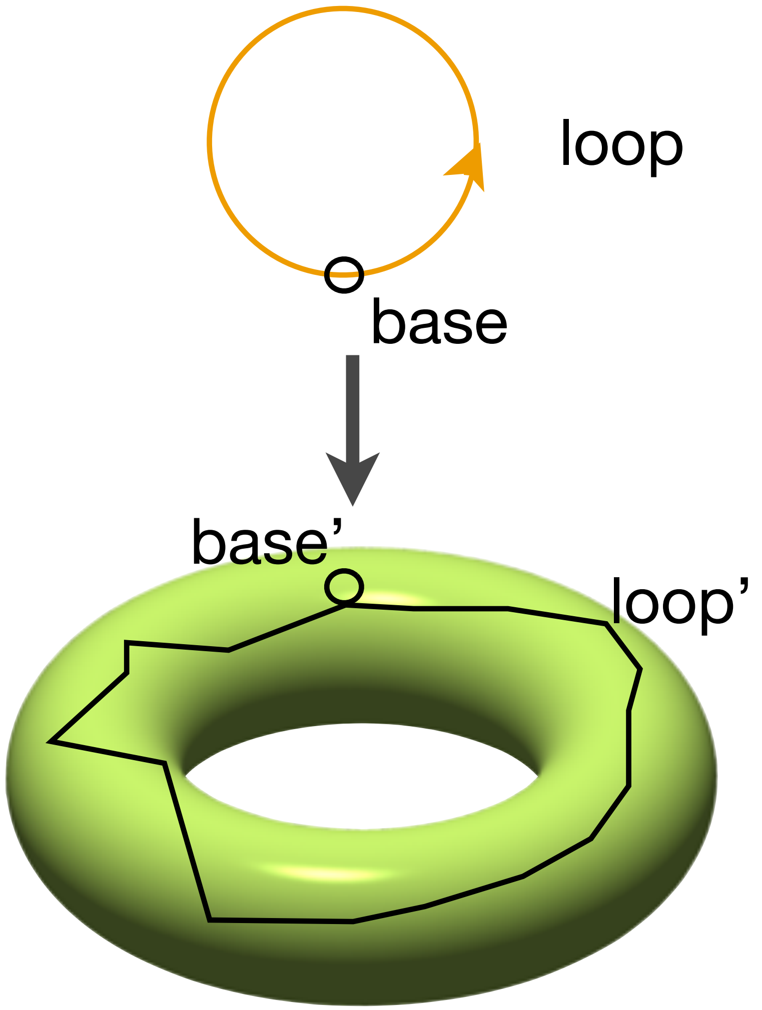
`loop o loop`

The Circle

Circle recursion:

function $S^1 \to X$ determined by

```
base' : X
loop' : base' = base'
```



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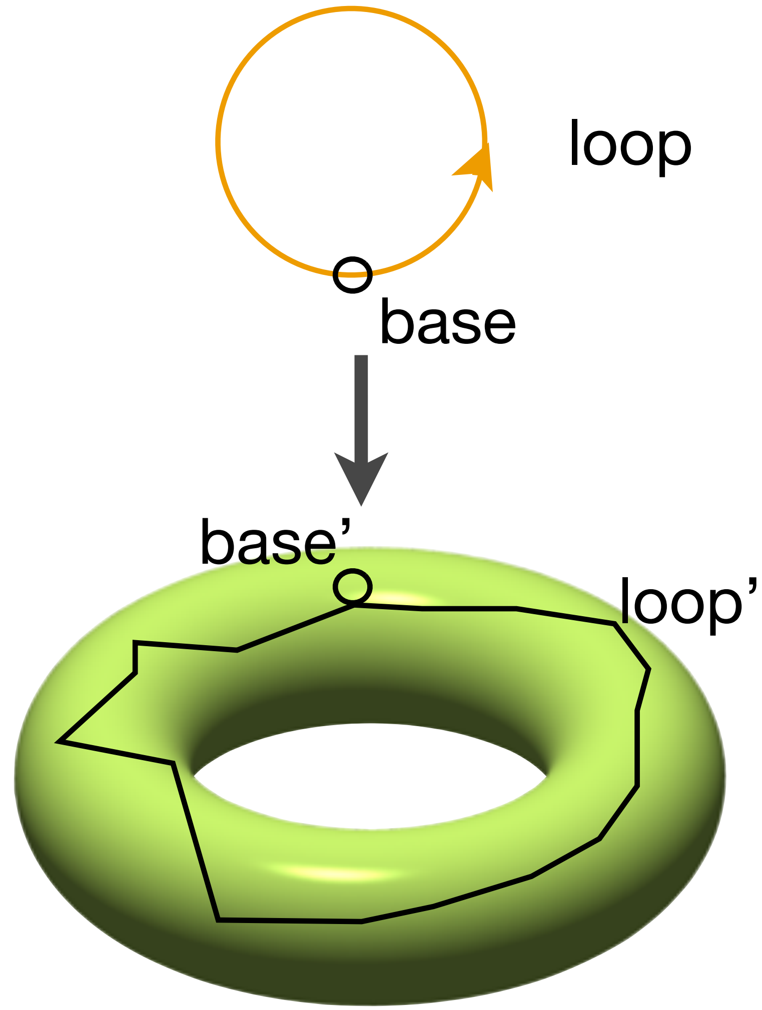
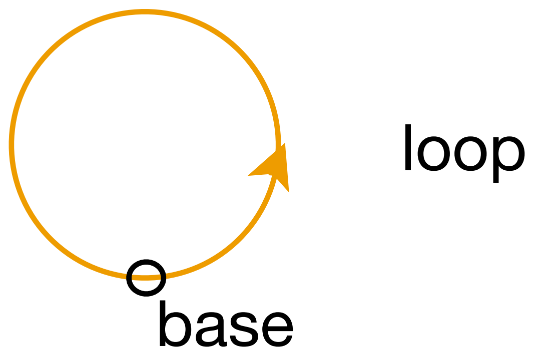


Circle induction: To prove a predicate P for all points on the circle, suffices to prove P(`base`), continuously in the loop

Fundamental group of circle

How many different loops are there on the circle, up to homotopy?



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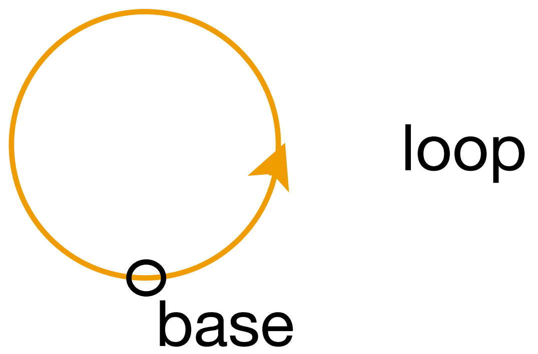

`id`

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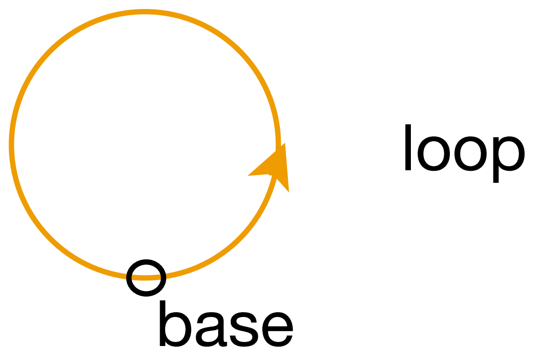


```
id
loop
```

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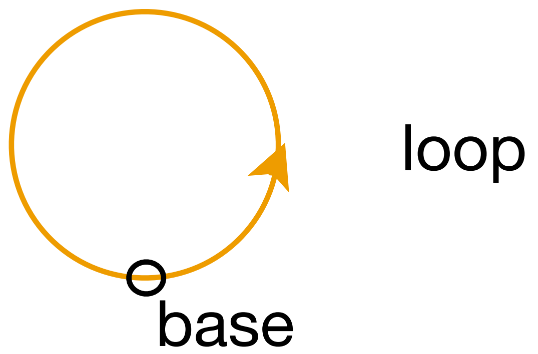


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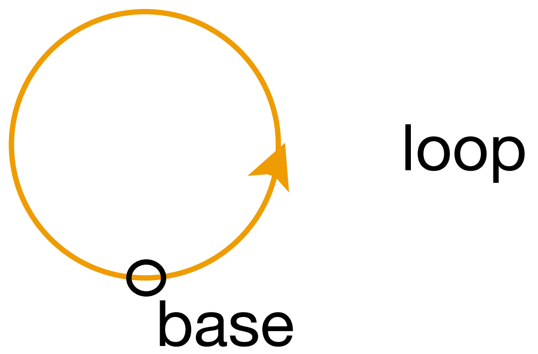


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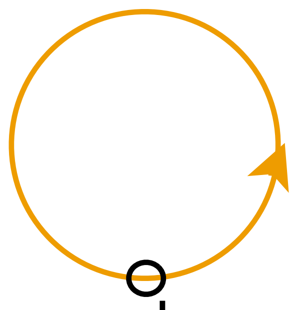


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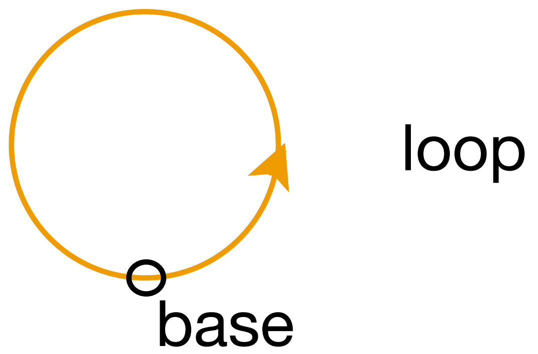


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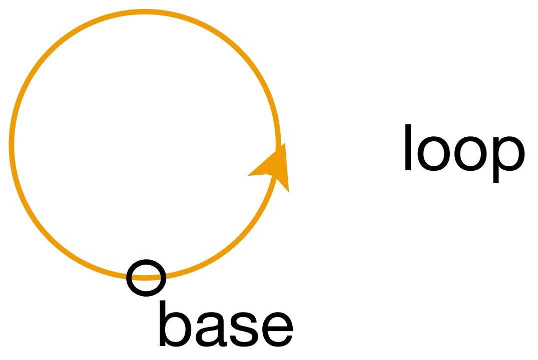
loop⁻¹
loop o loop
loop⁻¹ o loop⁻¹
loop o loop⁻¹
```

Fundamental group of circle

How many different loops are there on the circle, up to homotopy?




