A Cubical Type Theory

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Programming in HoTT

$\mathtt{transport}_{\mathtt{Monoid}}\ \mathtt{ua}(\mathtt{f},\mathtt{g},\ldots)\ (\odot,\mathtt{u})$

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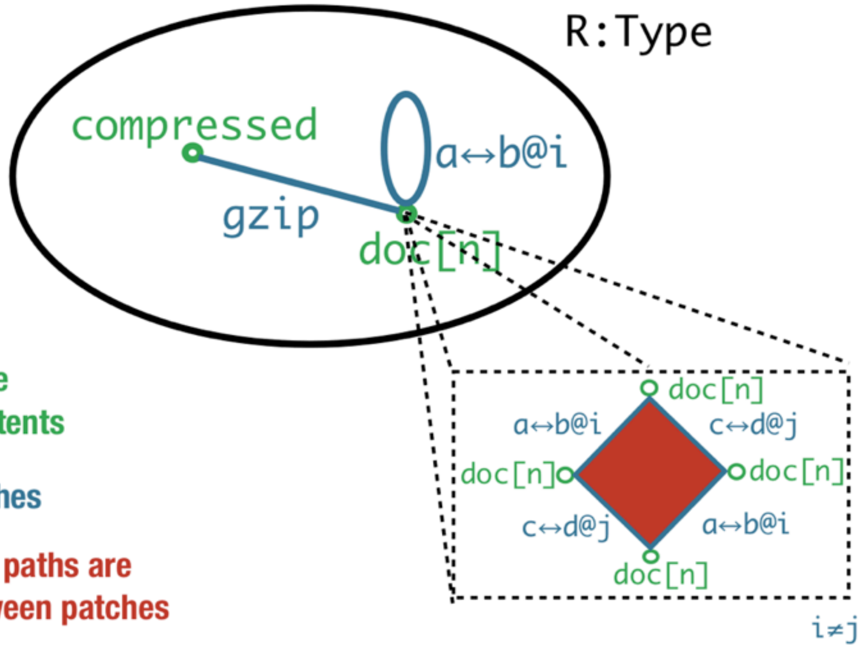
$$\mathtt{transport}_{\mathtt{Monoid}}\ \mathtt{ua}(\mathtt{f},\mathtt{g},\ldots)\ (\odot,\mathtt{u})$$



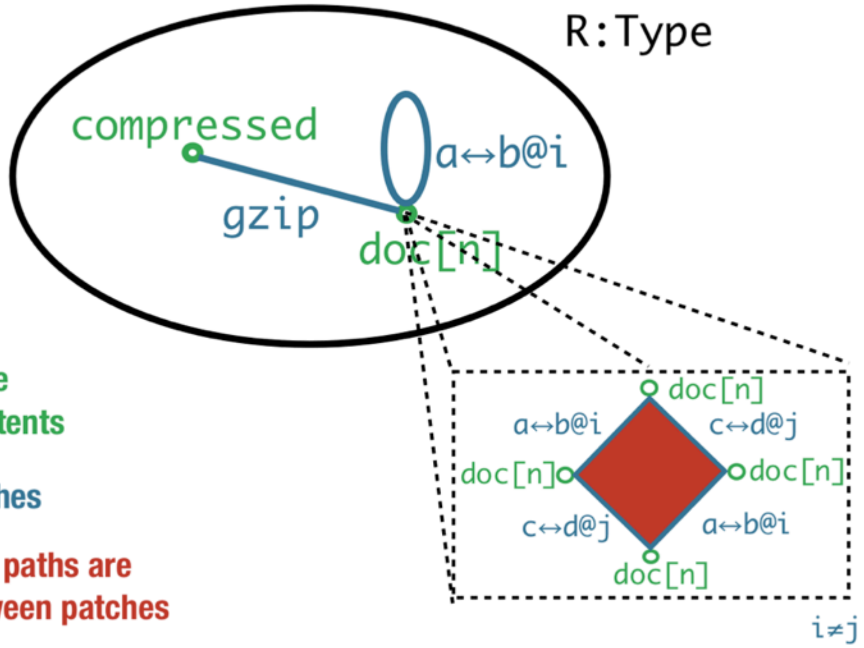
points describe repository contents

paths are patches

paths between paths are equations between patches

Programming in HoTT

$$\mathtt{transport}_{\mathtt{Monoid}}\ \mathtt{ua}(f,g,\ldots)\ (\odot,u)$$



points describe repository contents

paths are patches

paths between paths are equations between patches

$$\Sigma k.\pi_4(S^3) = \mathbb{Z}_k$$

Goal

HoTT as a programming language:

1. Type system
2. Decidable definitional equality on open terms
3. Operational semantics on closed terms satisfying preservation, canonicity for nat

Types are ∞-groupoids

id : $M = M$

α^{-1} : $N = M$ if $\alpha : M = N$

β o α : $M = P$ if $\alpha : M = N$ and $\beta : N = P$

unitl : id o $p = p$

invl : p^{-1} o p = id

assoc : r o $(q$ o $p) = (r$ o $q)$ o p

Types are ∞-groupoids

$\mathrm{id} : M = M$
$\alpha^{-1} : N = M$ if $\alpha : M = N$
$\beta \circ \alpha : M = P$ if $\alpha : M = N$ and $\beta : N = P$

$\mathrm{unitl} : \mathrm{id} \circ p = p$
$\mathrm{invl} : p^{-1} \circ p = \mathrm{id}$
$\mathrm{assoc} : r \circ (q \circ p) = (r \circ q) \circ p$

$\vdots$

Groupoid Structure

Reformulate identity, inverses, composition as:
identity + one ternary operation

Need to define this in each type

Groupoid Structure

Reformulate identity, inverses, composition as:
identity + one ternary operation

$$p : a = b$$
$$q : a = c$$
$$r : c = d$$

$$\begin{tikzcd}
a \arrow[r, "q"] \arrow[d, "p"'] & c \arrow[d, "r"] \\
b & d
\end{tikzcd}$$

Need to define this in each type

Groupoid Structure

Reformulate identity, inverses, composition as: identity + one ternary operation

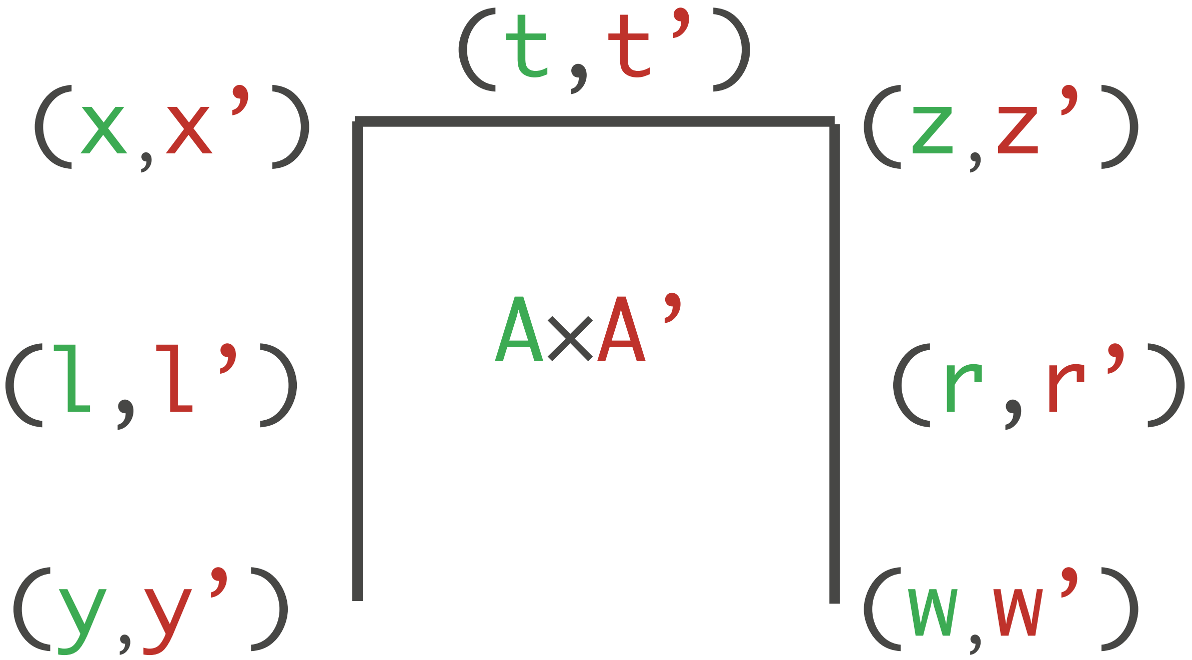
$$
\begin{aligned}
p &: a = b \\
q &: a = c \\
r &: c = d
\end{aligned}
$$

$$\begin{tikzcd}
a \arrow[r, "q"] \arrow[d, "p"'] & c \arrow[d, "r"] \\
b \arrow[r, dashed] & d
\end{tikzcd}$$

Need to define this in each type

$$\frac{p : x =_{A} y \qquad p' : x' =_{A'} y'}{(p,p') : (x,x') =_{A\times A'} (y,y')}$$

$$\frac{p : x =_{A} y \qquad p' : x' =_{A'} y'}{(p,p') : (x,x') =_{A\times A'} (y,y')}$$



(t,t')

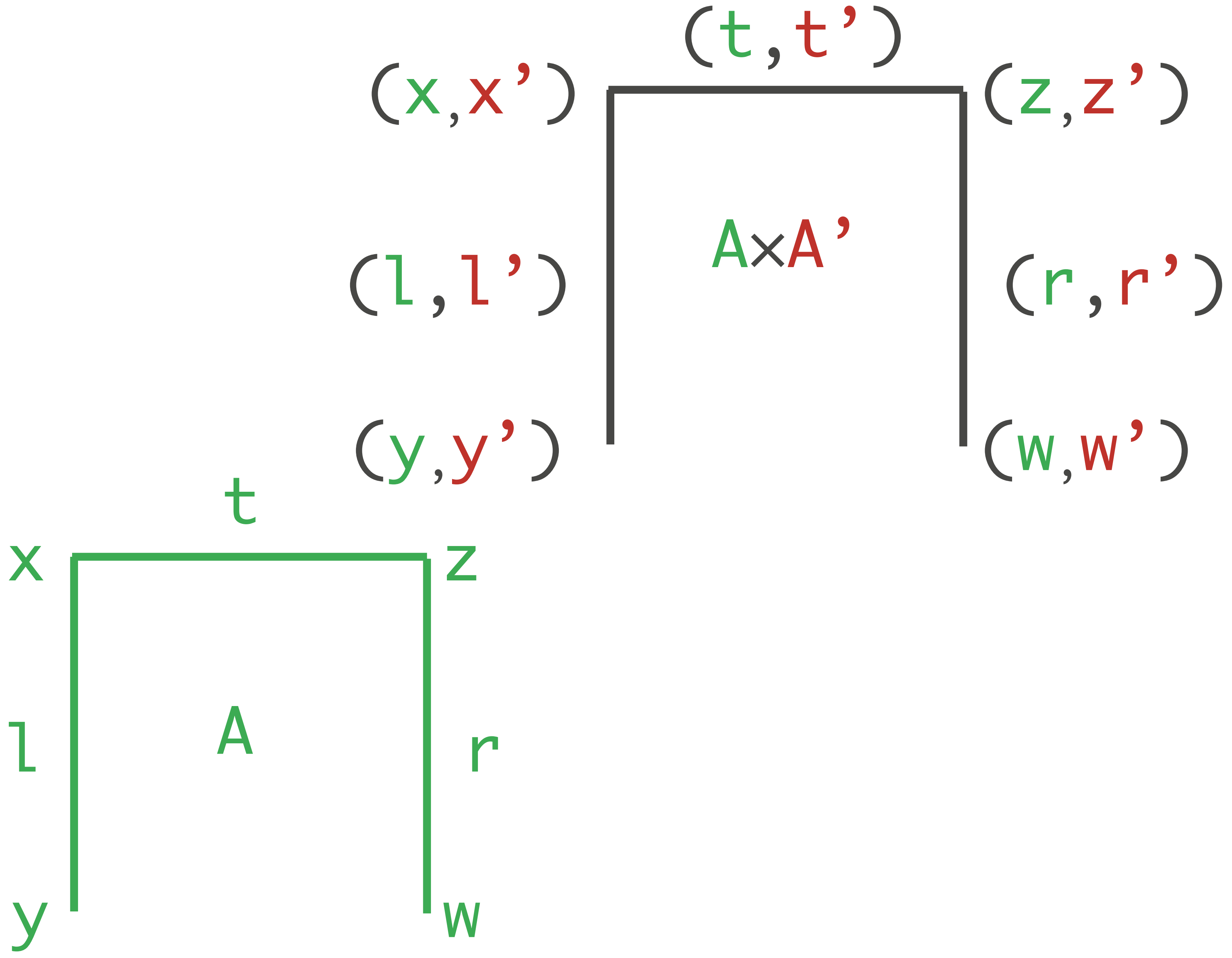
(x,x') (z,z')

$A\times A'$

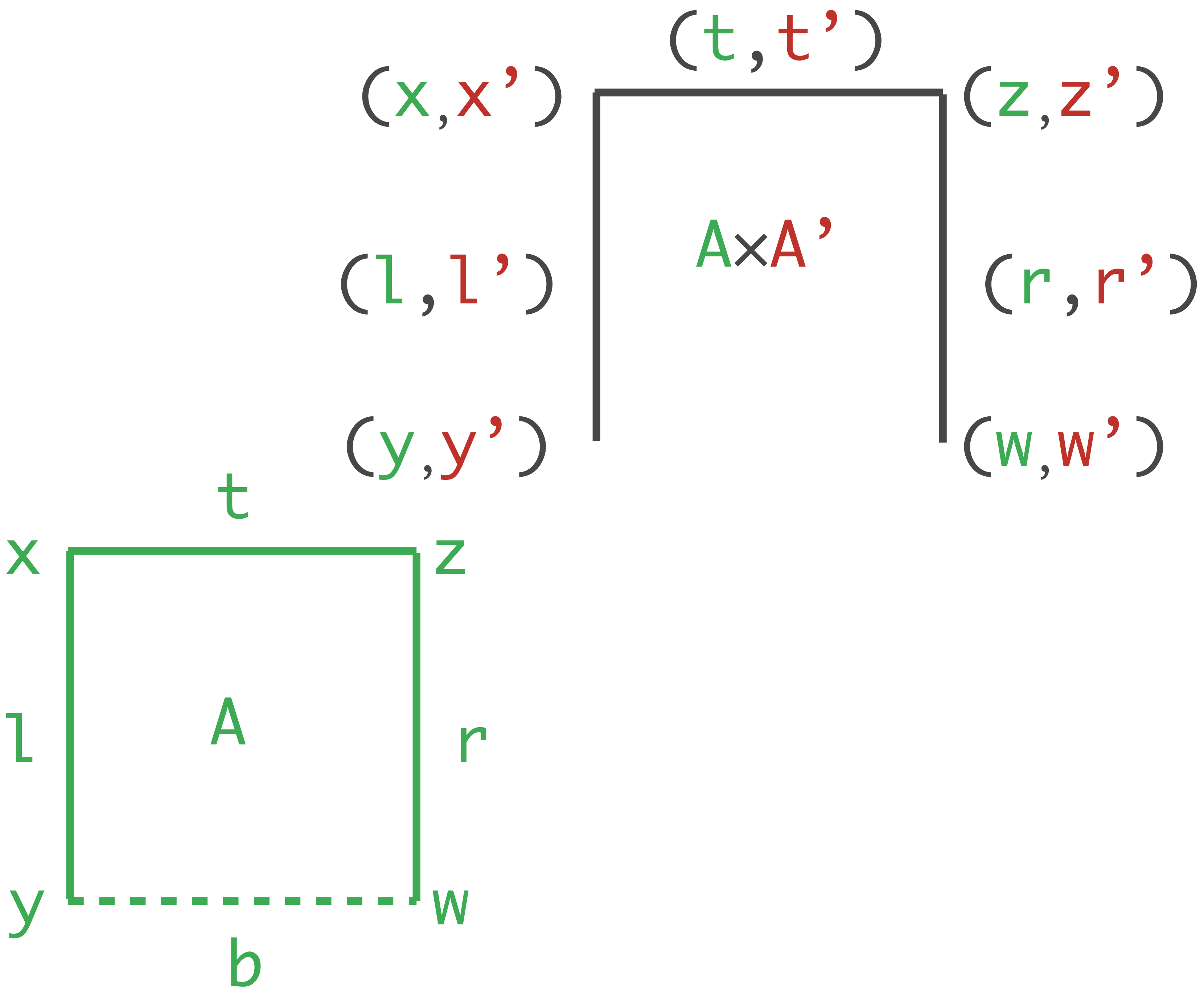
(l,l') (r,r')

(y,y') (w,w')

$$\frac{p : x =_{A} y \qquad p' : x' =_{A'} y'}{(p,p') : (x,x') =_{A\times A'} (y,y')}$$



$$\frac{p : x =_{A} y \qquad p' : x' =_{A'} y'}{(p,p') : (x,x') =_{A\times A'} (y,y')}$$



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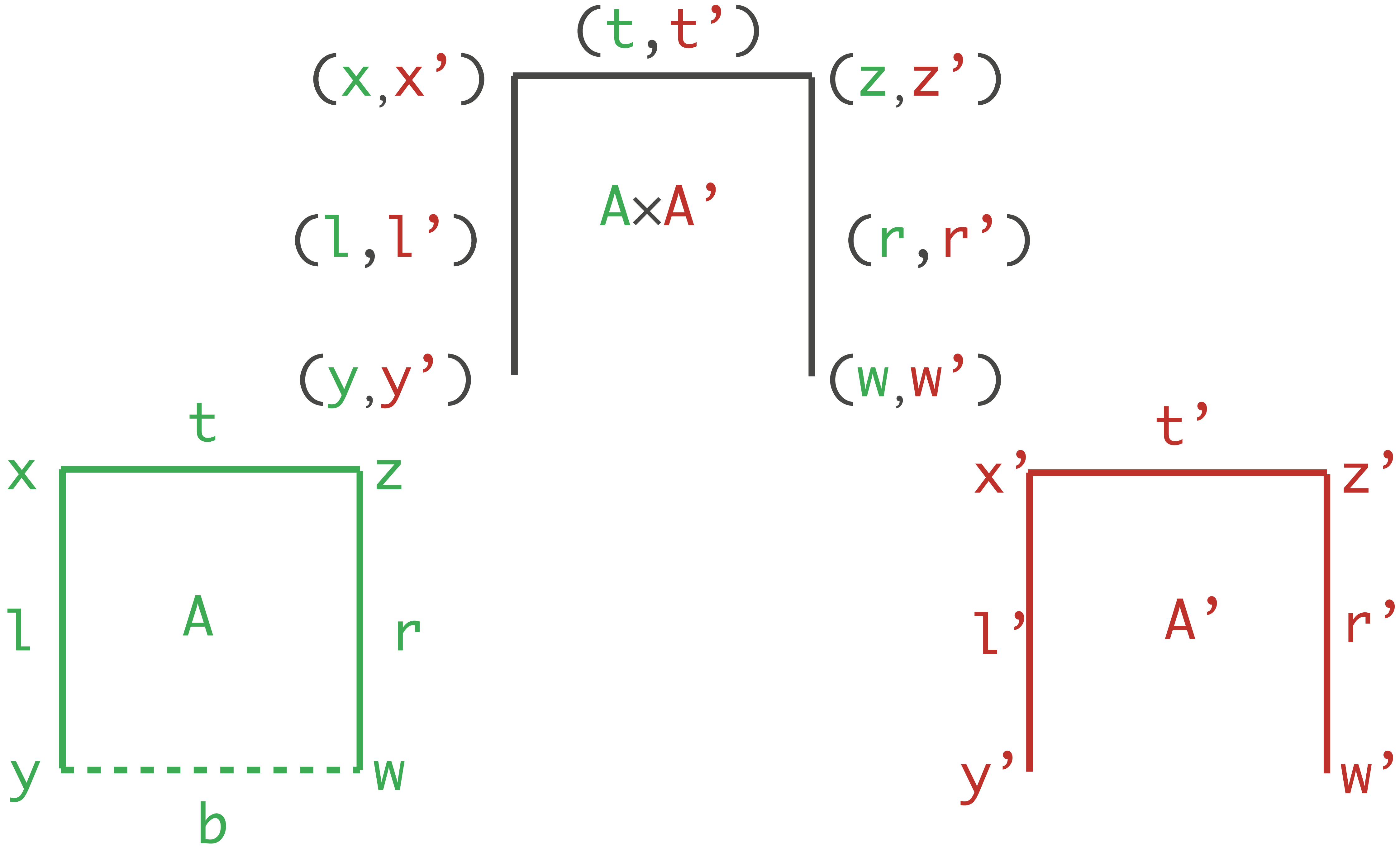

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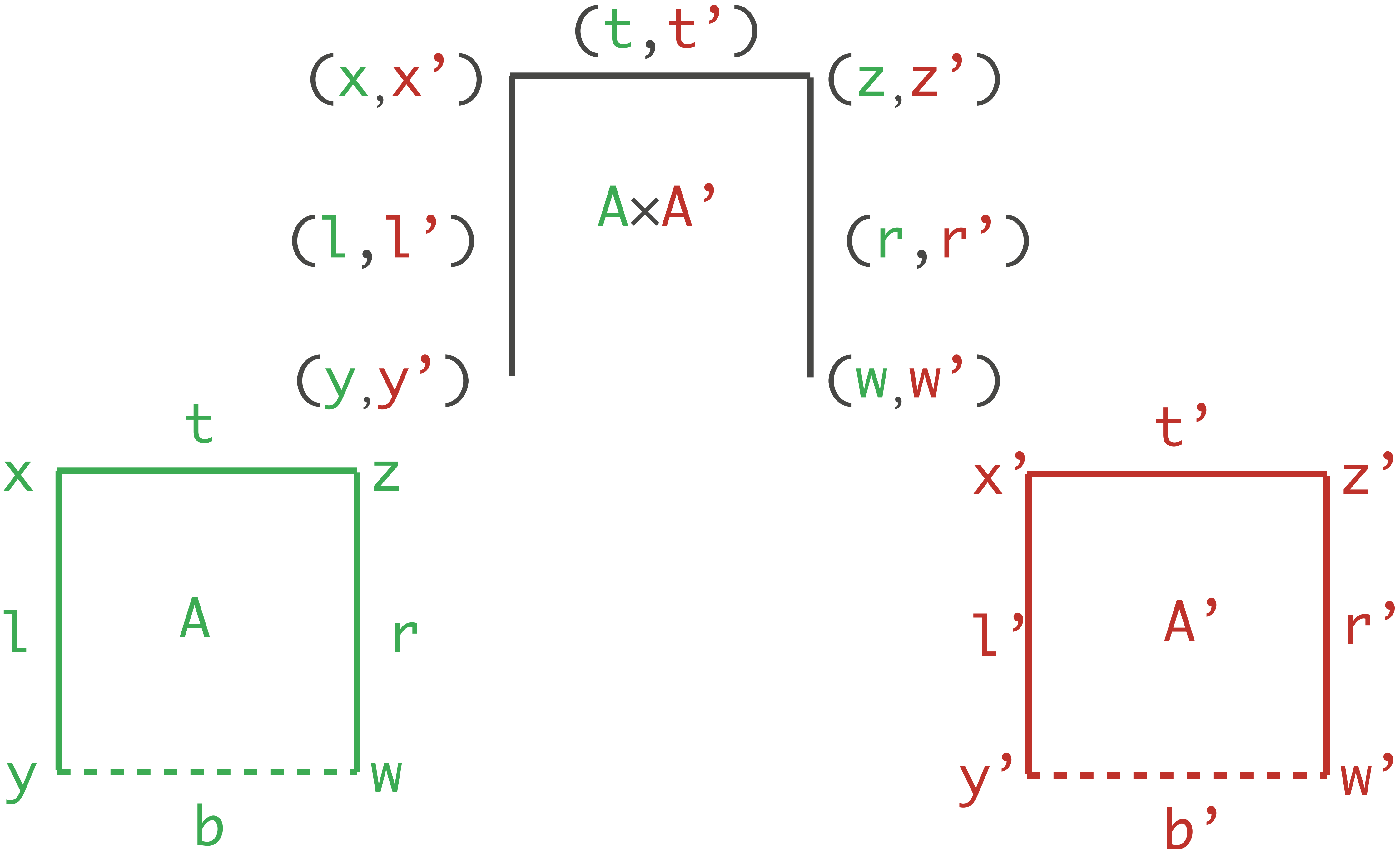

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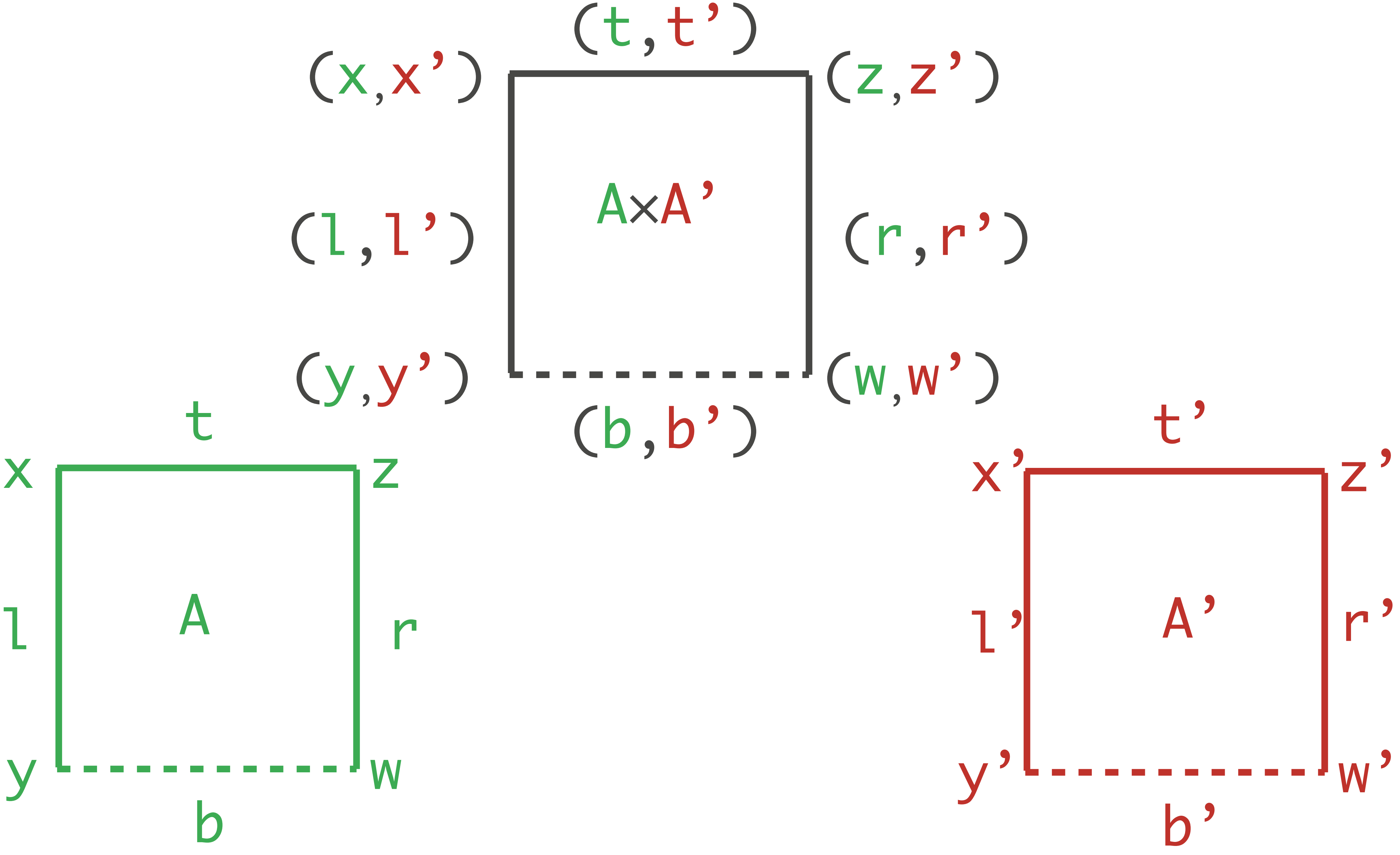
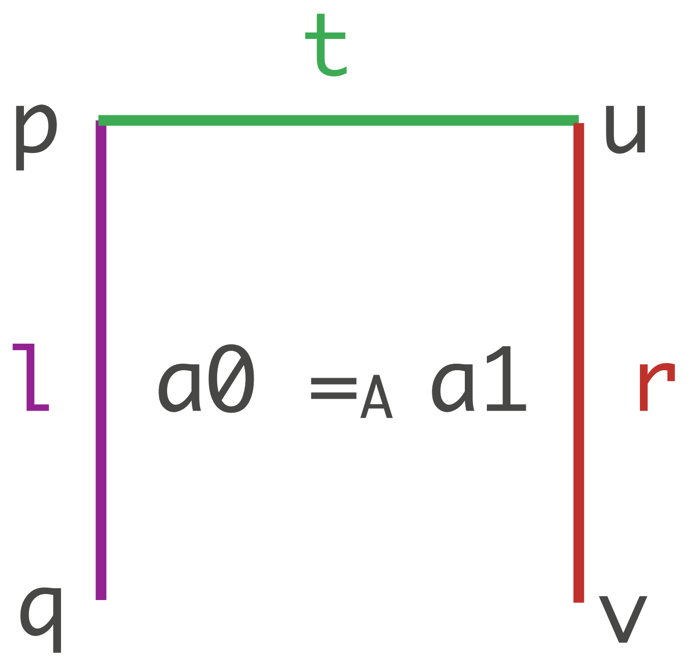
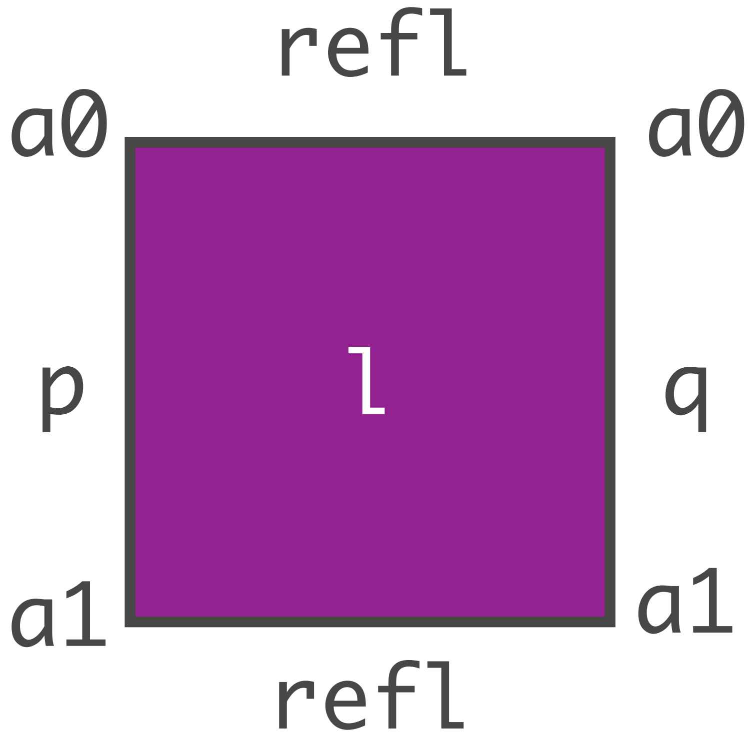
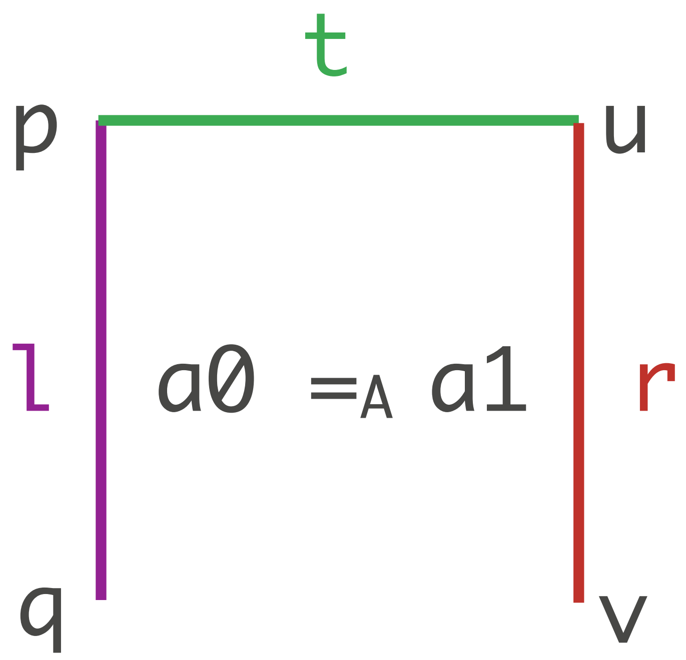