```
id
loop
loop⁻¹
loop o loop
loop⁻¹ o loop⁻¹
loop o loop⁻¹  = id
```

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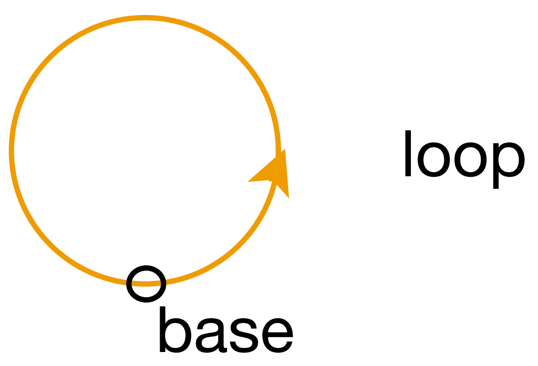


```
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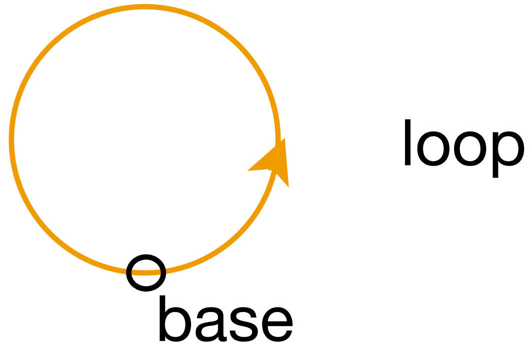

`id`	`0`
`loop`	`1`
`loop`$^{-1}$	
`loop o loop`	
`loop`$^{-1}$ `o loop`$^{-1}$	
`loop o loop`$^{-1}$ `= id`	

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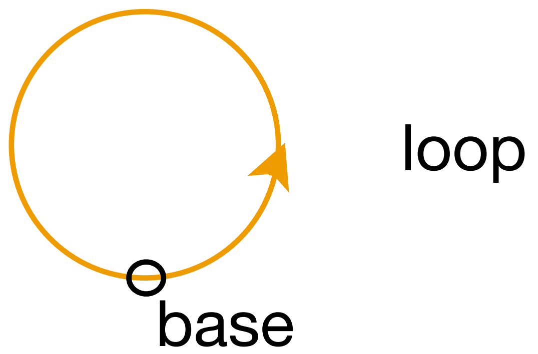

`id`	`0`
`loop`	`1`
`loop`$^{-1}$	`-1`
`loop o loop`	
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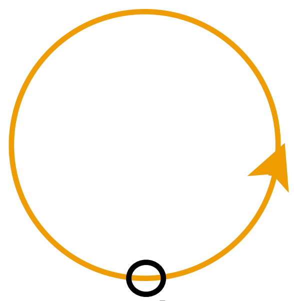

`id`	`0`
`loop`	`1`
`loop`$^{-1}$	`-1`
`loop o loop`	`2`
`loop`$^{-1}$ `o loop`$^{-1}$	
`loop o loop`$^{-1}$ `= id`	

Fundamental group of circle

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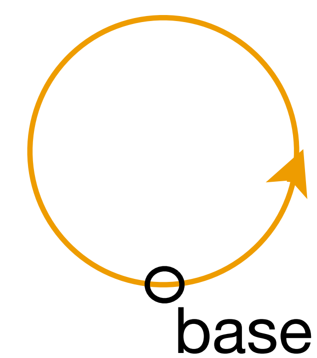


```
id                      0
loop                    1
loop⁻¹                  -1
loop o loop             2
loop⁻¹ o loop⁻¹         -2
loop o loop⁻¹   = id
```

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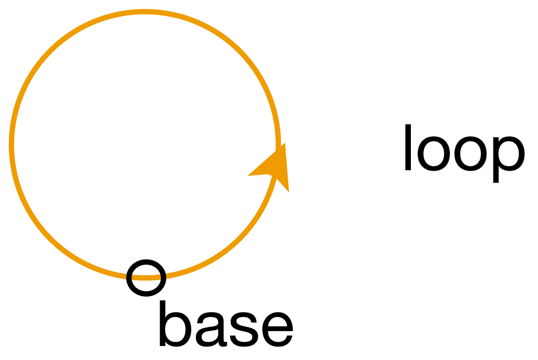

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`loop o loop`	`2`
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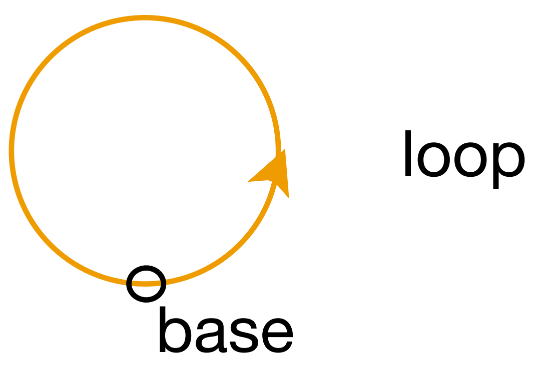

`id`	`0`
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`loop`$^{-1}$	`-1`
`loop o loop`	`2`
`loop`$^{-1}$ `o loop`$^{-1}$	`-2`
`loop o loop`$^{-1}$ `= id`	`0`

integers are "codes" for paths on the circle

Fundamental group of circle

Definition. $\Omega(S^1)$ is the **type** of loops at base
i.e. the type $(\mathit{base} = \mathit{base})$

Fundamental group of circle

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0-truncation (set of connected components) of $\Omega(S^1)$

Fundamental group of circle

Theorem. $\Omega(S^1)$ is equivalent to $\mathbb{Z}$

Proof: two mutually inverse functions

```
wind  : Ω(S¹) → ℤ

loop⁻ : ℤ → Ω(S¹)
```