$p\ q\ u\ v : a0 =_A a1$

$l : p =_{a0=a1} q$ $t : p =_{a0=a1} u$ $r : u =_{a0=a1} v$





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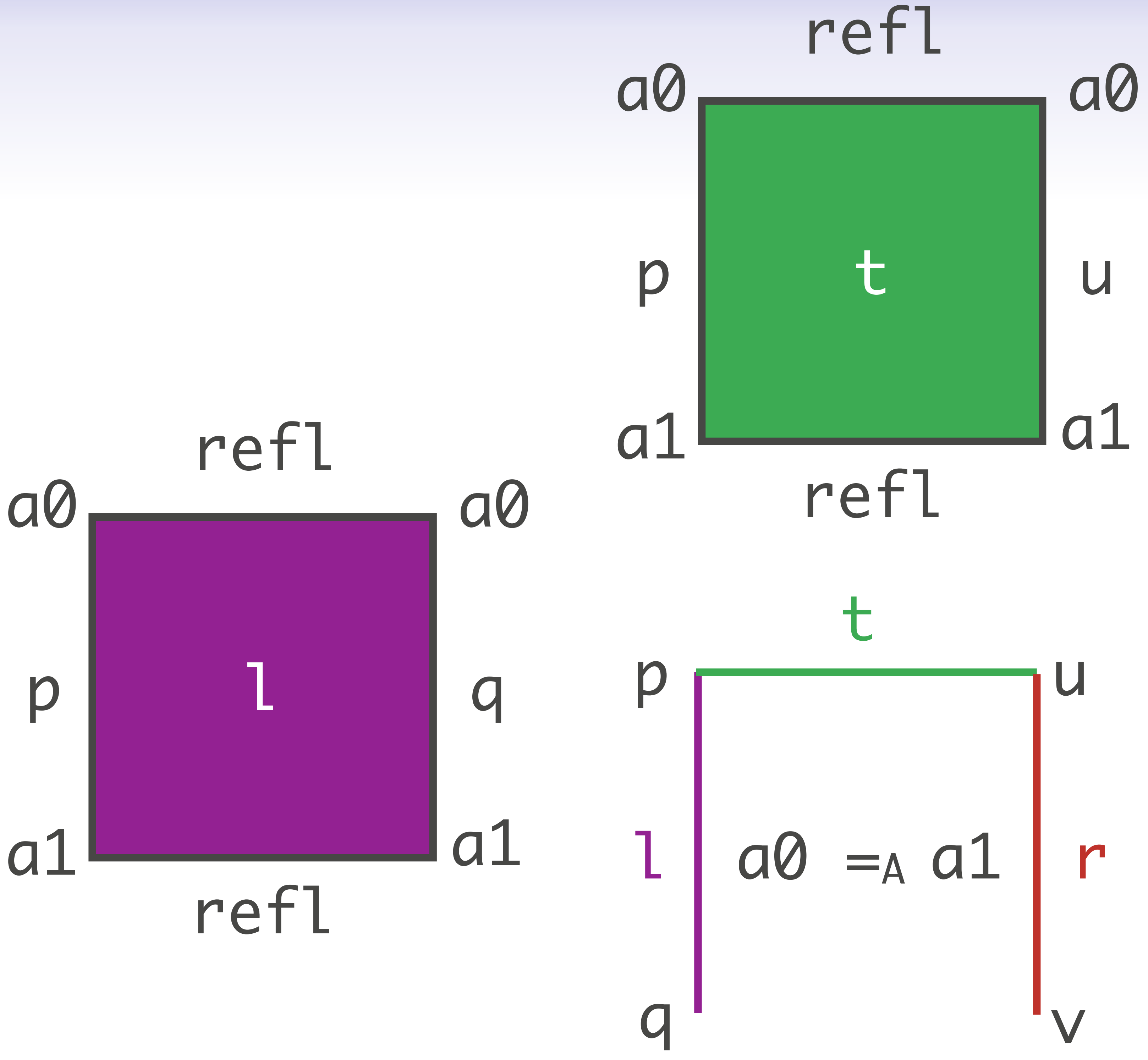

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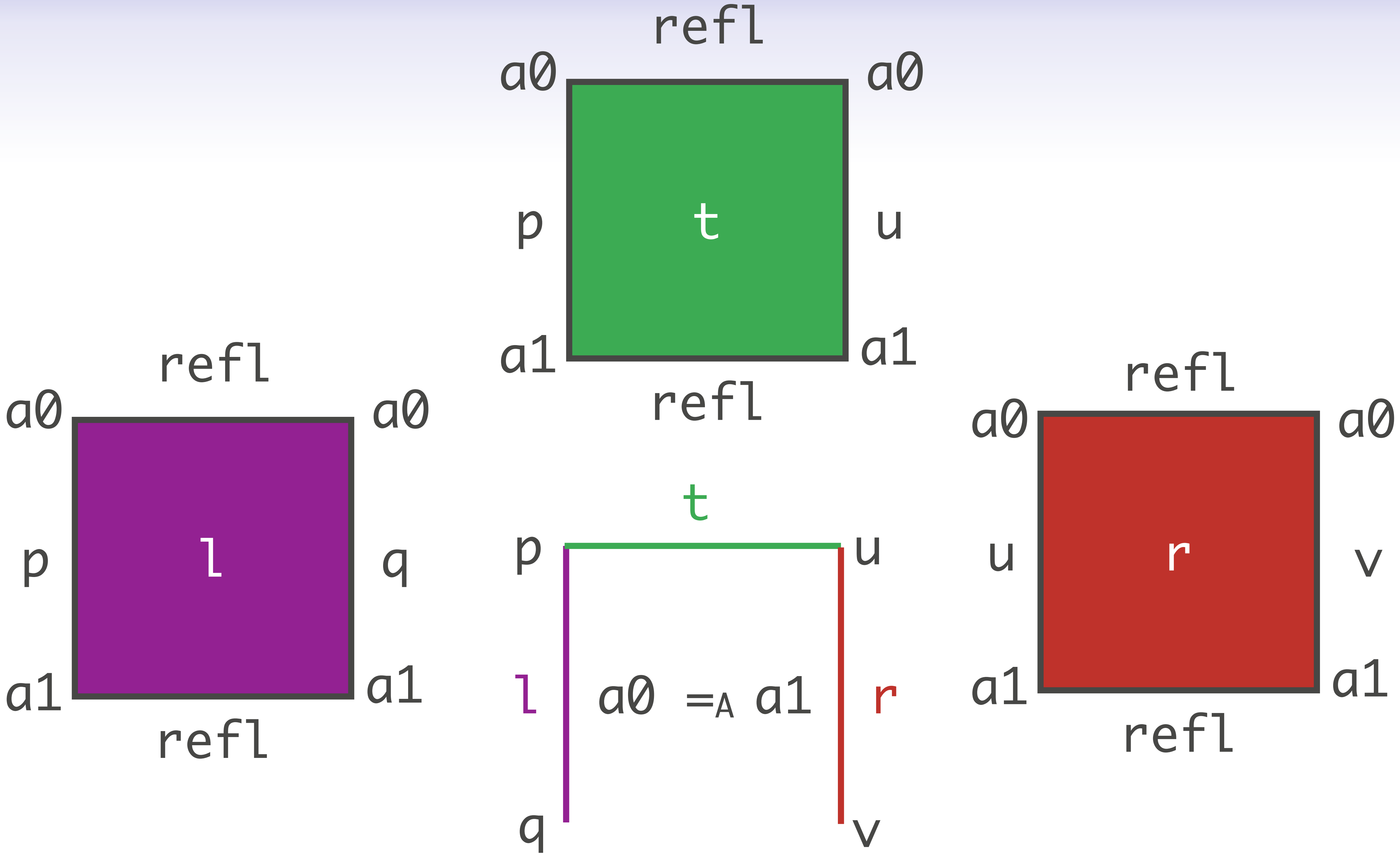

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p t u

a1 a1

refl

refl

a0 a0

p l q

a1 a1

refl

t

p u

l $a0 =_A a1$ r

q v

refl

a0 a0

u r v

a1 a1

refl

$p\ q\ u\ v : a0 =_A a1$

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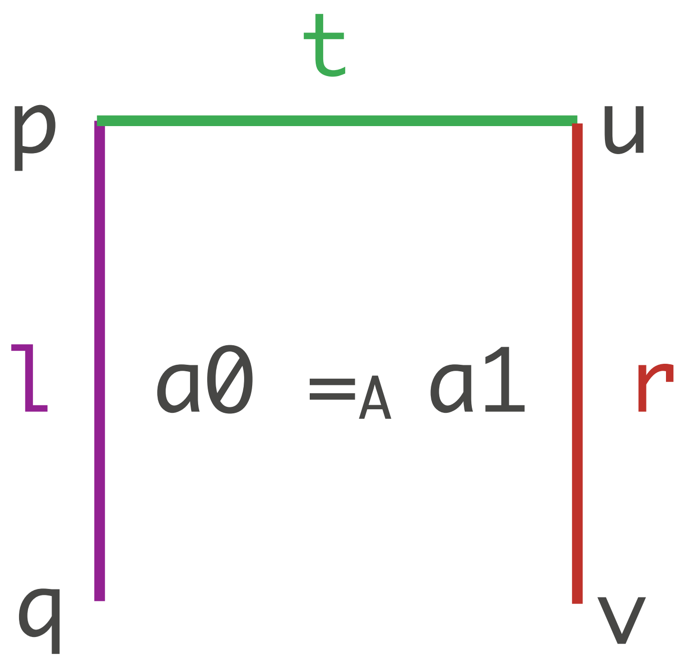
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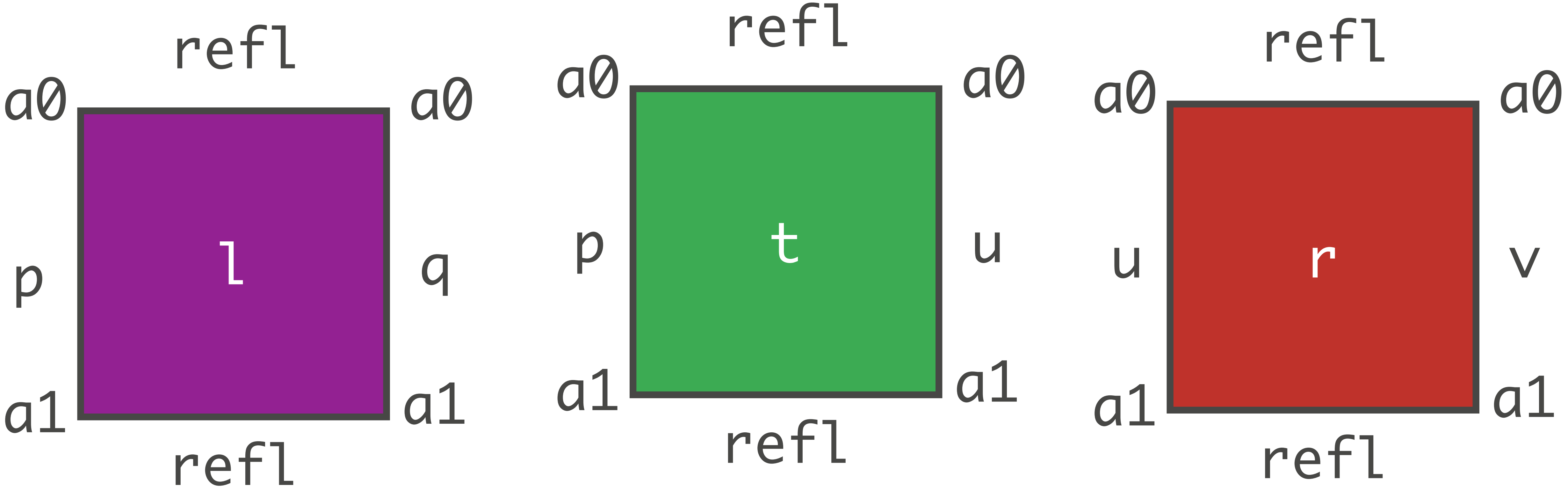
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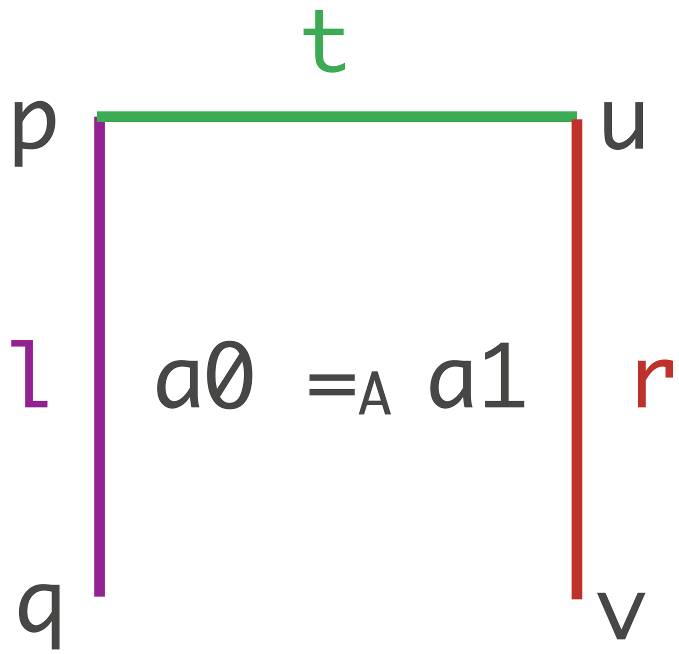
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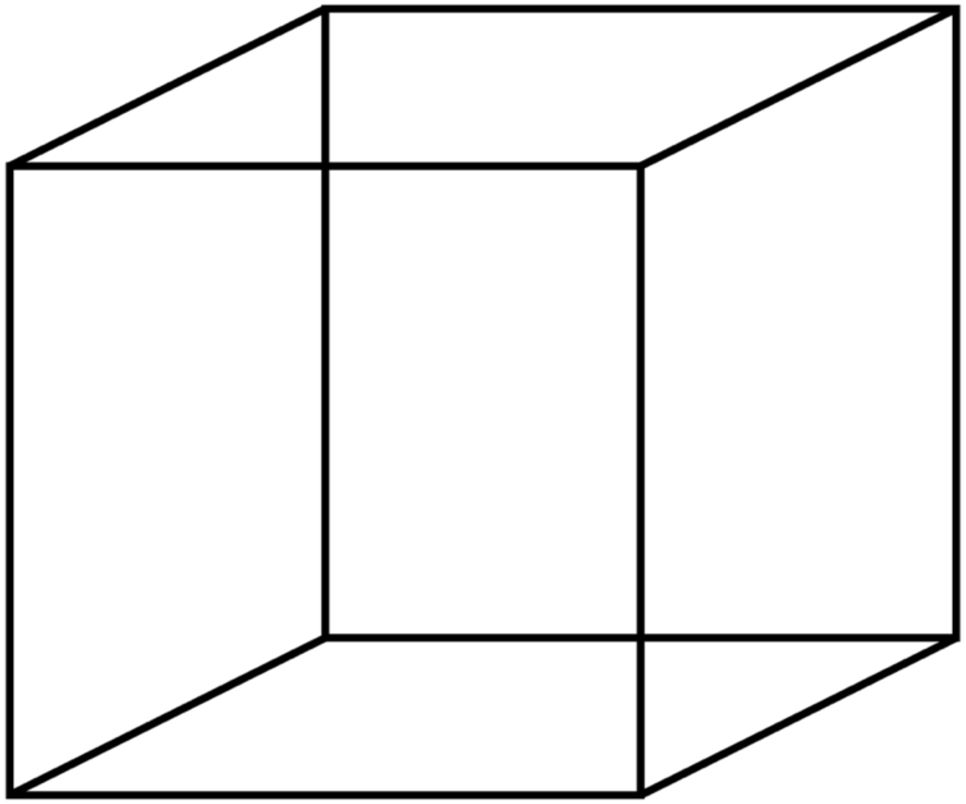

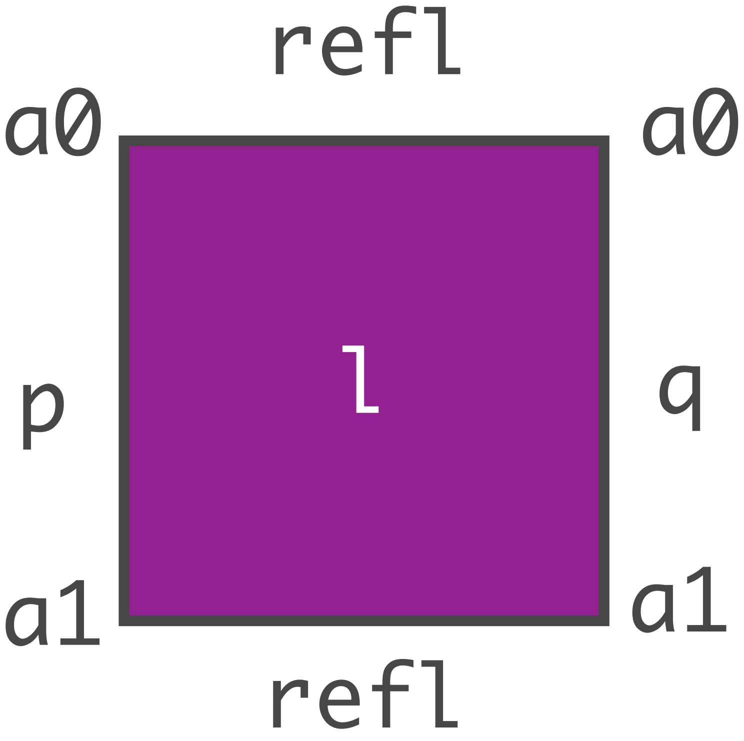
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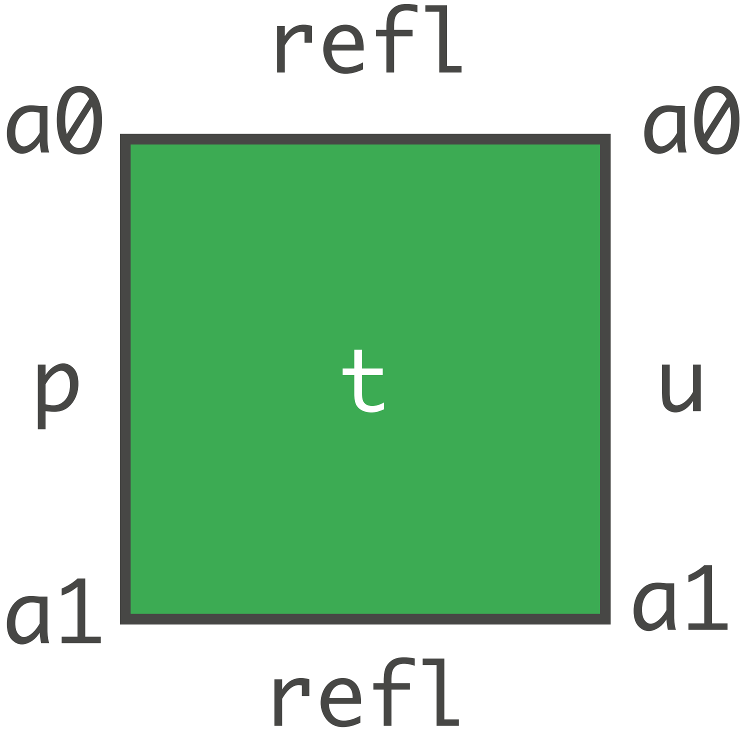
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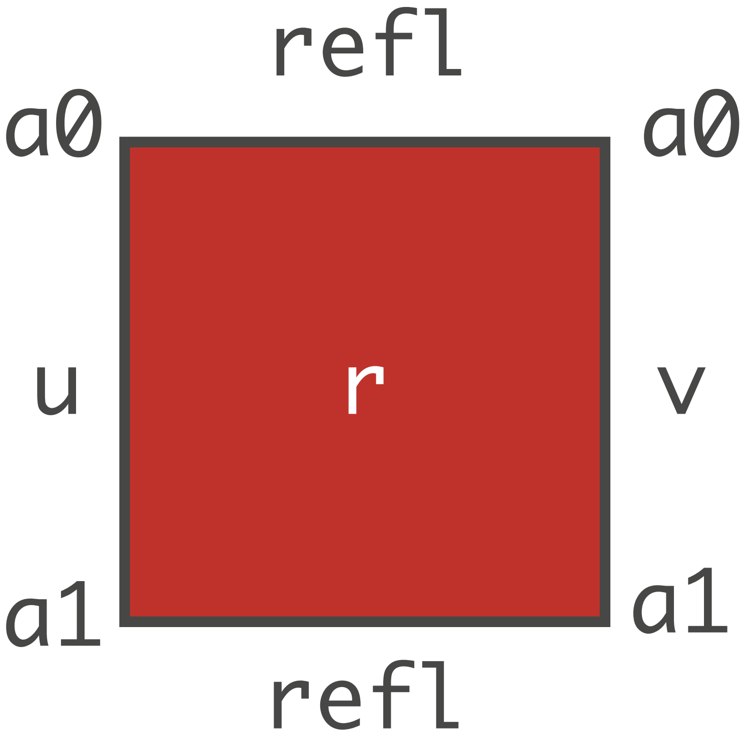
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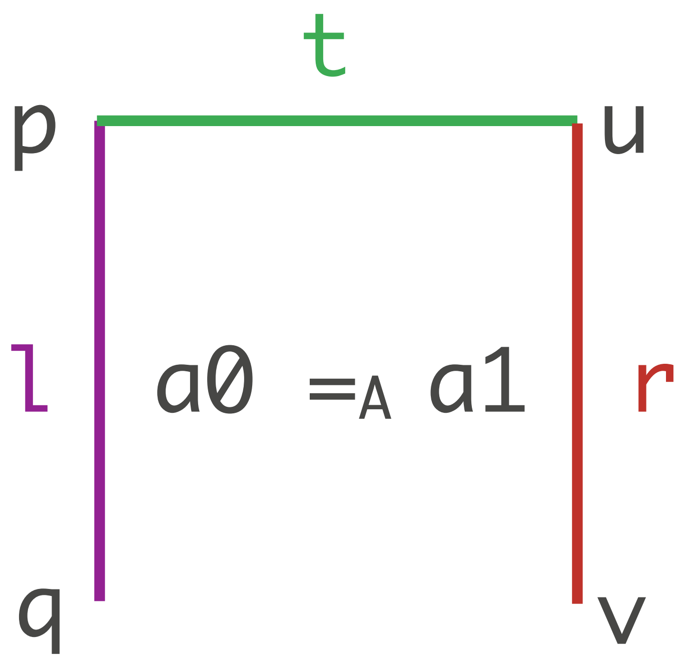
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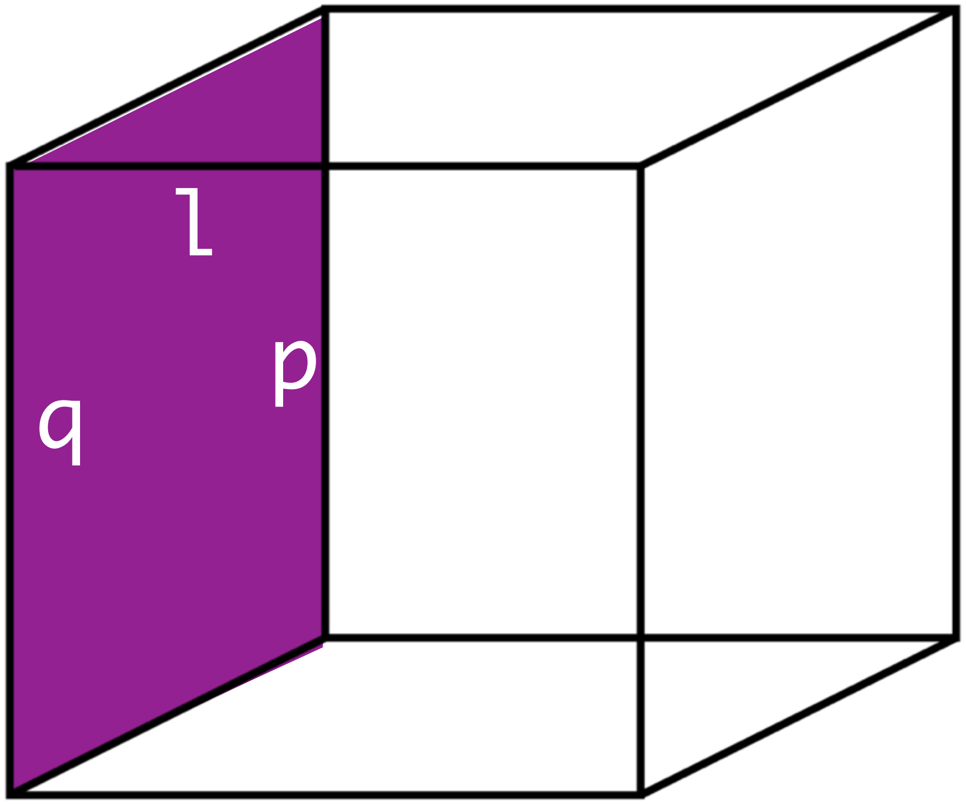
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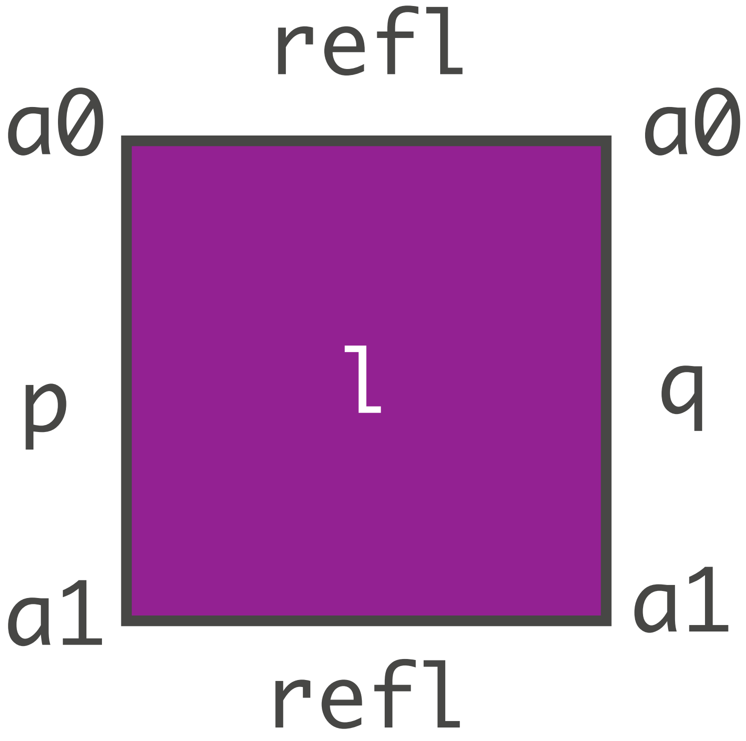
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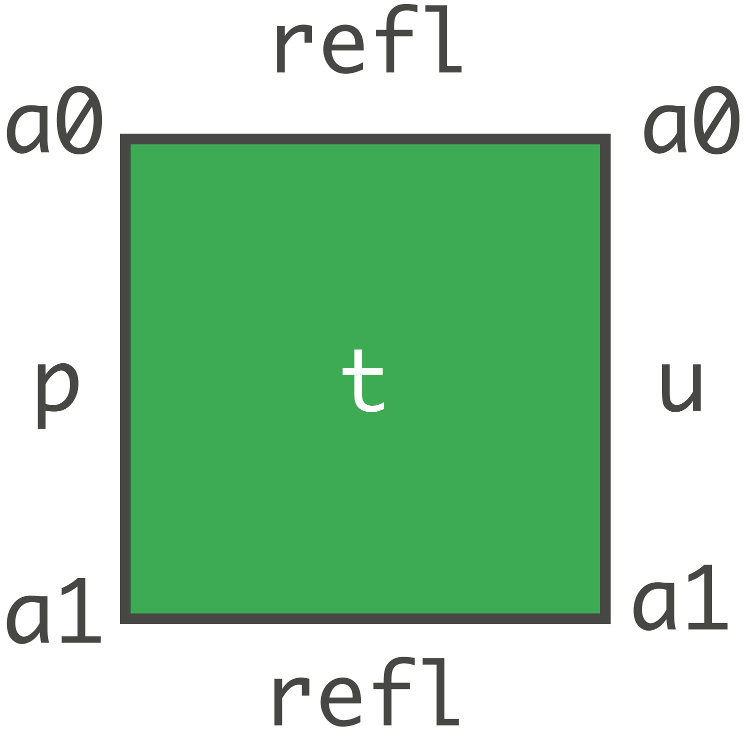
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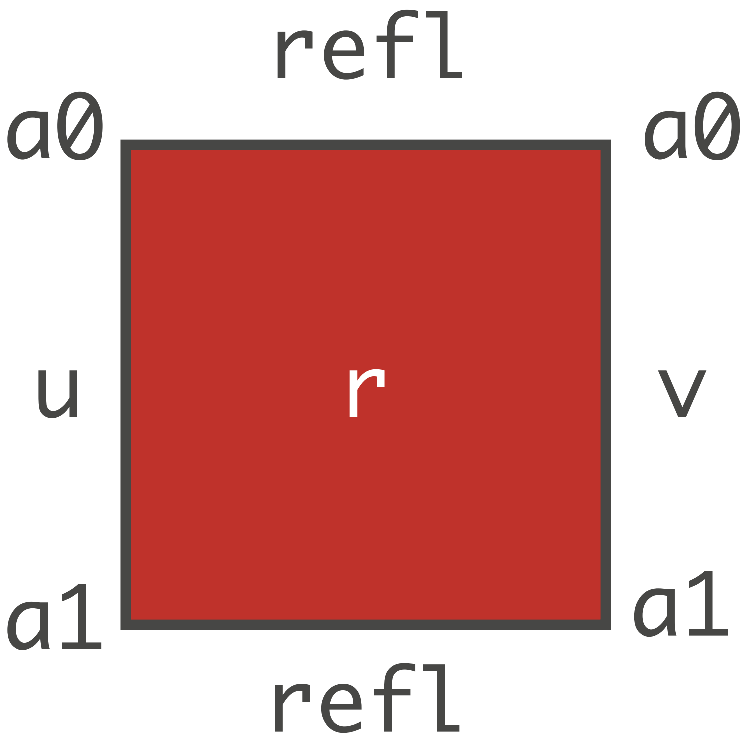
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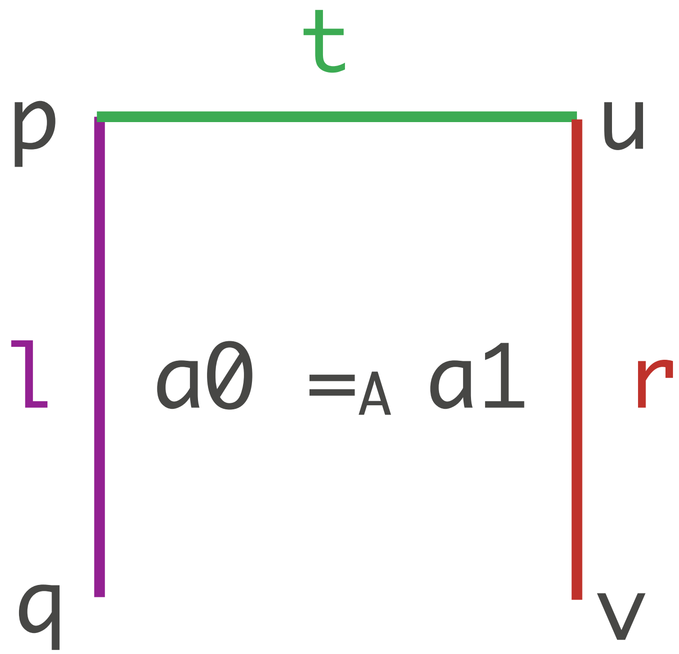
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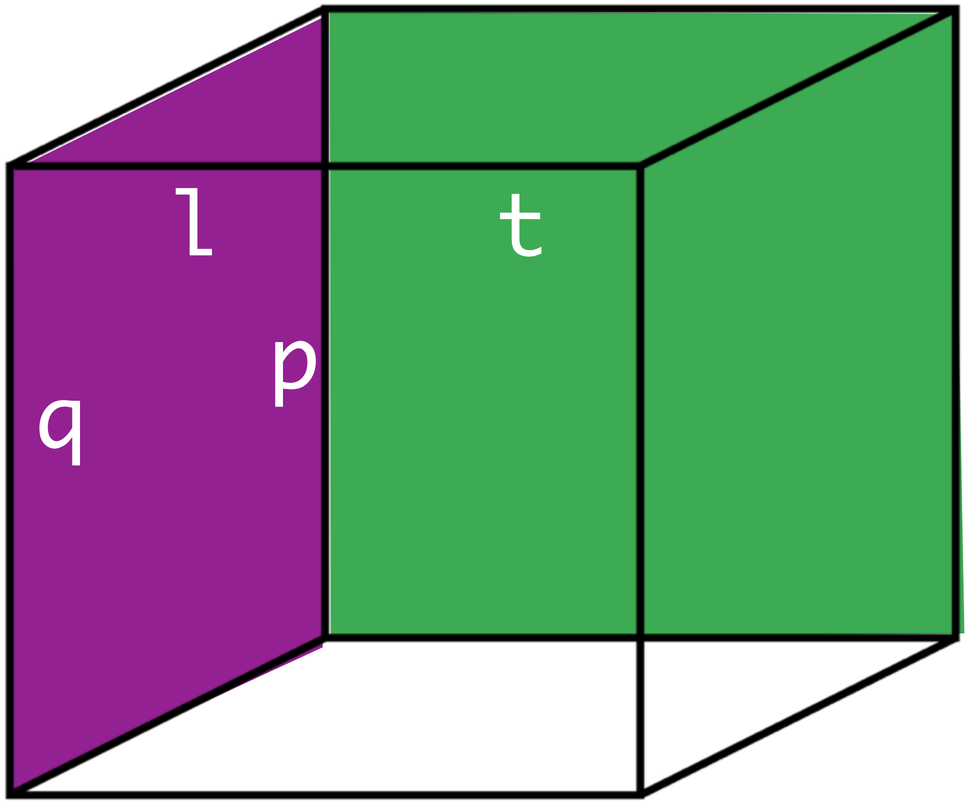
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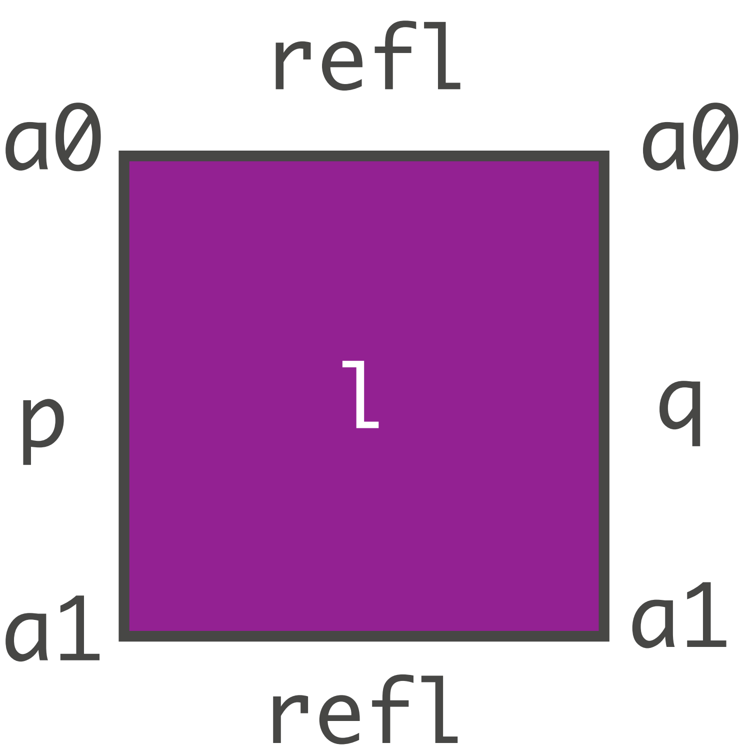
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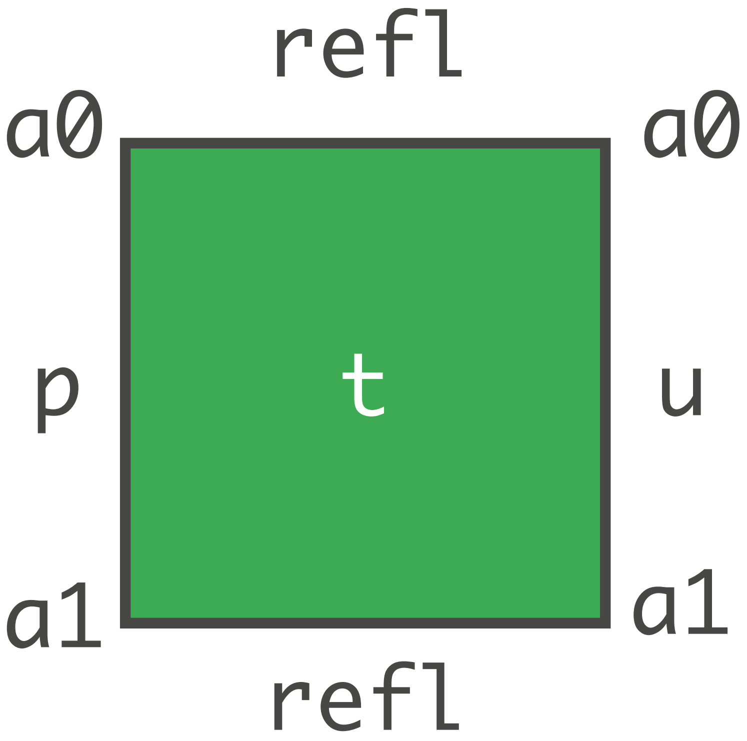