Fundamental group of circle

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```
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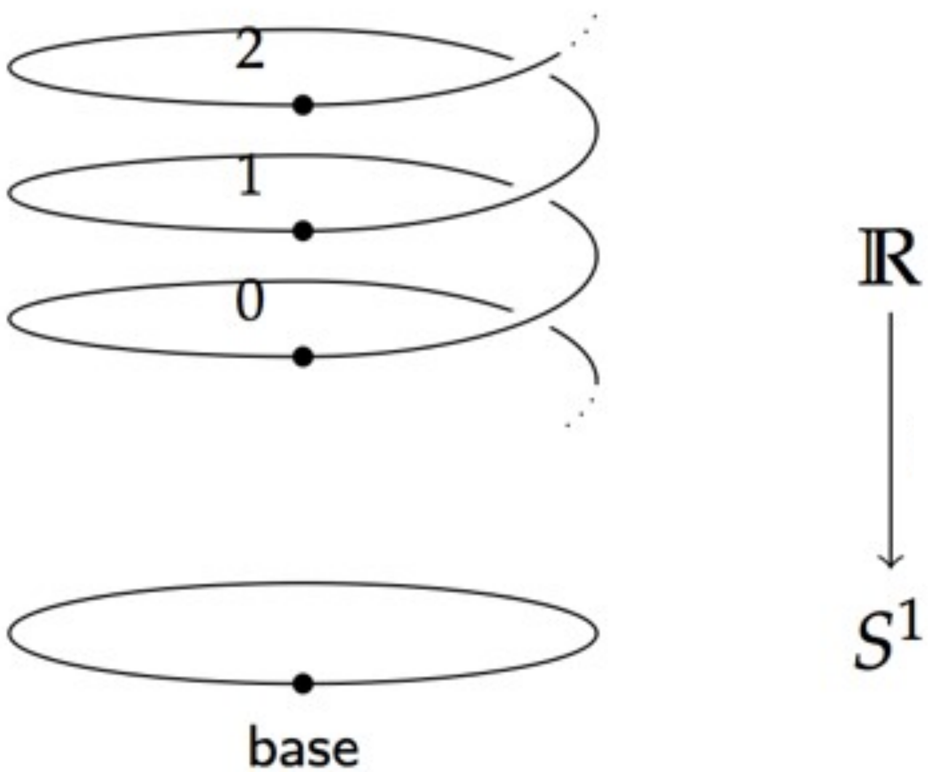
loop⁰   = id
loop⁺ⁿ  = loop o loop o … loop        (n times)
loop⁻ⁿ  = loop⁻¹ o loop⁻¹ o … loop⁻¹  (n times)
```

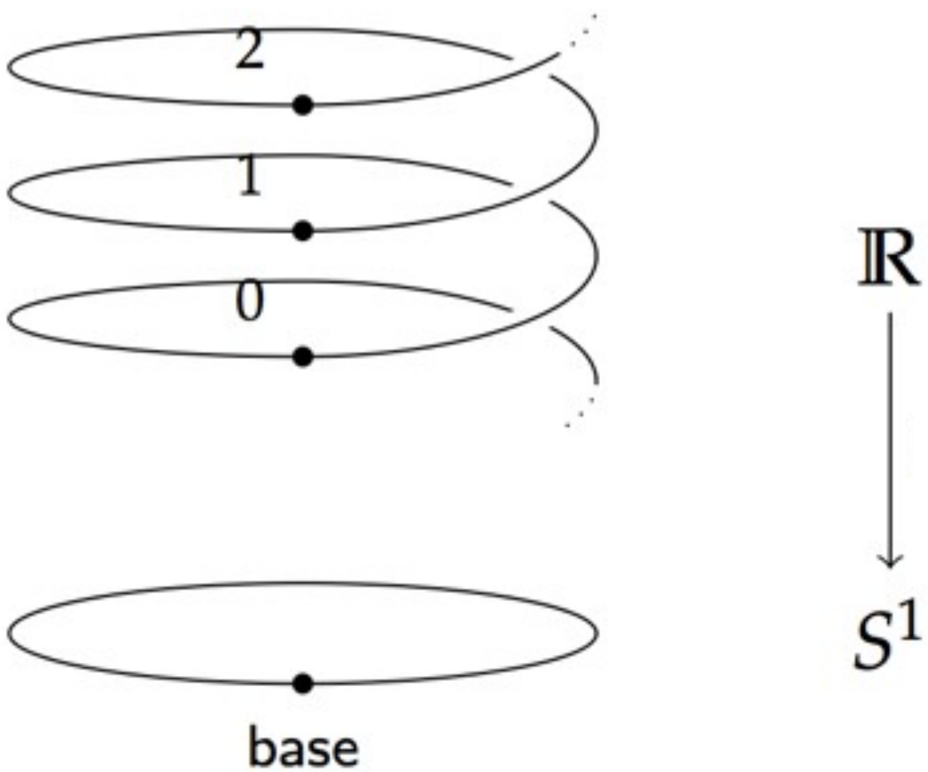
Universal Cover



$\texttt{wind} : \Omega(S^1) \to \mathbb{Z}$

defined by **lifting** a loop to the cover, and giving the other endpoint of 0

Universal Cover



lifting is functorial

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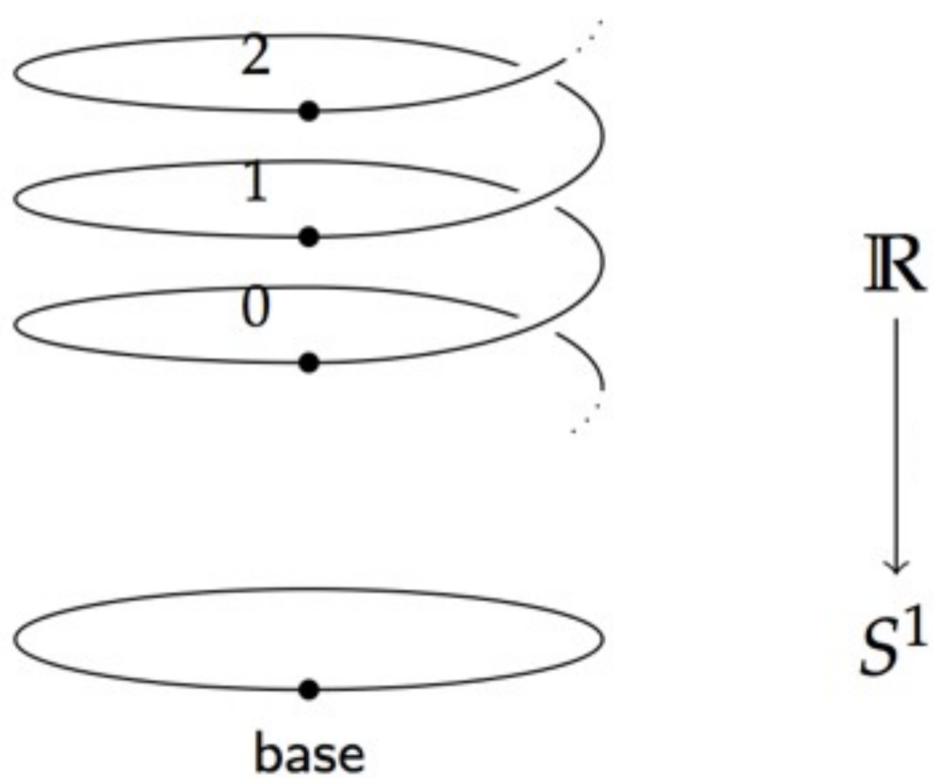

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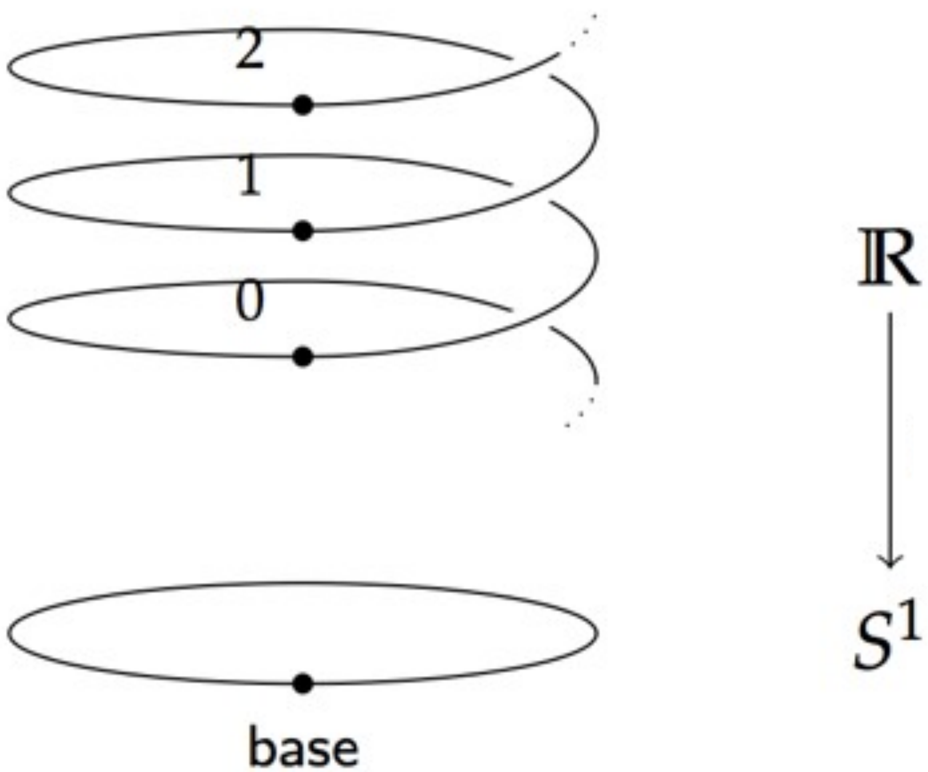
Universal Cover



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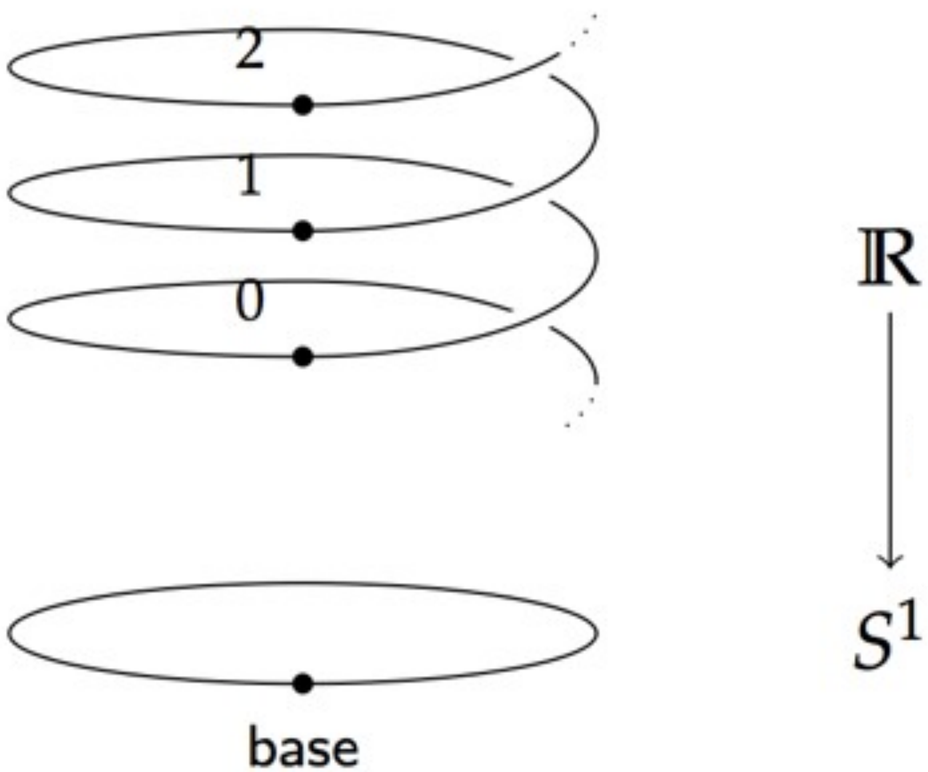

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lifting is functorial

lifting `loop` adds 1

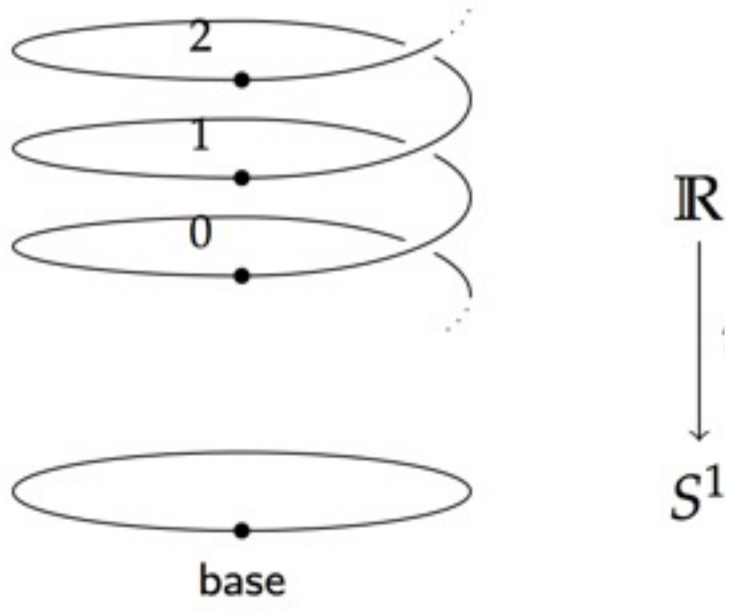
lifting `loop`$^{-1}$ subtracts 1

Example:

$$\mathtt{wind}(\mathtt{loop} \circ \mathtt{loop}^{-1})$$
$$= 0 + 1 - 1$$
$$= 0$$

Fibration = Family of types

Fibration (classically):
map $p: E \to B$ such that
any path from $p(e)$ to y
lifts to a path in E from e
to some point in $p^{-1}(y)$



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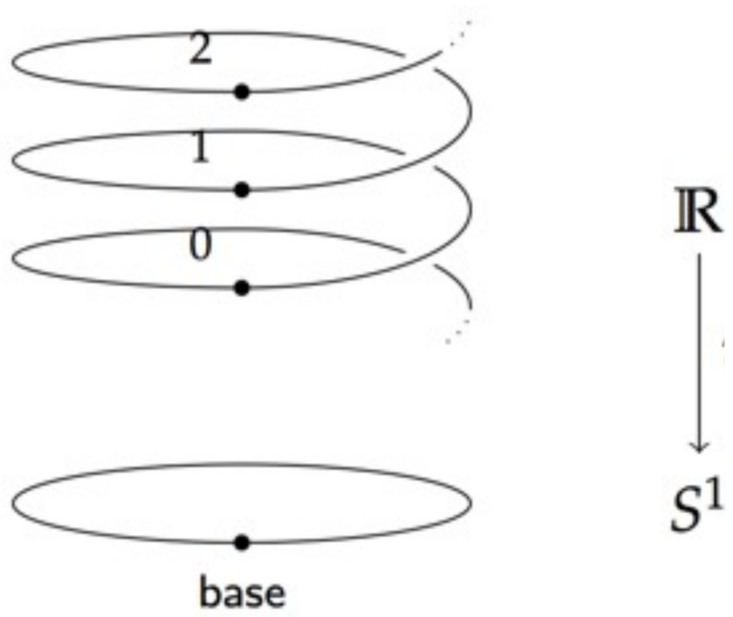

Family of types $(E(x))_{x:B}$

- *Fibers:* $E(b)$ is a type for all $b:B$
- *transport:* equivalence $E(b_1) \simeq E(b_2)$ for all $p: b_1 =_B b_2$

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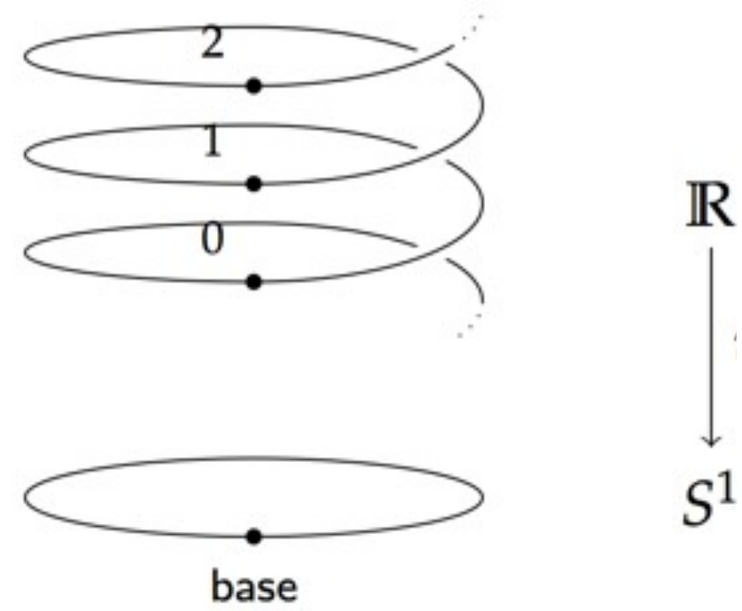

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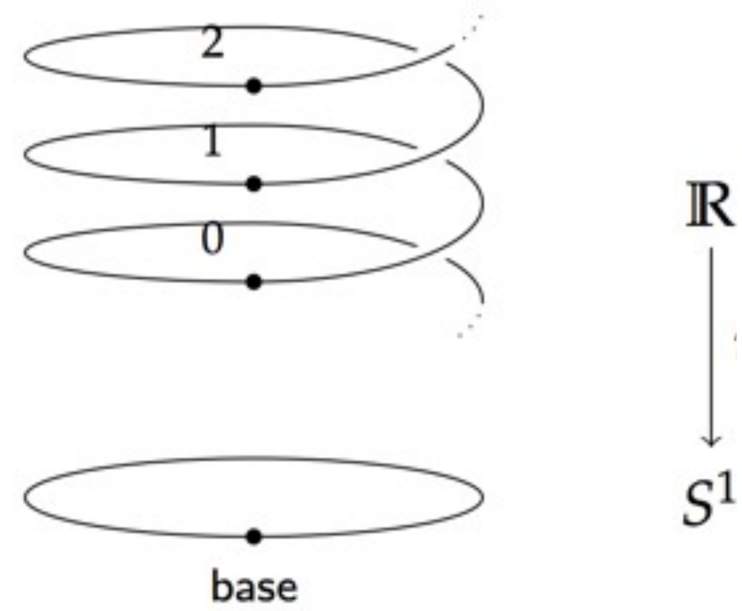

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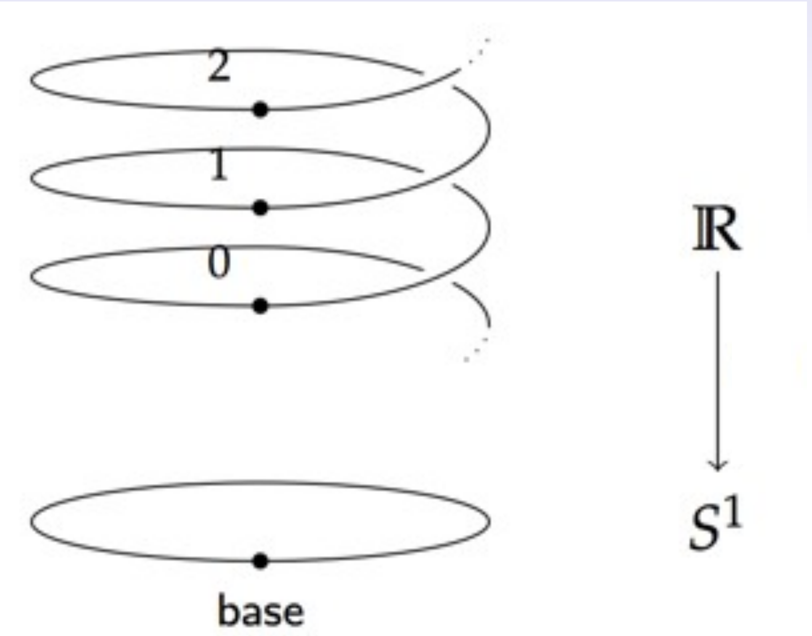
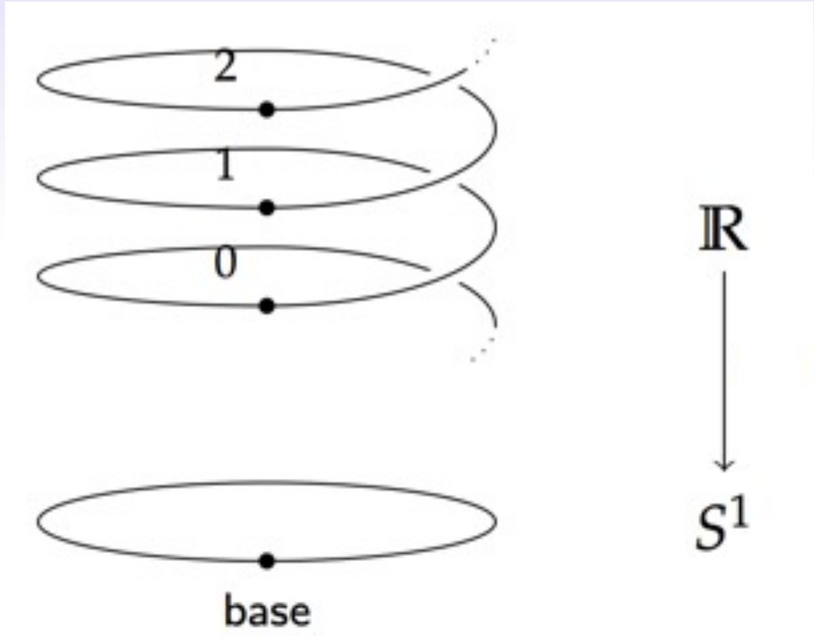
sends $e \in E(x)$ to other endpoint of lifting of p

Universal Cover

family of types $(\texttt{Cover(x)})_{\texttt{x:S1}}$



Universal Cover

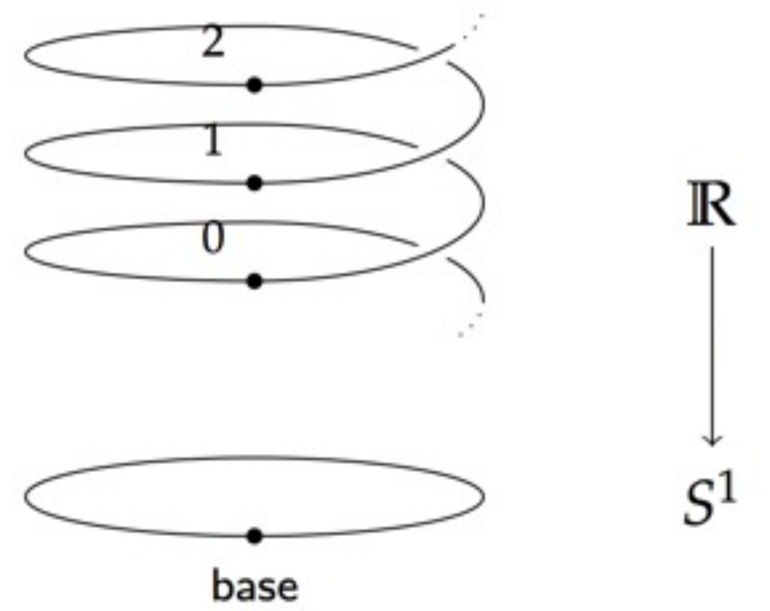


family of types $(\mathtt{Cover(x)})_{\mathtt{x:S1}}$

By circle recursion, it suffices to give

- Fiber over `base`: the type $\mathbb{Z}$
- Equivalence $\mathbb{Z} \simeq \mathbb{Z}$ as lifting of `loop`: `successor`

Universal Cover



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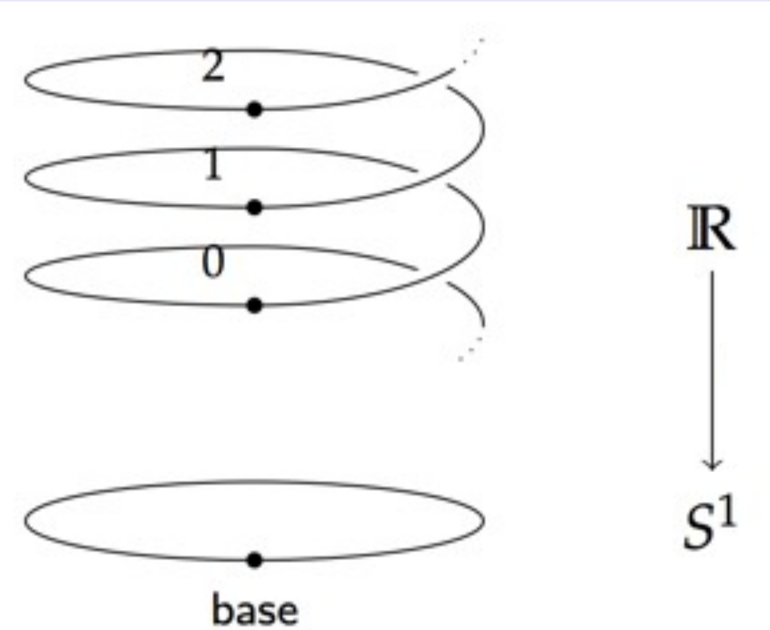

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- Equivalence $\mathbb{Z} \simeq \mathbb{Z}$ as lifting of `loop`: `successor`