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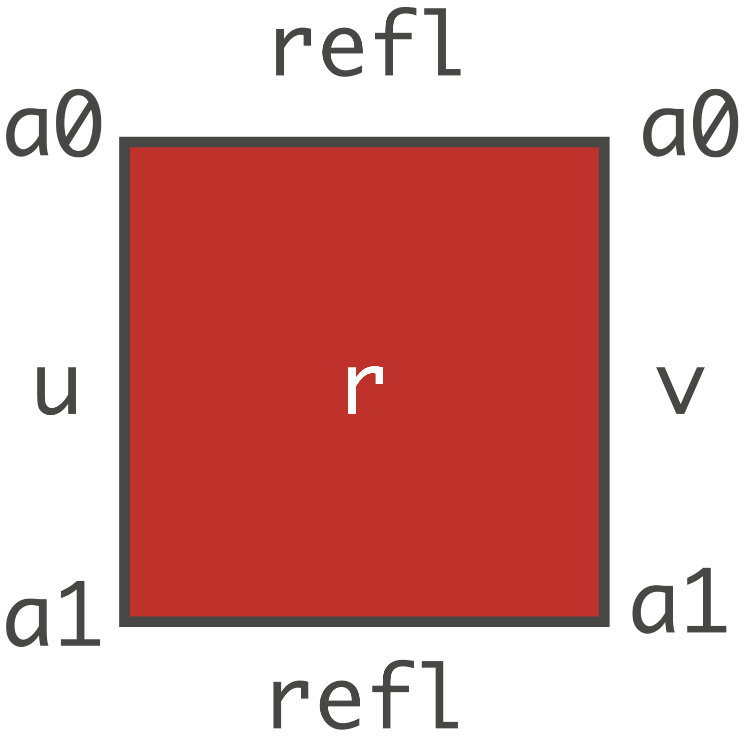
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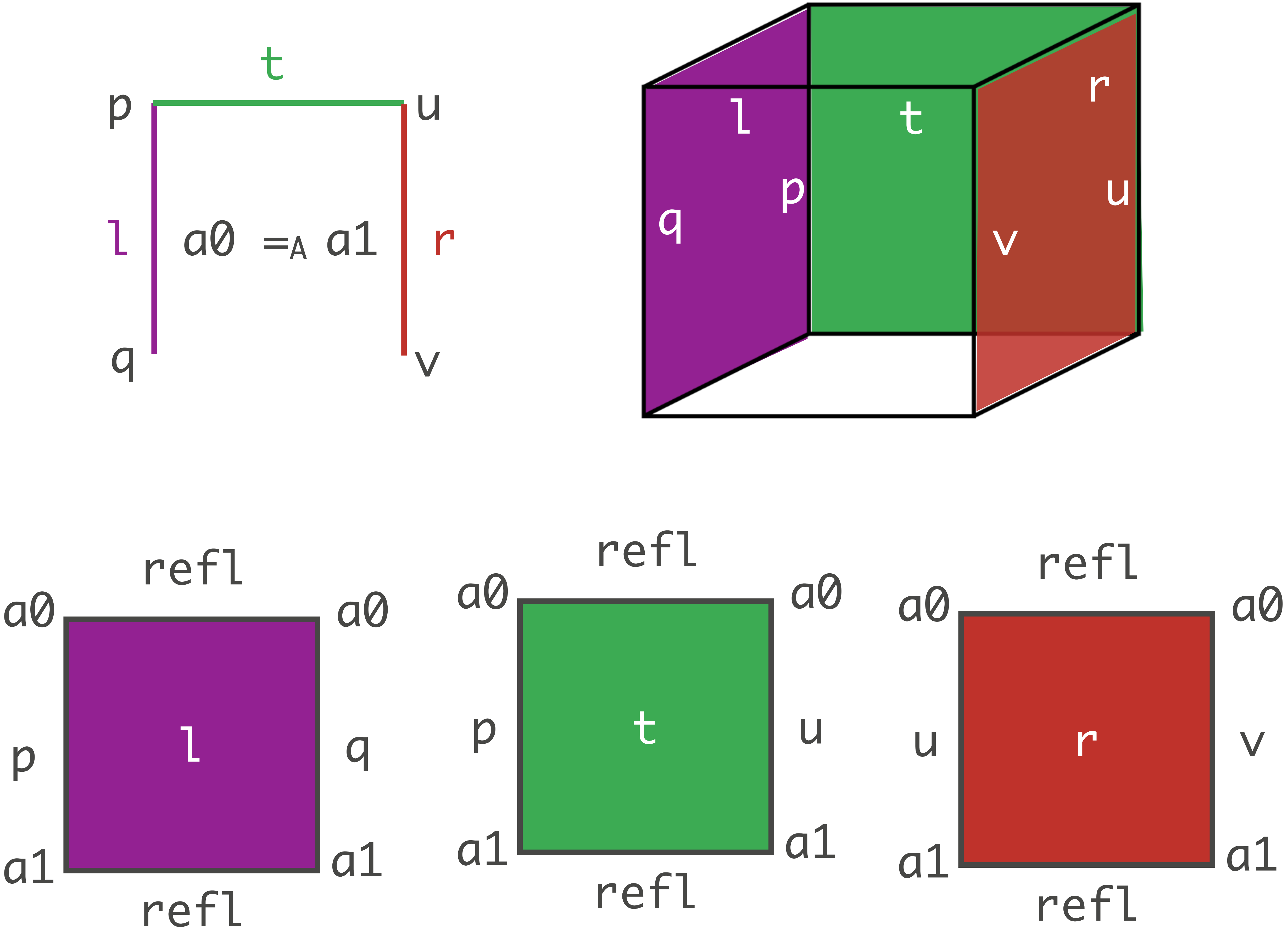
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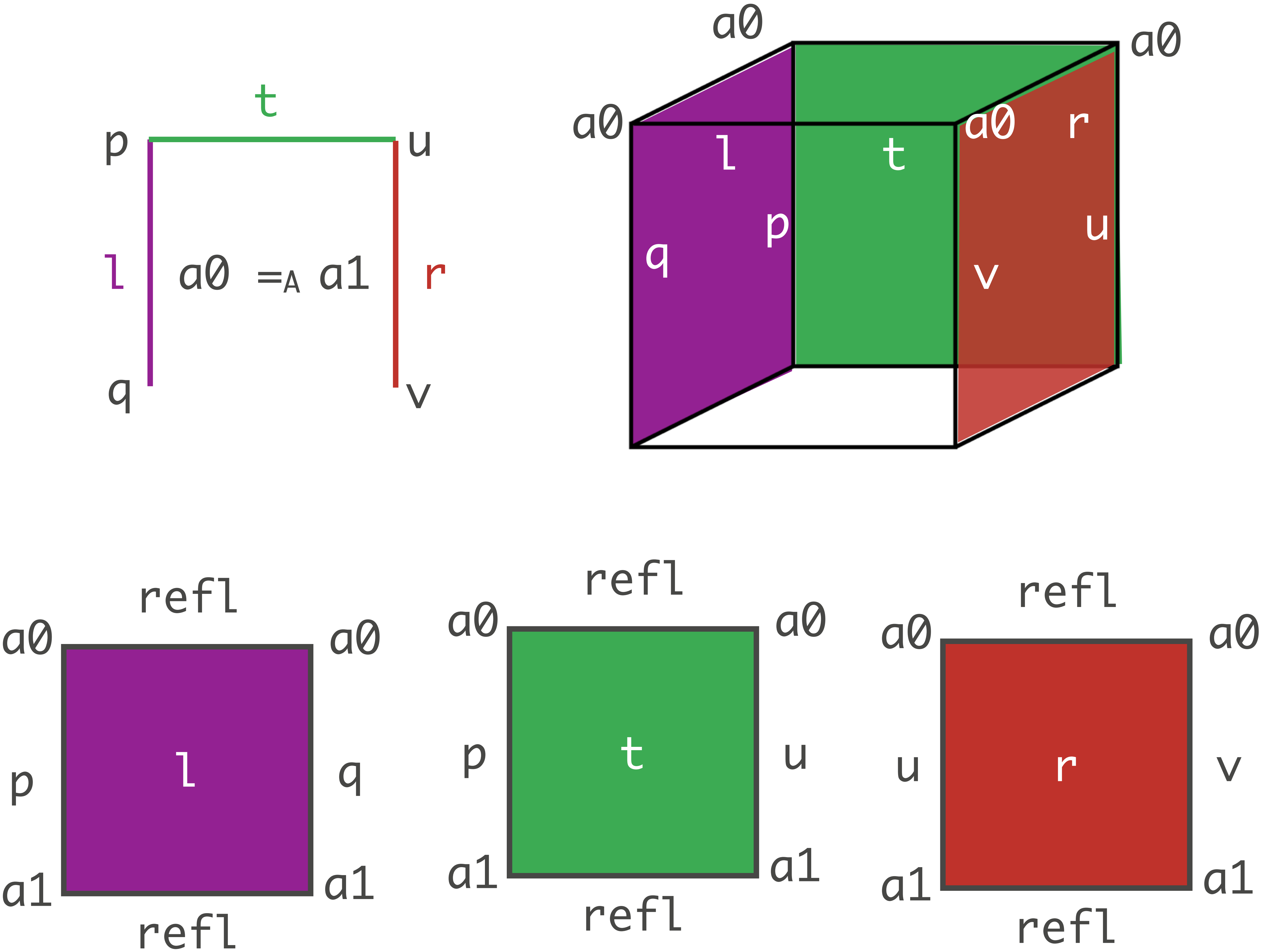
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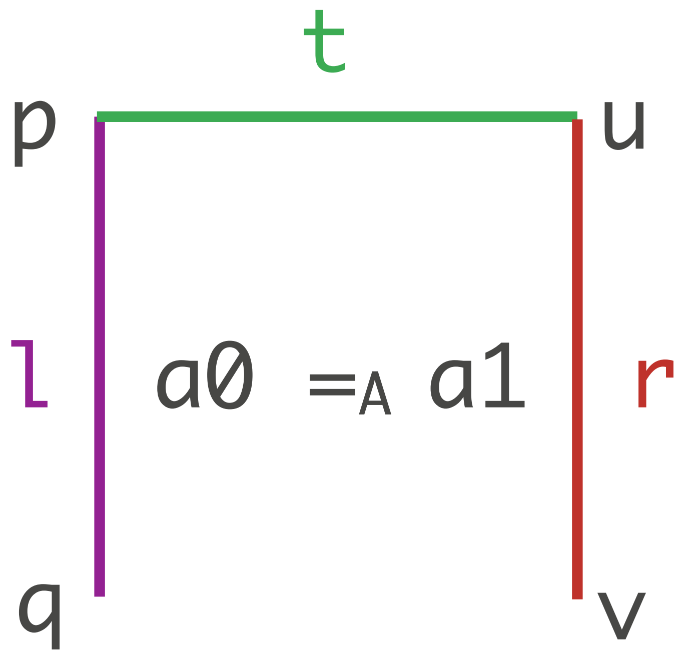
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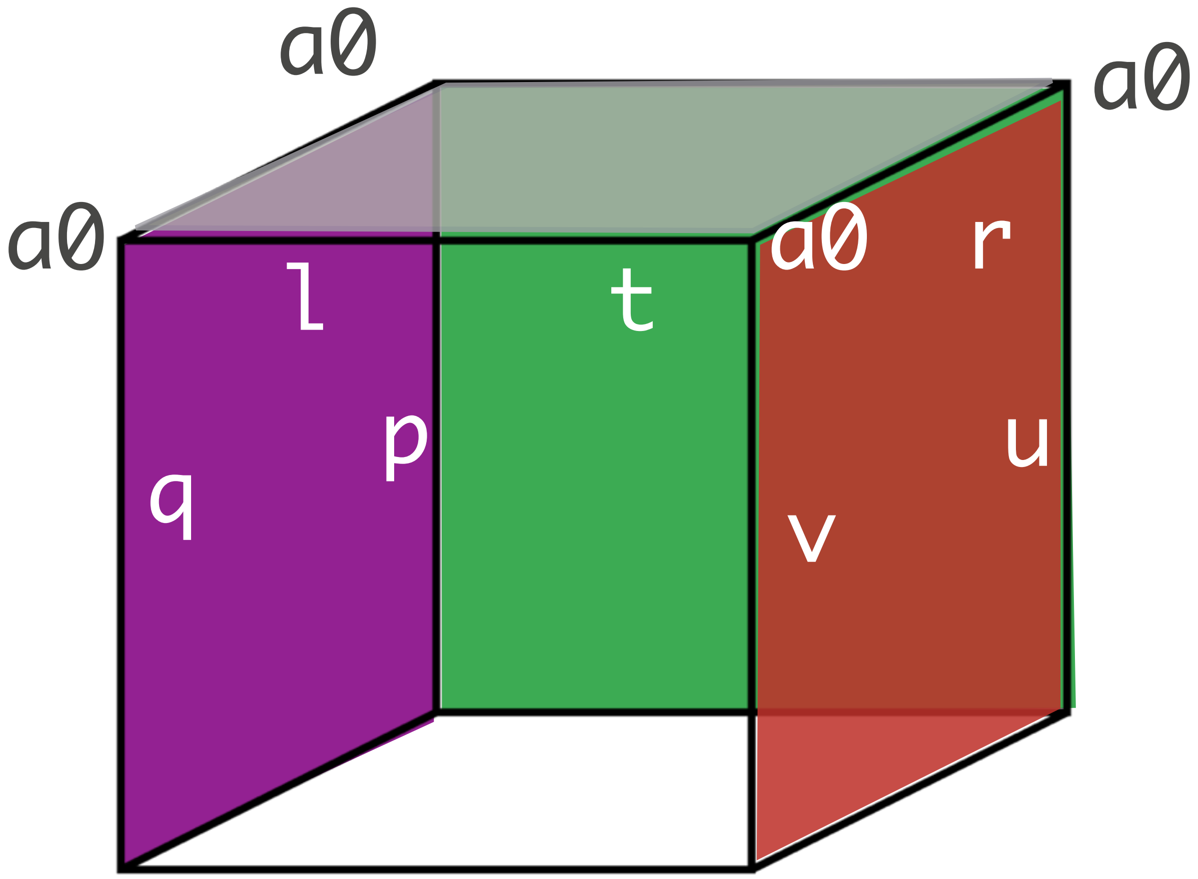
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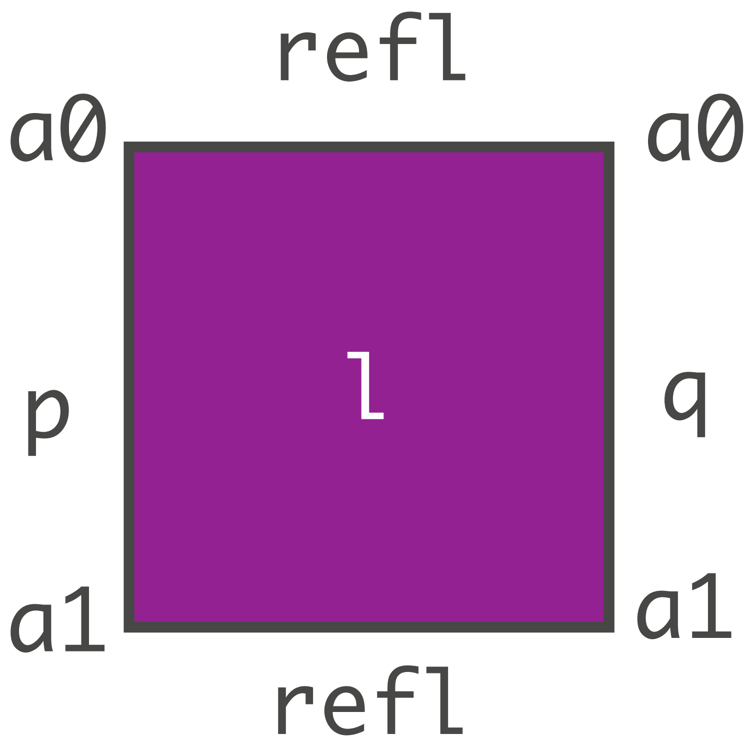
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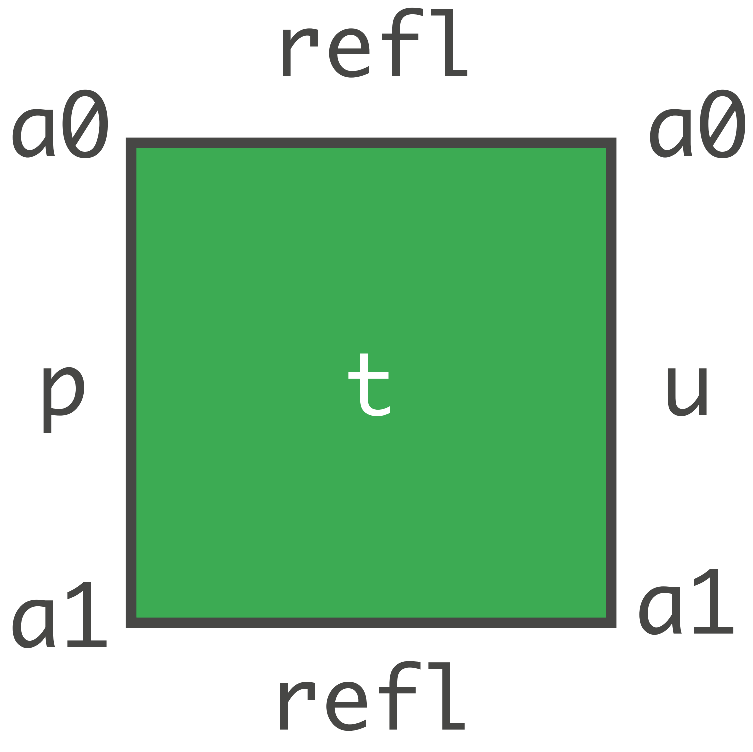
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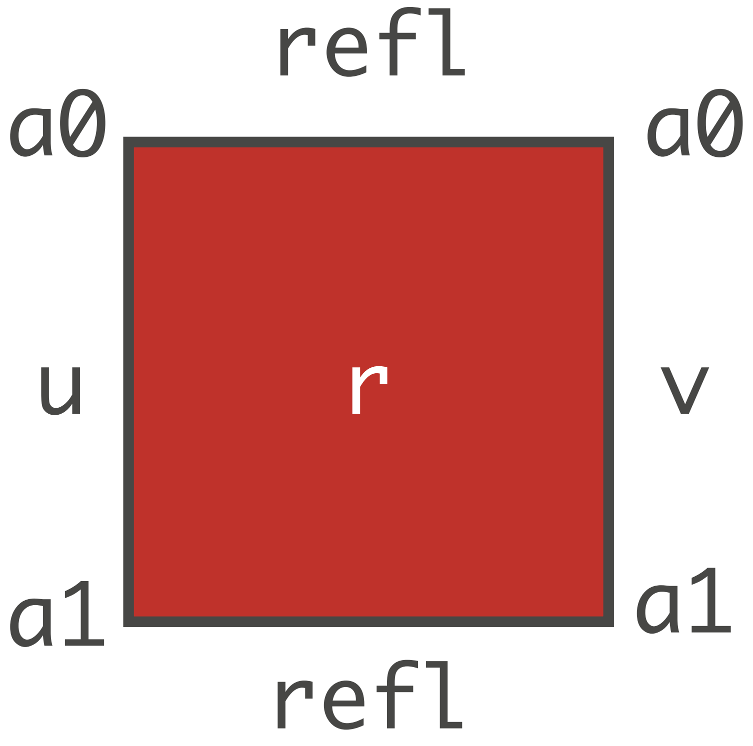
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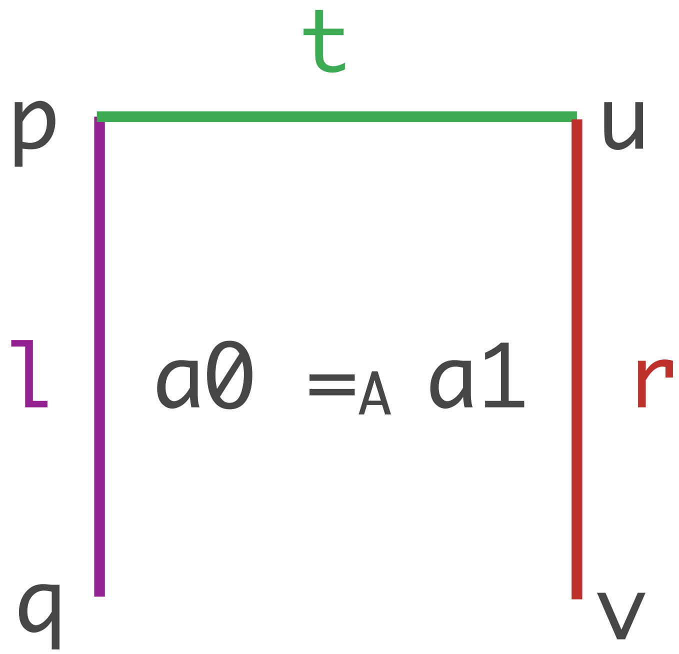
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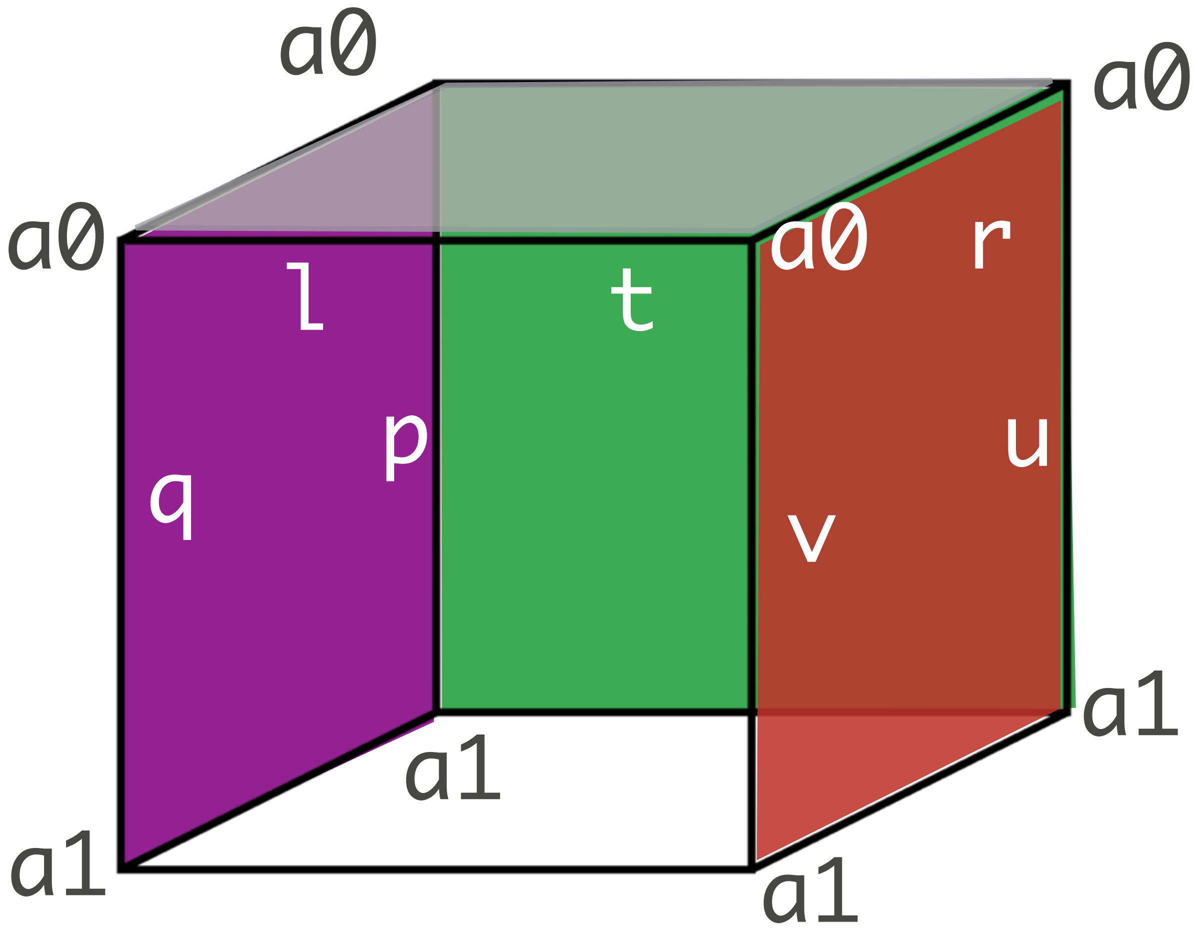
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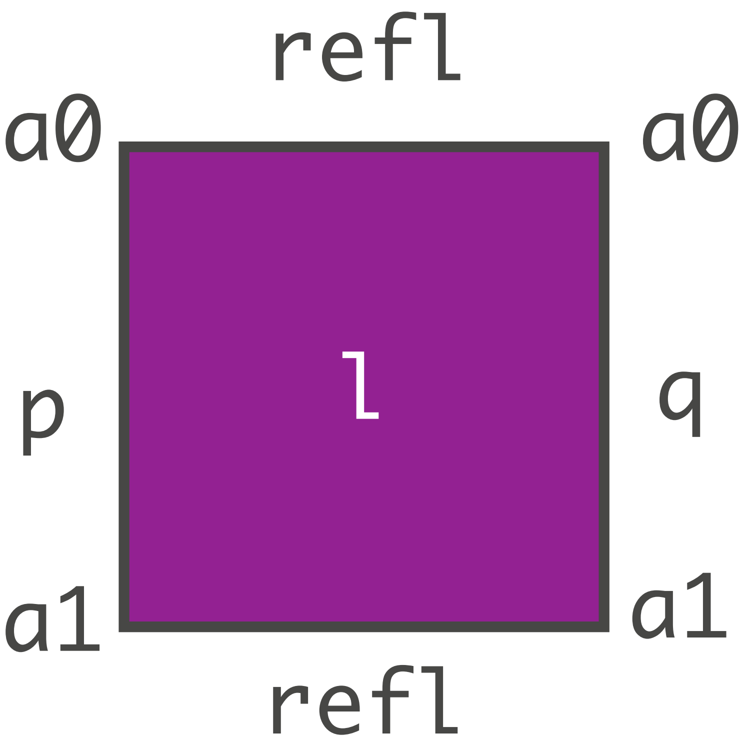