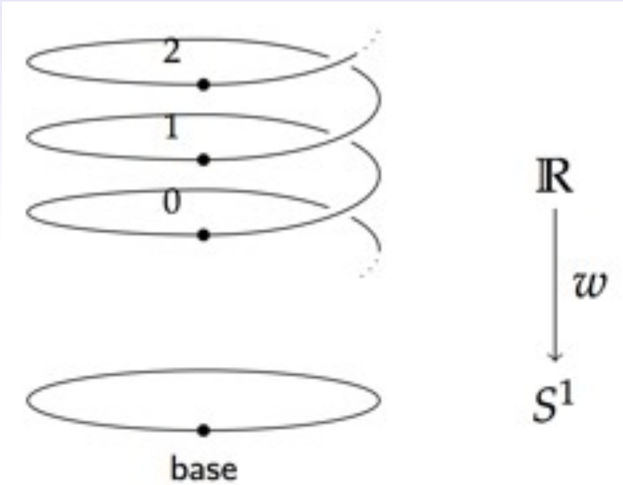
uses univalence

Defining equations:

$\mathtt{Cover(base)} := \mathbb{Z}$

$\mathtt{transport}_{\mathtt{Cover}}\mathtt{(loop)} := \mathtt{successor}$

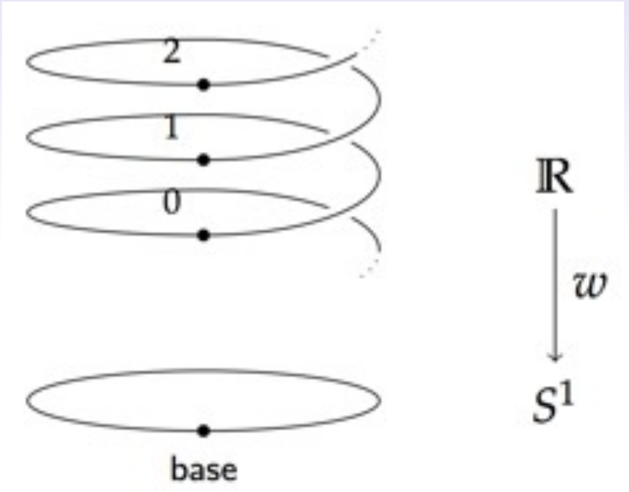
Winding number



$$\texttt{wind} : \Omega(S^1) \to \mathbb{Z}$$
$$\texttt{wind}(p) = \texttt{transport}_{\texttt{Cover}}(p,0)$$

lift p to cover, starting at 0

Winding number



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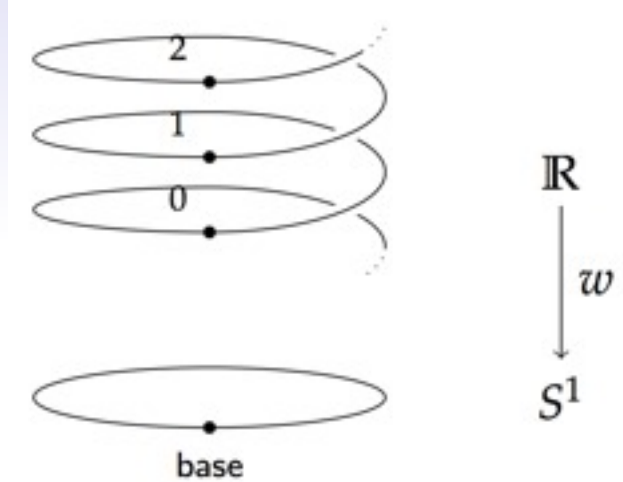
lift p to cover, starting at 0

$\texttt{wind}(\texttt{loop}^{-1} \circ \texttt{loop})$

Winding number



$\texttt{wind} : \Omega(S^1) \to \mathbb{Z}$

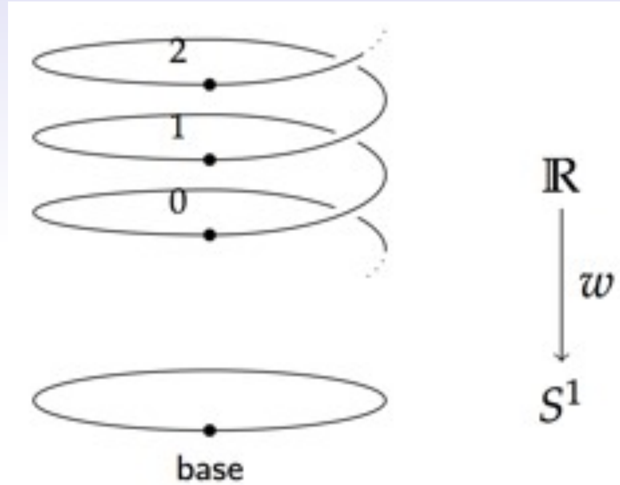
$\texttt{wind(p)} = \texttt{transport}_{\texttt{Cover}}\texttt{(p,0)}$

lift p to cover, starting at 0

$\texttt{wind(loop}^{-1}\texttt{ o loop)}$

$= \texttt{transport}_{\texttt{Cover}}\texttt{(loop}^{-1}\texttt{ o loop, 0)}$

Winding number



$$\text{wind} : \Omega(S^1) \to \mathbb{Z}$$
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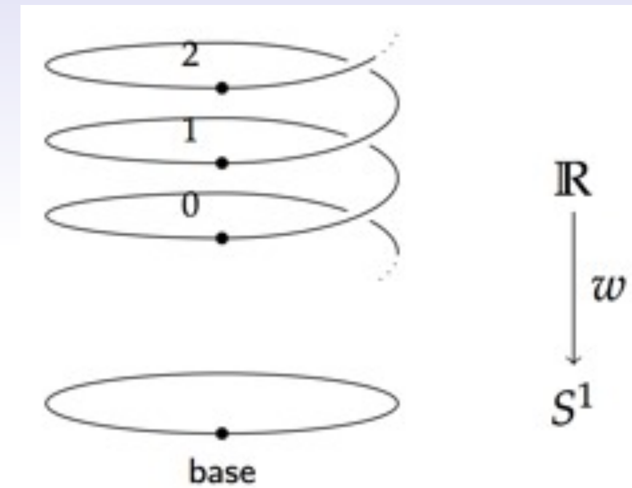
lift p to cover, starting at 0

$$\text{wind}(\text{loop}^{-1} \circ \text{loop})$$
$$= \text{transport}_{\text{Cover}}(\text{loop}^{-1} \circ \text{loop}, 0)$$
$$= \text{transport}_{\text{Cover}}(\text{loop}^{-1}, \text{transport}_{\text{Cover}}(\text{loop}, 0))$$

Winding number



$$\text{wind} : \Omega(S^1) \to \mathbb{Z}$$
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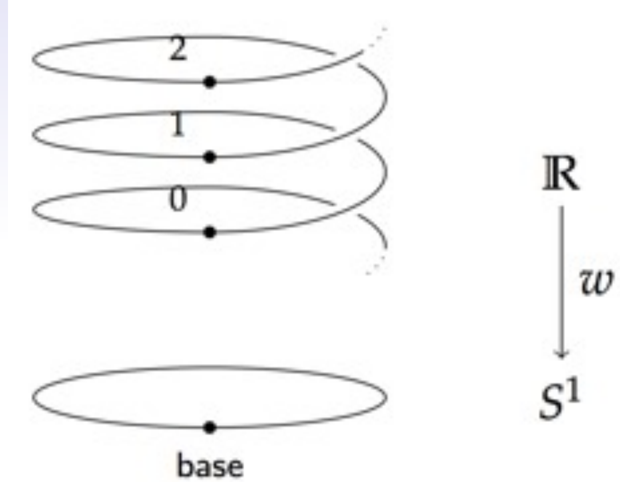
lift p to cover, starting at 0

$$\begin{aligned} &\text{wind}(\text{loop}^{-1} \circ \text{loop}) \\ = \; &\text{transport}_{\text{Cover}}(\text{loop}^{-1} \circ \text{loop}, 0) \\ = \; &\text{transport}_{\text{Cover}}(\text{loop}^{-1}, \text{transport}_{\text{Cover}}(\text{loop}, 0)) \\ = \; &\text{transport}_{\text{Cover}}(\text{loop}^{-1}, 1) \end{aligned}$$

Winding number



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lift p to cover, starting at 0

$\mathtt{wind(loop^{-1}\ o\ loop)}$

$= \mathtt{transport}_{\mathtt{Cover}}\mathtt{(loop^{-1}\ o\ loop,\ 0)}$

$= \mathtt{transport}_{\mathtt{Cover}}\mathtt{(loop^{-1},\ transport}_{\mathtt{Cover}}\mathtt{(loop,0))}$

$= \mathtt{transport}_{\mathtt{Cover}}\mathtt{(loop^{-1},\ 1)}$

$= 0$

So far

Theorem. $\Omega(S^1)$ is equivalent to $\mathbb{Z}$

Proof: two functions

```
wind    : Ω(S¹) → ℤ
wind(p) = transport_Cover(p,0)

loop⁻  : ℤ → Ω(S¹)
loop⁰   = id
loop⁺ⁿ  = loop o loop o … loop          (n times)
loop⁻ⁿ  = loop⁻¹ o loop⁻¹ o … loop⁻¹   (n times)
```

So far

Theorem. $\Omega(S^1)$ is equivalent to $\mathbb{Z}$

Proof: two **mutually inverse** functions

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wind    : Ω(S¹) → ℤ
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loop⁻ⁿ  = loop⁻¹ o loop⁻¹ o … loop⁻¹
```

$$\mathtt{wind} : \Omega(S^1) \to \mathbb{Z}$$
$$\mathtt{wind(p)} = \mathtt{transport}_{\mathtt{Cover}}\mathtt{(p,0)}$$

$$\mathtt{loop}^{-} : \mathbb{Z} \to \Omega(S^1)$$
$$\mathtt{loop}^{0} = \mathtt{id}$$
$$\mathtt{loop}^{+n} = \mathtt{loop} \circ \mathtt{loop} \circ \ldots \mathtt{loop} \quad \text{(n times)}$$
$$\mathtt{loop}^{-n} = \mathtt{loop}^{-1} \circ \mathtt{loop}^{-1} \circ \ldots \mathtt{loop}^{-1} \quad \text{(n times)}$$

Composite #1

Lemma. $\forall n.$ `wind(loop`^{n}`) = n`

Proof: induction on n. E.g.

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Proof: induction on n. E.g.

`wind(loop`$^{n+1}$`)`

Composite #1

Lemma. $\forall n.\ \mathtt{wind}(\mathtt{loop}^n) = n$

Proof: induction on n. E.g.

$$
\begin{aligned}
&\mathtt{wind}(\mathtt{loop}^{n+1}) \\
= \;&\mathtt{wind}(\mathtt{loop}\ \mathtt{o}\ \mathtt{loop}^n) && [\text{def. } \mathtt{loop}^{-}]
\end{aligned}
$$

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Lemma. $\forall n.\ \texttt{wind}(\texttt{loop}^n) = n$

Proof: induction on n. E.g.

$$
\begin{aligned}
& \texttt{wind}(\texttt{loop}^{n+1}) \\
&= \texttt{wind}(\texttt{loop o loop}^n) && [\texttt{def. loop}^-] \\
&= \texttt{transport}_{\texttt{Cover}}(\texttt{loop o loop}^n\texttt{,0}) && [\texttt{def. wind}]
\end{aligned}
$$

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$$
\begin{aligned}
& \mathtt{wind}(\mathtt{loop}^{n+1}) \\
= \ & \mathtt{wind}(\mathtt{loop} \circ \mathtt{loop}^n) && [\mathtt{def.\ loop}^-] \\
= \ & \mathtt{transport}_{\mathtt{Cover}}(\mathtt{loop} \circ \mathtt{loop}^n, 0) && [\mathtt{def.\ wind}] \\
= \ & \mathtt{transport}_{\mathtt{Cover}}(\mathtt{loop}, \\
& \qquad \mathtt{transport}_{\mathtt{Cover}}(\mathtt{loop}^n, 0)) && [\mathtt{functorial}]
\end{aligned}
$$

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Proof: induction on n. E.g.

$$
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&= \mathtt{wind}(\mathtt{loop} \circ \mathtt{loop}^{n}) && [\text{def. } \mathtt{loop}^{-}] \\
&= \mathtt{transport}_{\mathtt{Cover}}(\mathtt{loop} \circ \mathtt{loop}^{n}, 0) && [\text{def. } \mathtt{wind}] \\
&= \mathtt{transport}_{\mathtt{Cover}}(\mathtt{loop}, \\
&\qquad \mathtt{transport}_{\mathtt{Cover}}(\mathtt{loop}^{n}, 0)) && [\text{functorial}] \\
&= \mathtt{transport}_{\mathtt{Cover}}(\mathtt{loop}, n) && [\text{IH}]
\end{aligned}
$$