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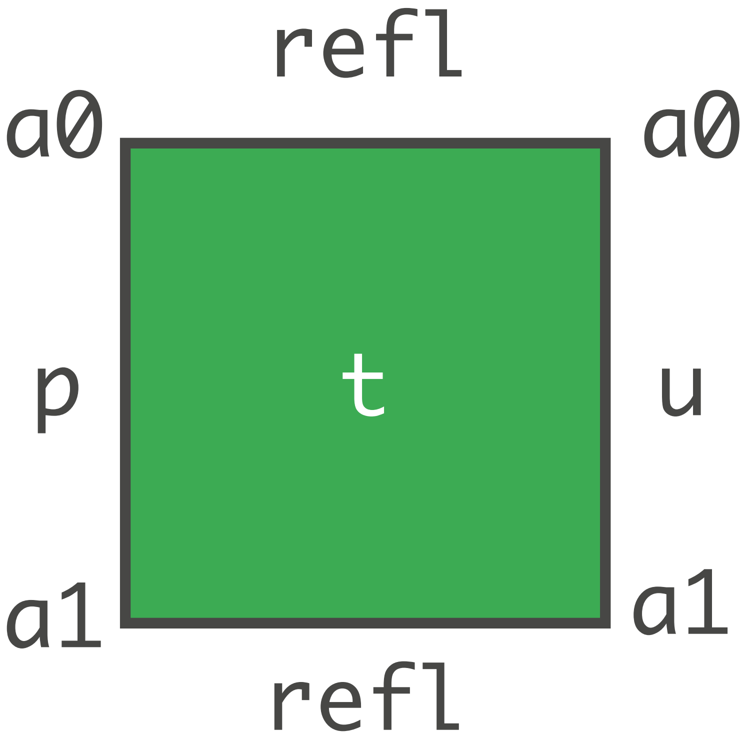
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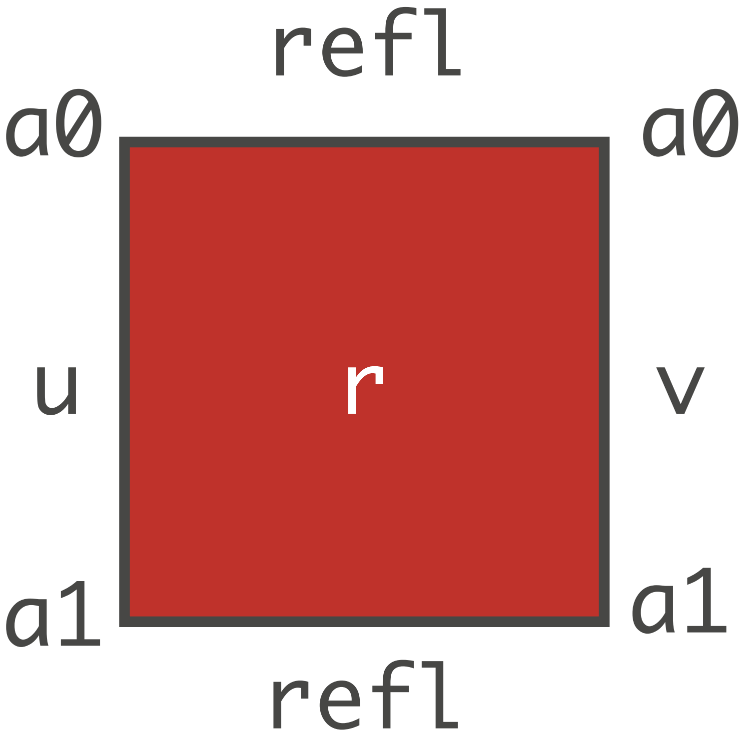
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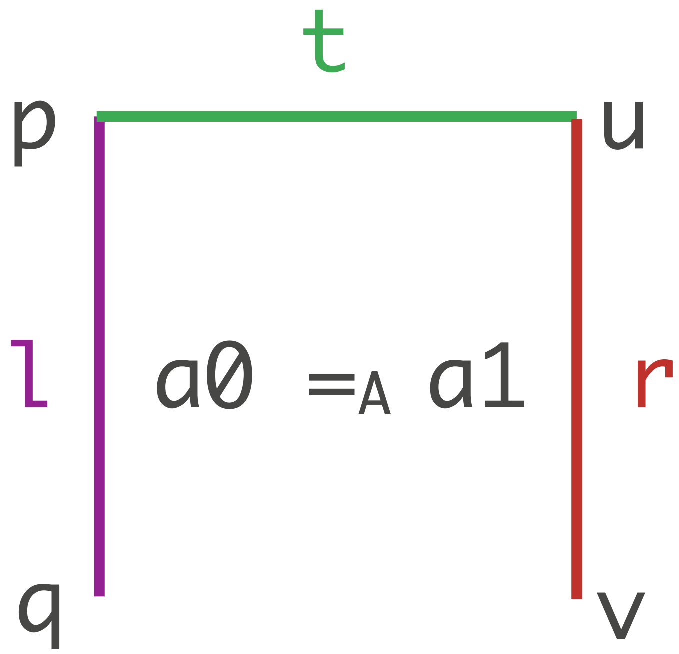
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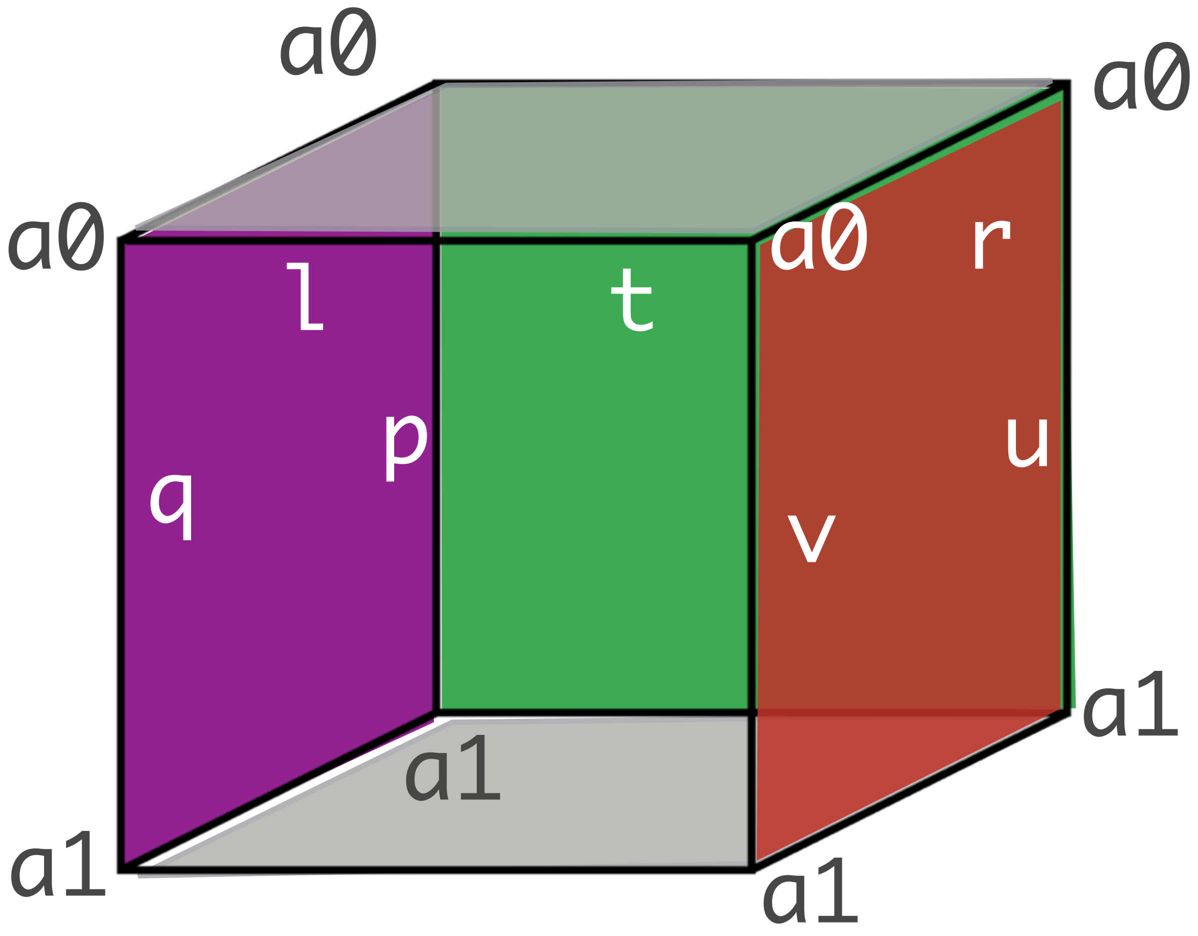
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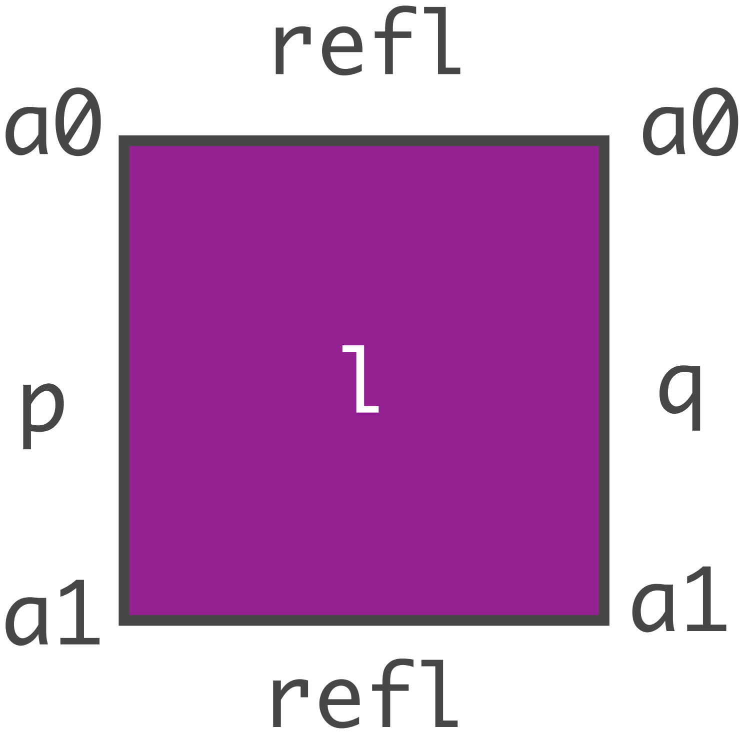
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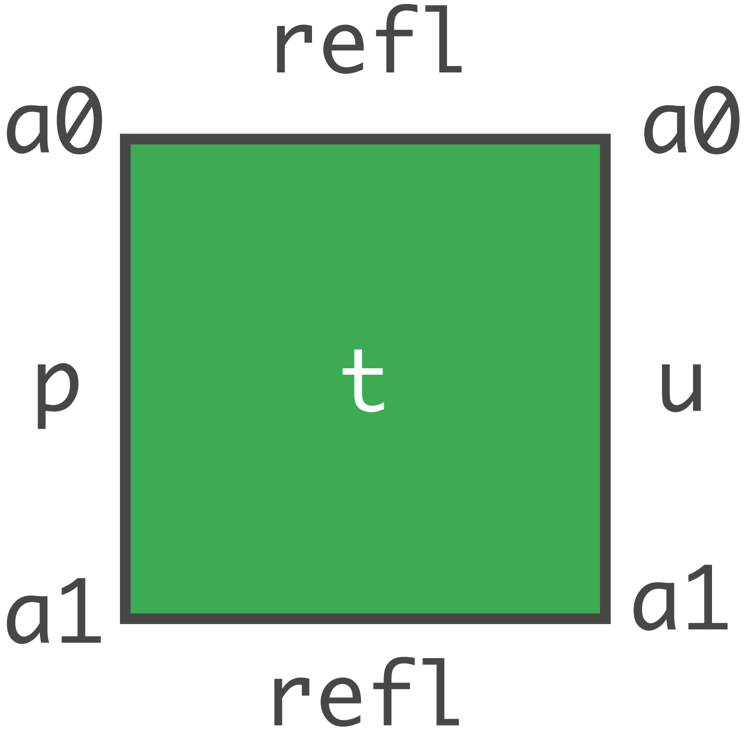
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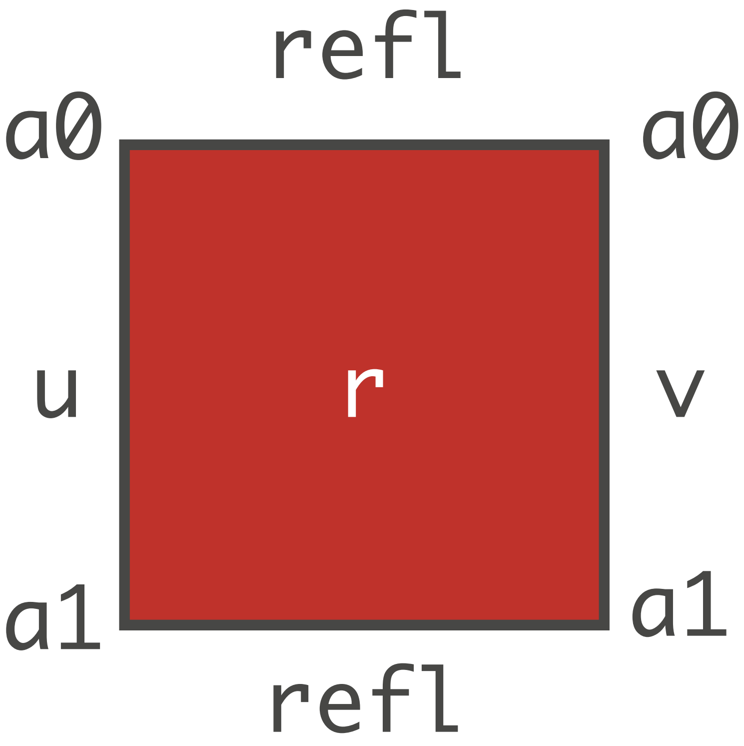
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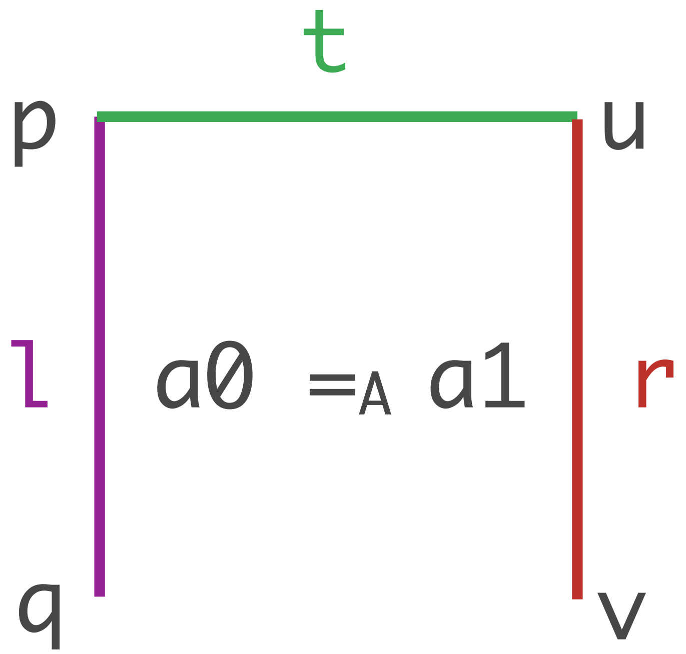
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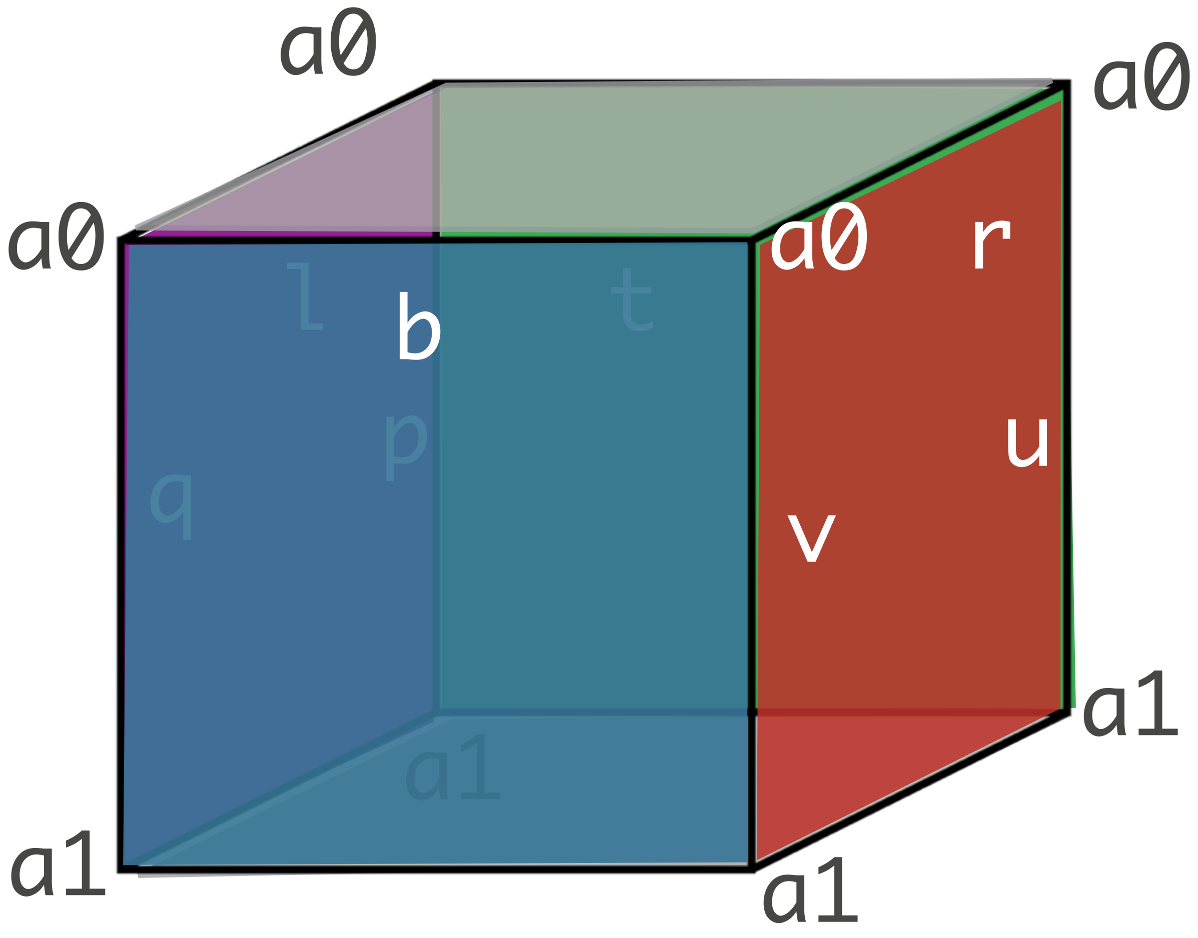
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b
t
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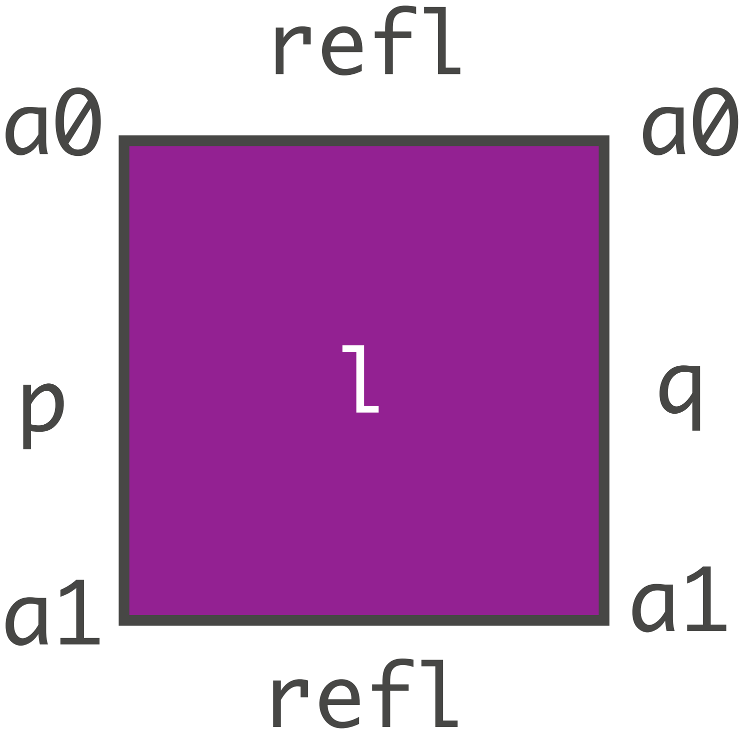
a1
a1
a1
a1