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Proof: induction on n. E.g.

$$
\begin{aligned}
& \mathtt{wind(loop}^{n+1}\mathtt{)} \\
&= \mathtt{wind(loop\ o\ loop}^{n}\mathtt{)} && \mathtt{[def.\ loop}^{-}\mathtt{]} \\
&= \mathtt{transport}_{\mathtt{Cover}}\mathtt{(loop\ o\ loop}^{n}\mathtt{,0)} && \mathtt{[def.\ wind]} \\
&= \mathtt{transport}_{\mathtt{Cover}}\mathtt{(loop,} \\
&\qquad \mathtt{transport}_{\mathtt{Cover}}\mathtt{(loop}^{n}\mathtt{,0))} && \mathtt{[functorial]} \\
&= \mathtt{transport}_{\mathtt{Cover}}\mathtt{(loop,n)} && \mathtt{[IH]} \\
&= \mathtt{n+1} && \mathtt{[def.\ Cover]}
\end{aligned}
$$

Composite #1

```
wind-loop^ : (n : Int) → Path (wind (loop^ n)) n
wind-loop^ Zero = id
wind-loop^ (Pos One) = ap≃ transport-Cover-loop
wind-loop^ (Pos (S n)) =
  transport Cover (loop ∙ loop^ (Pos n)) Zero        ≃⟨ ap≃ (transport-∙ Cover loop (loop^ (Pos n))) ⟩
  transport Cover loop
           (transport Cover (loop^ (Pos n)) Zero)  ≃⟨ ap (transport Cover loop) (wind-loop^ (Pos n))  ⟩
  transport Cover loop (Pos n)                      ≃⟨ ap≃ transport-Cover-loop ⟩
  succ (Pos n) ∎
wind-loop^ (Neg One) = ap≃ transport-Cover-!loop
wind-loop^ (Neg (S n)) =
  transport Cover (! loop ∙ loop^ (Neg n)) Zero      ≃⟨ ap≃ (transport-∙ Cover (! loop) (loop^ (Neg n))) ⟩
  transport Cover (! loop)
           (transport Cover (loop^ (Neg n)) Zero)   ≃⟨ ap≃ transport-Cover-!loop ⟩
  pred (transport Cover (loop^ (Neg n)) Zero)        ≃⟨ ap pred (wind-loop^ (Neg n)) ⟩
  pred (Neg n) ∎
```

Composite #2

Lem. $\forall p{:}\mathtt{base}{=}\mathtt{base}.\ \mathtt{loop}^{\mathtt{wind}(p)} = p$

Composite #2

Lem. $\forall$`p:base=base. loop`$^{\texttt{wind(p)}}$ `= p`

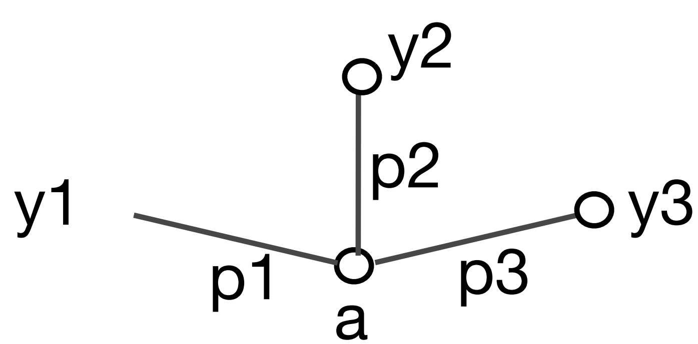
Proof: want to apply path induction
but path induction does not apply to loops

Composite #2

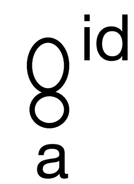
Lem. $\forall p{:}\mathtt{base}{=}\mathtt{base}.\ \mathtt{loop}^{\mathtt{wind}(p)} = p$

Proof: want to apply path induction
but path induction does not apply to loops

Type of paths from a to somewhere



is inductively generated by



Composite #2

Lem. $\forall y:S^1,\ p:\mathsf{base}=y.\ \mathsf{loop}^{\mathsf{wind}(p)} = p$

Composite #2

Lem. $\forall y{:}S^1,\ p{:}\mathtt{base}{=}y.\ \mathtt{loop}^{\mathtt{wind}(p)} = p$

Proof:

$\mathtt{wind} \quad : \mathtt{base}{=}\mathtt{base} \to \mathbb{Z}$

$\mathtt{wind}(p) = \mathtt{transport}_{\mathtt{Cover}}(p,0)$

$\mathtt{loop}^{-} \quad : \mathbb{Z} \to \Omega(S^1)$

Composite #2

Lem. $\forall y{:}S^1,\ p{:}\mathtt{base}{=}y.\ \mathtt{loop}^{\mathtt{wind}(p)} = p$

Proof: need to generalize wind

$$\mathtt{wind} : \mathtt{base}{=}\mathtt{base} \to \mathbb{Z}$$

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$$\mathtt{loop}^{-} : \mathbb{Z} \to \Omega(S^1)$$

Composite #2

Lem. $\forall y{:}S^1,\ p{:}\mathrm{base}{=}y.\ \mathtt{loop}^{\mathtt{wind}(p)} = p$

Proof:

$$\mathtt{encode} \;:\; \forall y{:}S^1.\ \mathtt{base}{=}y \to \mathtt{Cover}(y)$$
$$\mathtt{encode}(p) = \mathtt{transport}_{\mathtt{Cover}}(p,0)$$

$$\mathtt{loop}^{-} \;:\; \mathbb{Z} \to \Omega(S^1)$$

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$$\mathtt{decode} \ : \ \forall y{:}S^1.\ \mathtt{Cover}(y) \to \mathtt{base}{=}y$$

Composite #2

Lem. $\forall y{:}S^1,\ p{:}\mathrm{base}{=}y.\ \mathrm{decode}_y(\mathrm{encode}_y(p)) = p$

Proof: need to generalize wind and loop$^-$

$$\mathrm{encode} \quad : \forall y{:}S^1.\ \mathrm{base}{=}y \to \mathrm{Cover}(y)$$
$$\mathrm{encode}(p) = \mathrm{transport}_{\mathrm{Cover}}(p, 0)$$

$$\mathrm{decode} \quad : \forall y{:}S^1.\ \mathrm{Cover}(y) \to \mathrm{base}{=}y$$

Decode

$$\texttt{decode} \;:\; \forall y{:}S^1.\ \texttt{Cover}(y) \to \texttt{base}{=}y$$

Defined by circle induction:

```
decode(base) := loop⁻
decode(loop) := …
```

Decode

$$\texttt{decode} \;:\; \forall y{:}S^1.\ \texttt{Cover}(y) \to \texttt{base}=y$$

Defined by circle induction:

Cover(base) → base=base

```
decode(base) := loop⁻
decode(loop) := …
```

Decode

`decode : ∀y:S¹. Cover(y) → base=y`

Defined by circle induction:

Cover(base) → base=base

```
decode(base) := loop⁻
decode(loop) := …
```

"`loop⁻` is invariant under going around the `loop` in the fibration `Cover(y) → base=y`"

Decode

```
decode : {x : S¹} → Cover x → Path base x
decode {x} =
 S¹-induction
  (λ x' → Cover x' → Path base x')
  loop^
  loop^-respects-loop
  x where
    loop^-respects-loop : transport (λ x' →  Cover x' → Path base x') loop loop^ ≡ (λ n → loop^ n)
    loop^-respects-loop =
       (transport (λ x' →  Cover x' → Path base x') loop loop^  ≃〈 transport-→ Cover (Path base) loop loop^ 〉

          transport (λ x' → Path base x') loop
        o loop^
        o transport Cover (! loop)                             ≃〈 λ≃ (λ y → transport-Path-right loop (loop^ (transport Cover (! loop) y))) 〉

          (λ p → loop · p)
        o loop^
        o transport Cover (! loop)                             ≃〈 λ≃ (λ y → ap (λ x' → loop · loop^ x') (ap≃ transport-Cover-!loop)) 〉

          (λ p → loop · p)
        o loop^
        o pred                                                 ≃〈 id 〉

        (λ n → loop · (loop^ (pred n)))                        ≃〈 λ≃ (λ y → move-left-! _ loop (loop^ y) (loop^-preserves-pred y)) 〉

        (λ n → loop^ n)
        ∎)
```

Composite #2

Lem. $\forall y{:}S^1,\ p{:}base{=}y.\ \mathrm{decode}_y(\mathrm{encode}_y(p)) = p$

$\mathrm{encode} \ :\ \forall y{:}S^1.\ base{=}y \to \mathrm{Cover}(y)$

$\mathrm{encode}_y(p) = \mathrm{transport}_{\mathrm{Cover}}(p, 0)$

$\mathrm{decode} \ :\ \forall y{:}S^1.\ \mathrm{Cover}(y) \to base{=}y$

$\mathrm{decode}_{base}(n) = \mathrm{loop}^n$

Composite #2

Lem. $\forall y{:}S^1,\ p{:}\mathrm{base}{=}y.\ \ \mathrm{decode}_y(\mathrm{encode}_y(p)) = p$

Proof: By **path induction**, suffices to show

$$\mathrm{decode}_{\mathrm{base}}(\mathrm{encode}_{\mathrm{base}}(\mathrm{id}))$$

$$= \mathrm{id}$$

$\mathrm{encode} \ :\ \forall y{:}S^1.\ \mathrm{base}{=}y \to \mathrm{Cover}(y)$

$\mathrm{encode}_y(p) = \mathrm{transport}_{\mathrm{Cover}}(p,0)$

$\mathrm{decode} \ :\ \forall y{:}S^1.\ \mathrm{Cover}(y) \to \mathrm{base}{=}y$

$\mathrm{decode}_{\mathrm{base}}(n) = \mathrm{loop}^n$

Composite #2

Lem. $\forall y{:}S^1,\ p{:}base{=}y.\ \ decode_y(encode_y(p)) = p$

Proof: By **path induction**, suffices to show

$$decode_{base}(encode_{base}(id))$$

$$= decode_{base}(0)$$

$$= id$$

$encode\ :\ \forall y{:}S^1.\ base{=}y \rightarrow Cover(y)$

$encode_y(p) = transport_{Cover}(p,0)$

$decode\ :\ \forall y{:}S^1.\ Cover(y) \rightarrow base{=}y$

$decode_{base}(n) = loop^n$

Composite #2

Lem. $\forall y{:}S^1,\ p{:}\mathrm{base}{=}y.\ \ \mathrm{decode}_y(\mathrm{encode}_y(p)) = p$