refl
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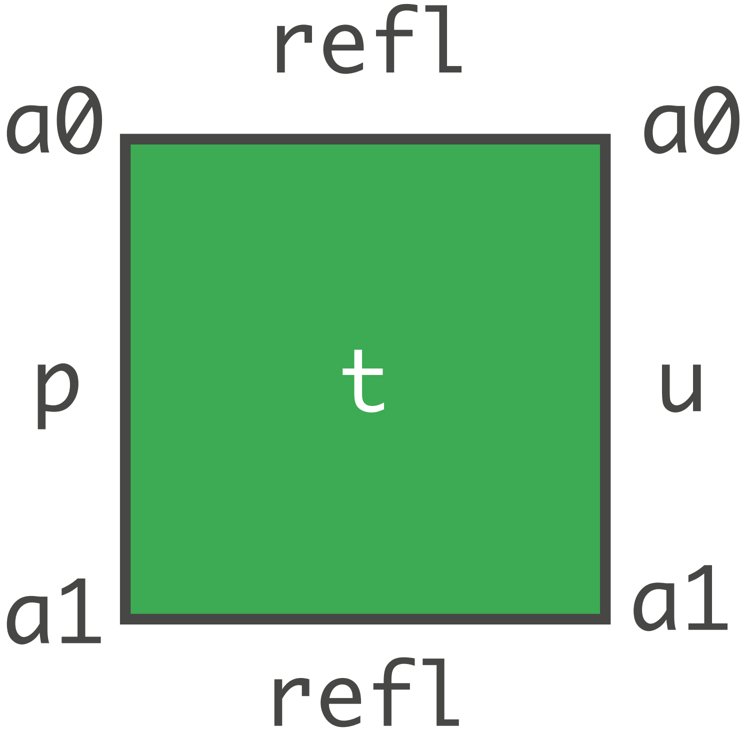
refl