Proof: By **path induction**, suffices to show

$$
\begin{aligned}
&\mathrm{decode}_{\mathrm{base}}(\mathrm{encode}_{\mathrm{base}}(\mathrm{id}))\\
&= \mathrm{decode}_{\mathrm{base}}(0)\\
&= \mathrm{loop}^0\\
&= \mathrm{id}
\end{aligned}
$$

$\mathrm{encode} \ : \ \forall y{:}S^1.\ \mathrm{base}{=}y \to \mathrm{Cover}(y)$

$\mathrm{encode}_y(p) = \mathrm{transport}_{\mathrm{Cover}}(p,0)$

$\mathrm{decode} \ : \ \forall y{:}S^1.\ \mathrm{Cover}(y) \to \mathrm{base}{=}y$

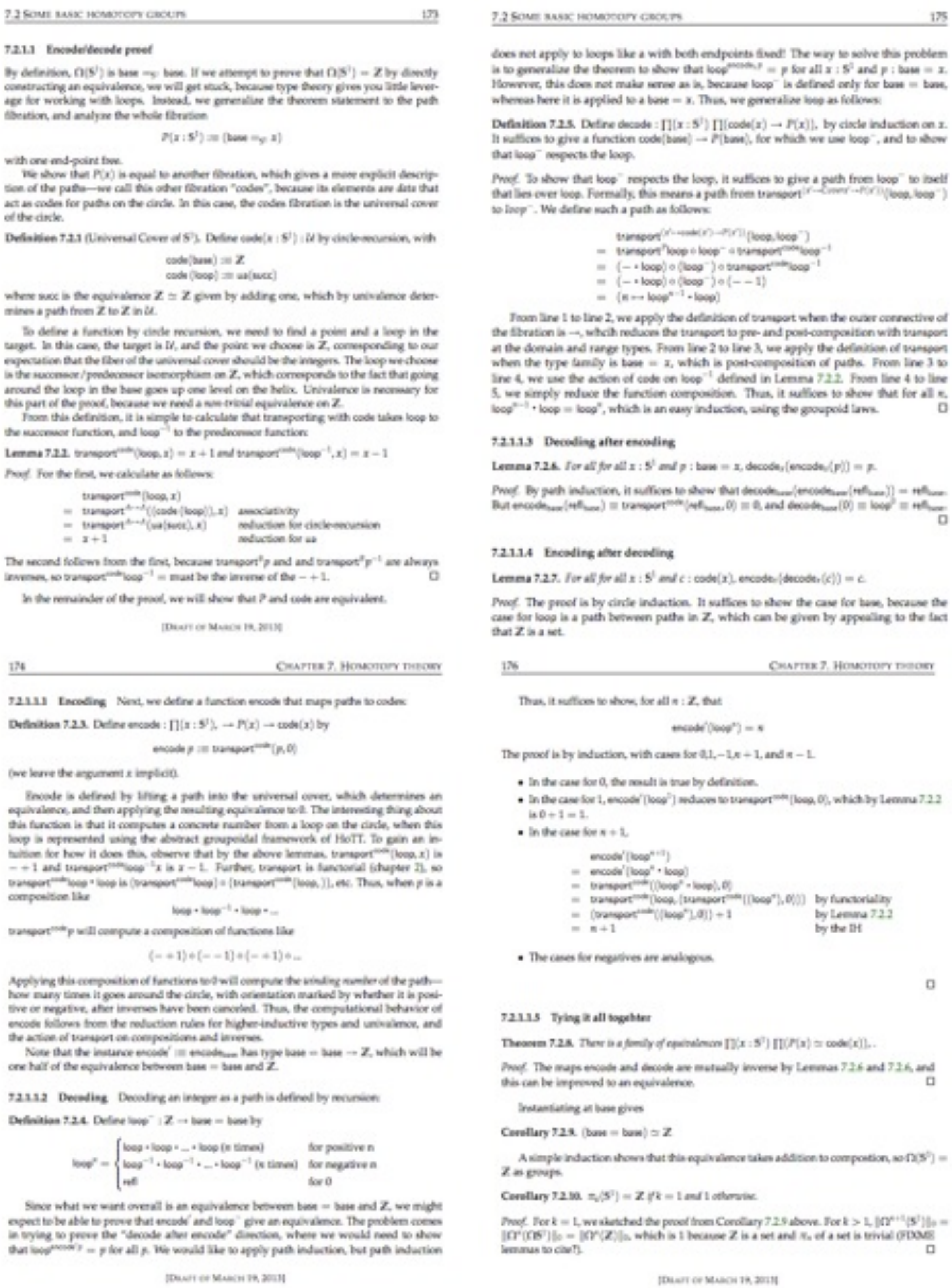
$\mathrm{decode}_{\mathrm{base}}(n) = \mathrm{loop}^n$

Composite #2

```
decode-encode  : {x : S¹} (α : Path base x)
                 → Path (decode (encode α)) α
decode-encode id = id
```

Fundamental group of the circle

The book



Computer-checked

Conclusion

We can do **computer-checked proofs** in **synthetic** homotopy theory

- Proofs are constructive*: can run them
- Results apply in a variety of settings, from simplicial sets (hence topological spaces) to Quillen model categories and ∞-topoi*
- New type-theoretic proofs/methods

*work in progress

Homotopy in HoTT

$\pi_1(S^1) = \mathbb{Z}$

$\pi_{k<n}(S^n) = 0$

Hopf fibration

$\pi_2(S^2) = \mathbb{Z}$

$\pi_3(S^2) = \mathbb{Z}$

James Construction

$\pi_4(S^3) = \mathbb{Z}_?$

Freudenthal

$\pi_n(S^n) = \mathbb{Z}$

K(G,n)

Cohomology axioms

Blakers-Massey

Van Kampen

Covering spaces

Whitehead for n-types

[Brunerie, Finster, Hou, Licata, Lumsdaine, Shulman]

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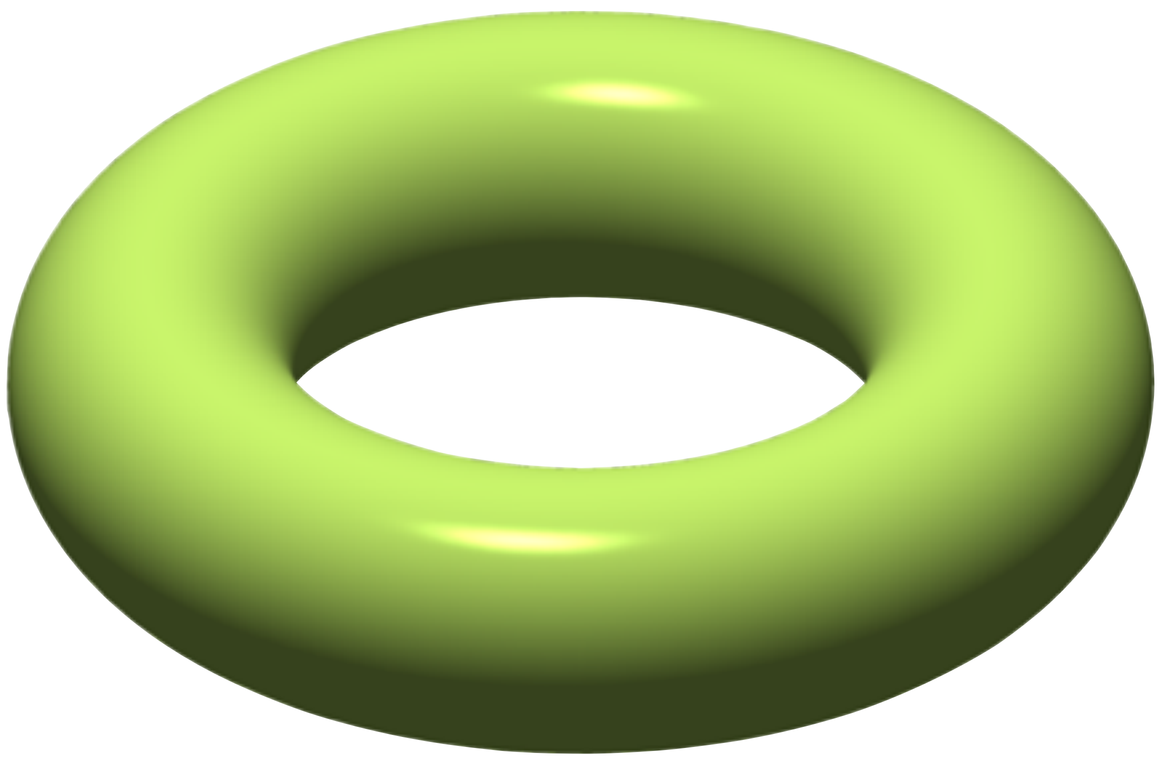
[Brunerie, Finster, Hou, Licata, Lumsdaine, Shulman]

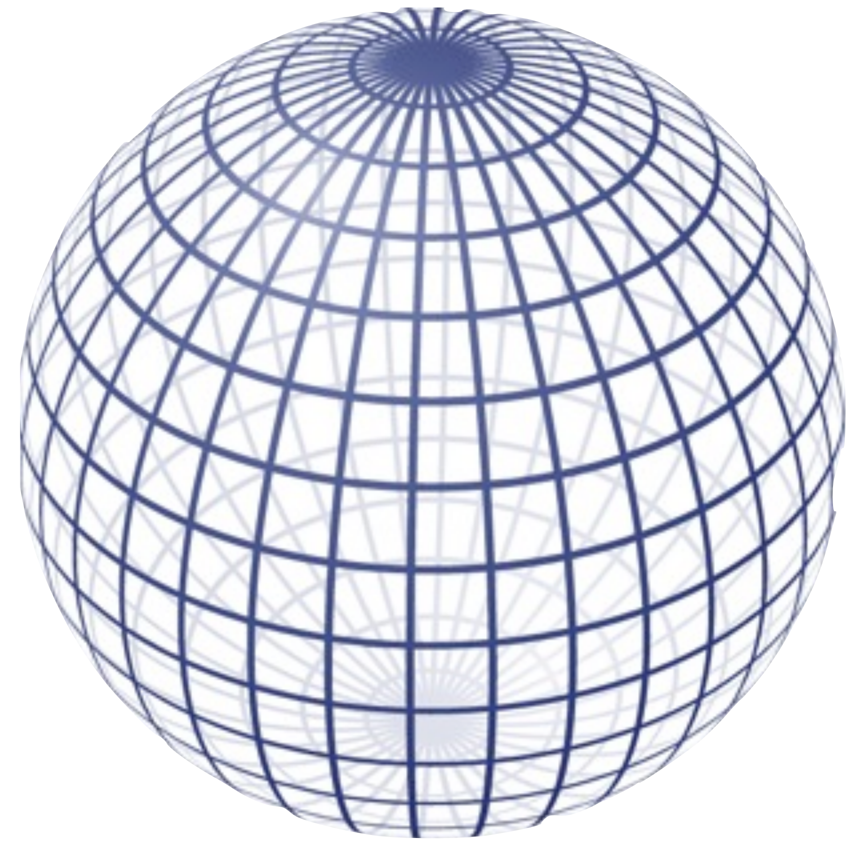
Reading list

- $\pi_1(S^1) = \mathbb{Z}$ [Licata and Shulman, LICS'13]
- Other results: forthcoming Homotopy Type Theory book
- Blog: `homotopytypetheory.org`
- Formalizations:
  `github.com/dlicata335/hott-agda`
  `github.com/hott/hott-agda`
  `github.com/hott/hott [Coq]`

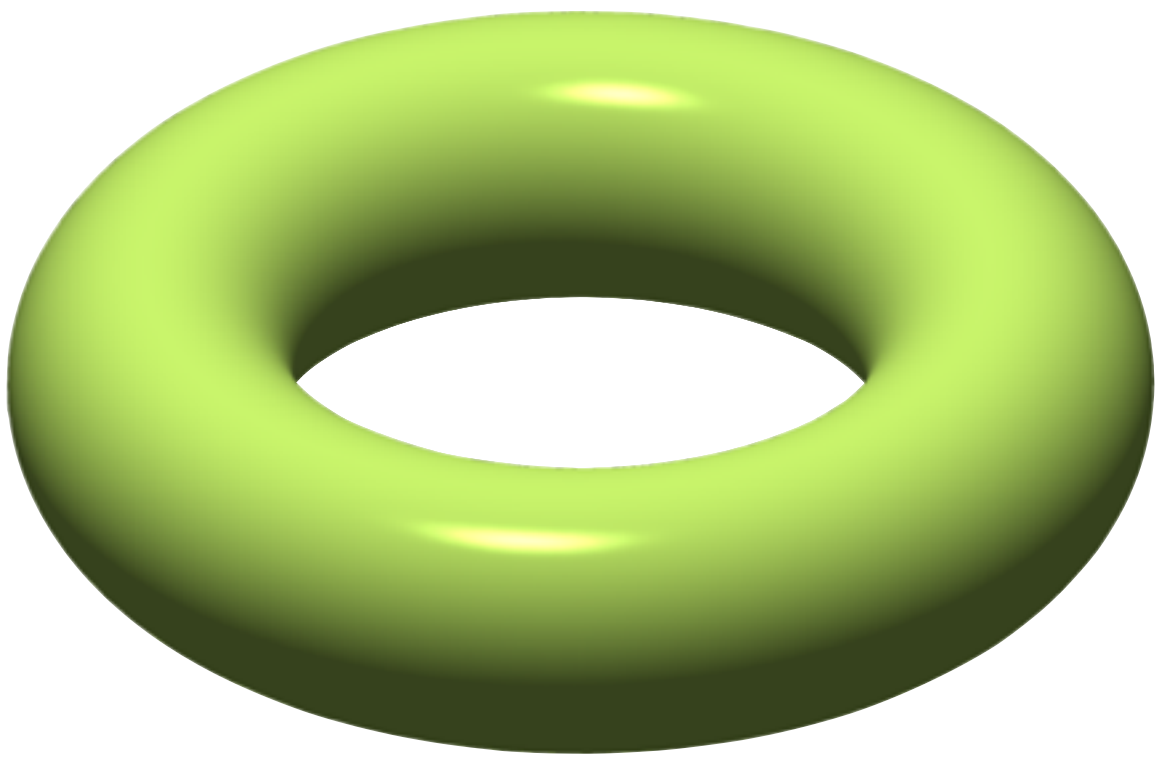
Algebraic invariants



$\neq$

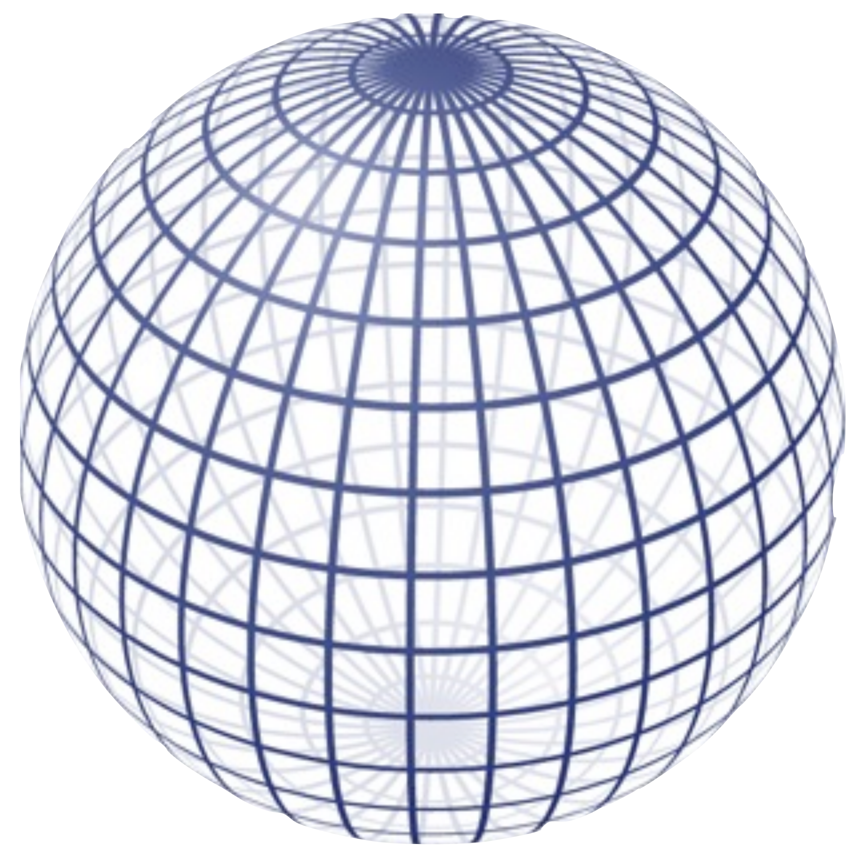


Algebraic invariants



fundamental group
is non-trivial ($\mathbb{Z} \times \mathbb{Z}$)

$\neq$



fundamental group
is trivial

Homotopy groups

Homotopy groups

- Fundamental group π_1: group of loops

Homotopy groups

- Fundamental group π_1: group of loops
- π_2: group of homotopies

Homotopy groups

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- π_3: group of homotopies between homotopies

Homotopy groups

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- π_2: group of homotopies
- π_3: group of homotopies between homotopies
- …

Homotopy groups

k^{th} homotopy group

n-dimensional sphere

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

[image from wikipedia]

$\pi_k(S^n)$ in HoTT

k^{th} homotopy group

n-dimensional sphere

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$									
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$							
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0					
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0			
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	

[image from wikipedia]

$\pi_k(S^n)$ in HoTT

k^{th} homotopy group

n-dimensional sphere

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	$\mathbf{Z}$	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	$\mathbf{Z}$	Z_2	Z_2									
S^5	0	0	0	0	$\mathbf{Z}$	Z_2	Z_2	Z_{24}							
S^6	0	0	0	0	0	$\mathbf{Z}$	Z_2	Z_2	Z_{24}	0					
S^7	0	0	0	0	0	0	$\mathbf{Z}$	Z_2	Z_2	Z_{24}	0	0			
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	Z_2	Z_2	Z_{24}	0	0	Z_2	

[image from wikipedia]

$\pi_n(S^n) = \mathbb{Z}$ for $n \geq 1$

Proof: Induction on n

- Base case: $\pi_1(S^1) = \mathbb{Z}$
- Inductive step: $\pi_{n+1}(S^{n+1}) = \pi_n(S^n)$

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n-truncation:
best approximation of a type such that all (n+1)-paths are equal

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n-truncation:
best approximation of a type such that all (n+1)-paths are equal

higher inductive type generated by
$\texttt{base}_n : S^n$
$\texttt{loop}_n : \Omega^n(S^n)$

$\tau_n(S^n) = \tau_n(\Omega(S^{n+1}))$

n-truncation of S^n is the type of "codes" for loops on S^{n+1}

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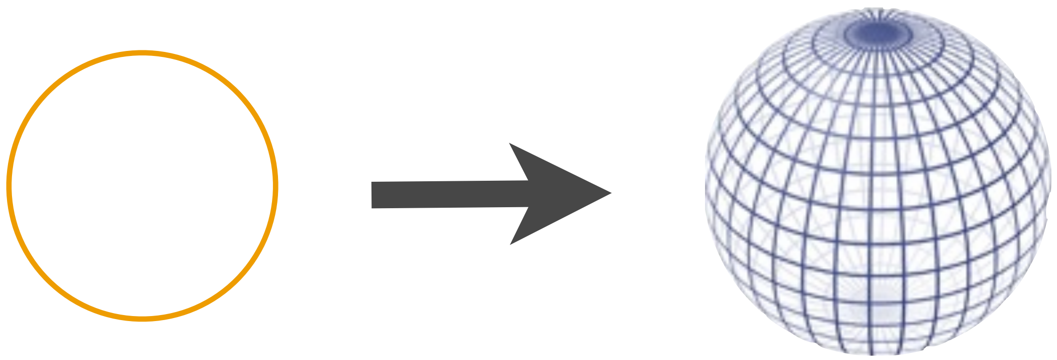
- Decode: promote n-dimensional loop on S^n to n+1-dimensional loop on S^{n+1}



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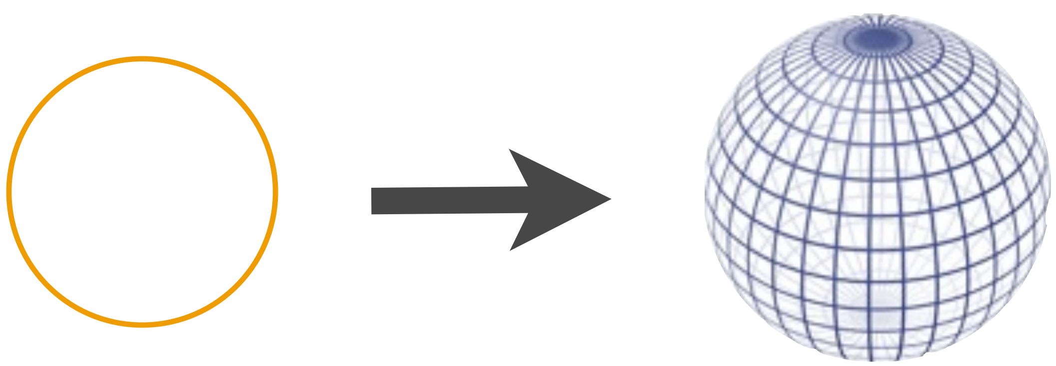

- Encode: define fibration $\texttt{Code}(x{:}S^{n+1})$ with
  $\texttt{Code}(\texttt{base}_{n+1}) := \tau_n(S^n)$
  $\texttt{Code}(\texttt{loop}_{n+1}) :=$ equivalence $\tau_n(S^n) \simeq \tau_n(S^n)$
  “rotating by $\texttt{loop}_n$”

$\pi_2(S^2)$: Hopf fibration