refl
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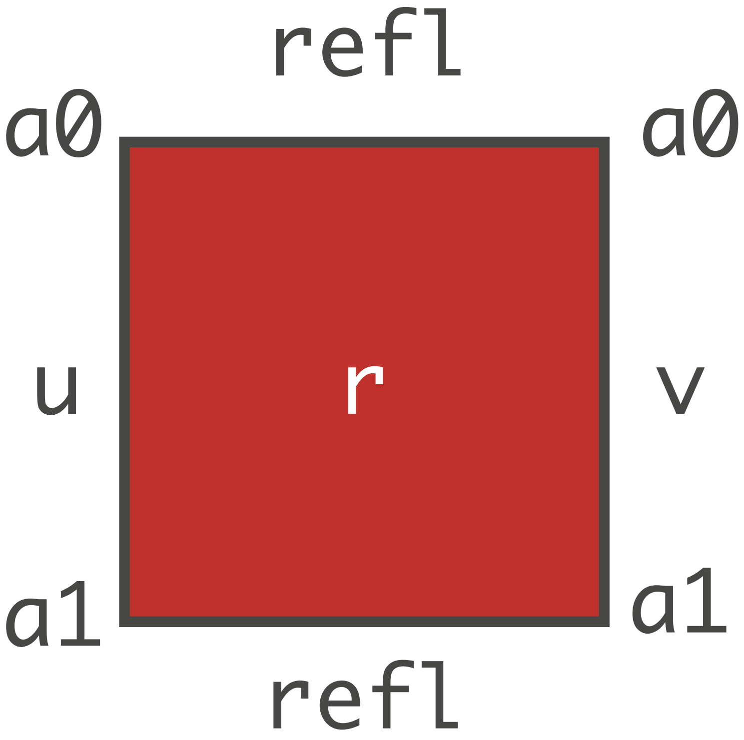
refl
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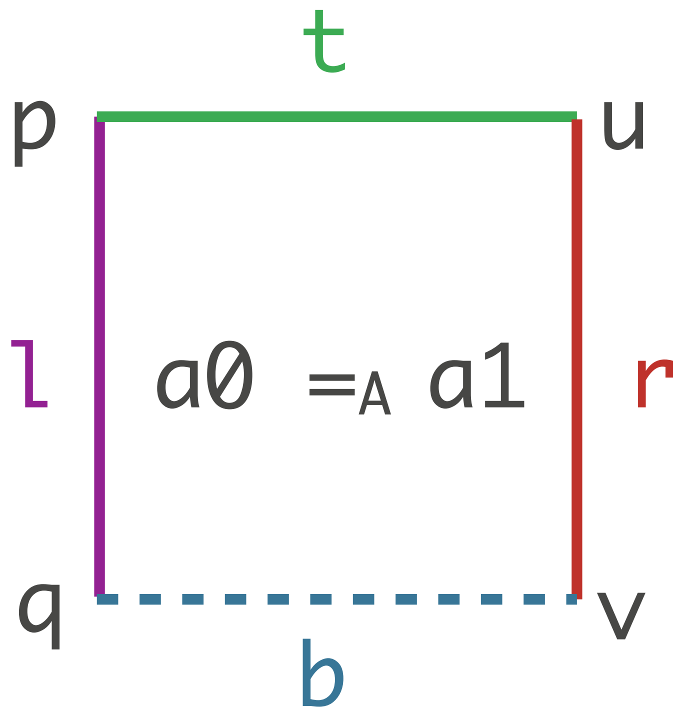
t
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a0 =A a1
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v
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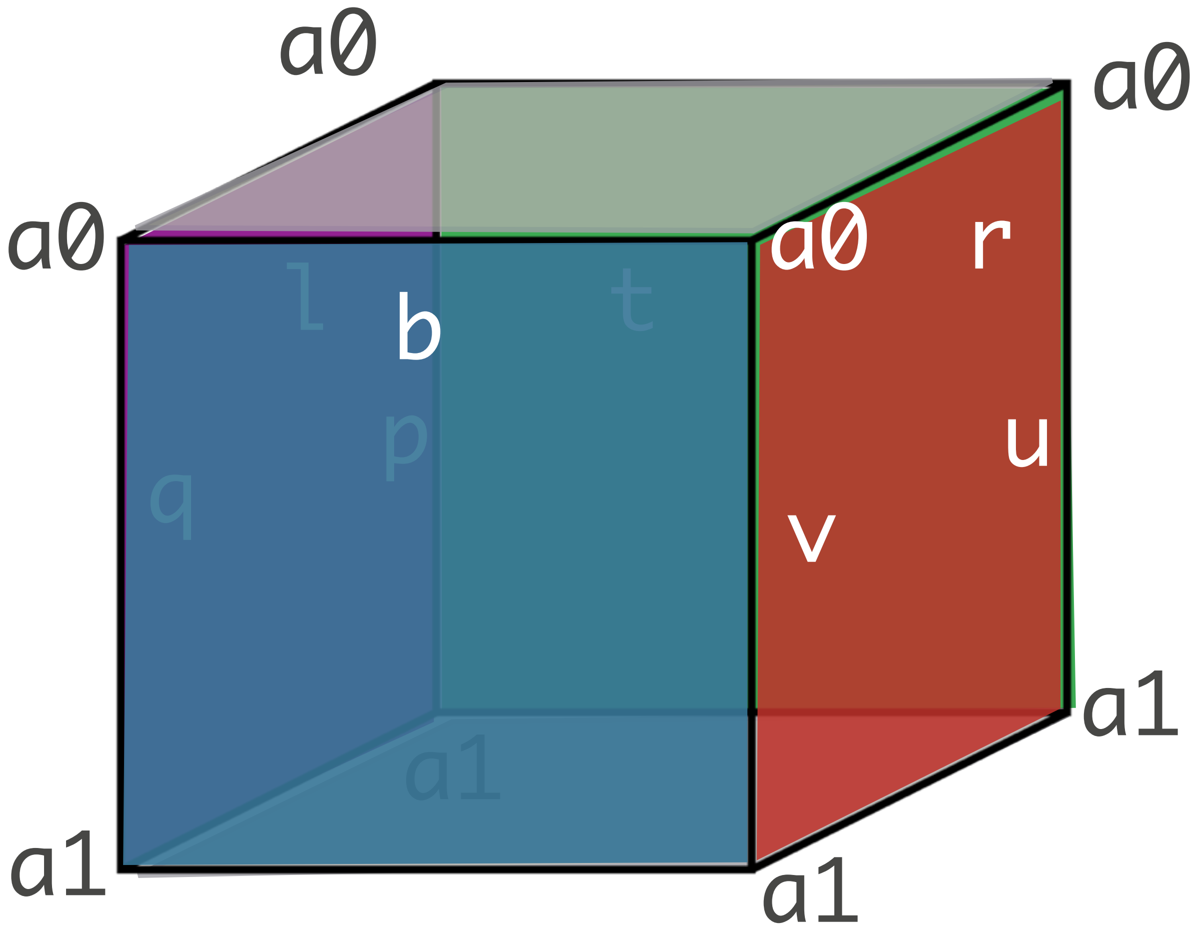
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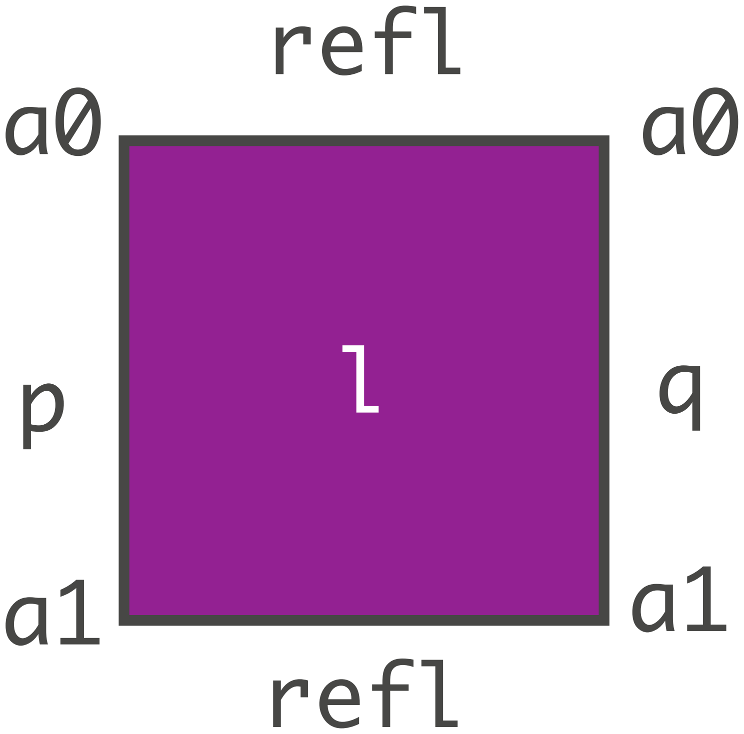
refl
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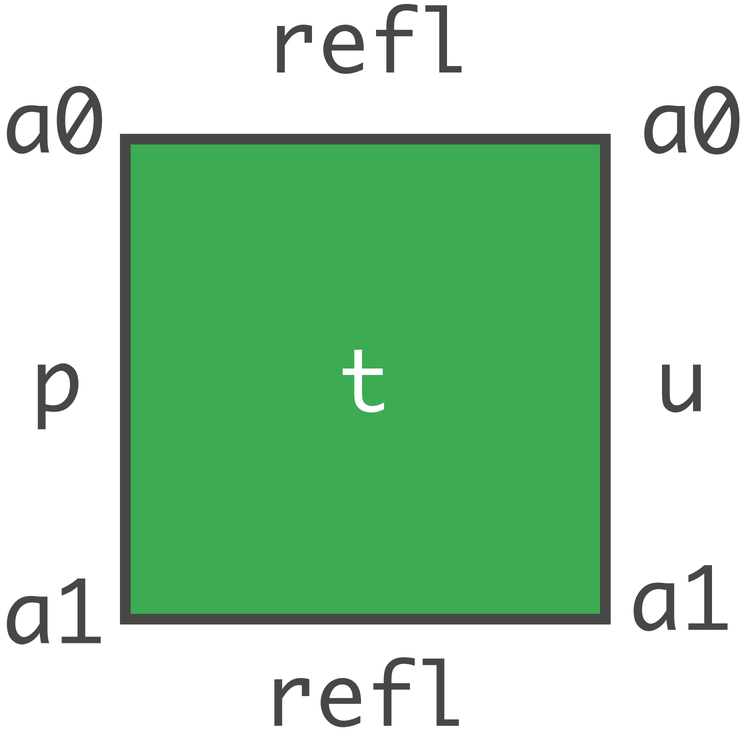
refl
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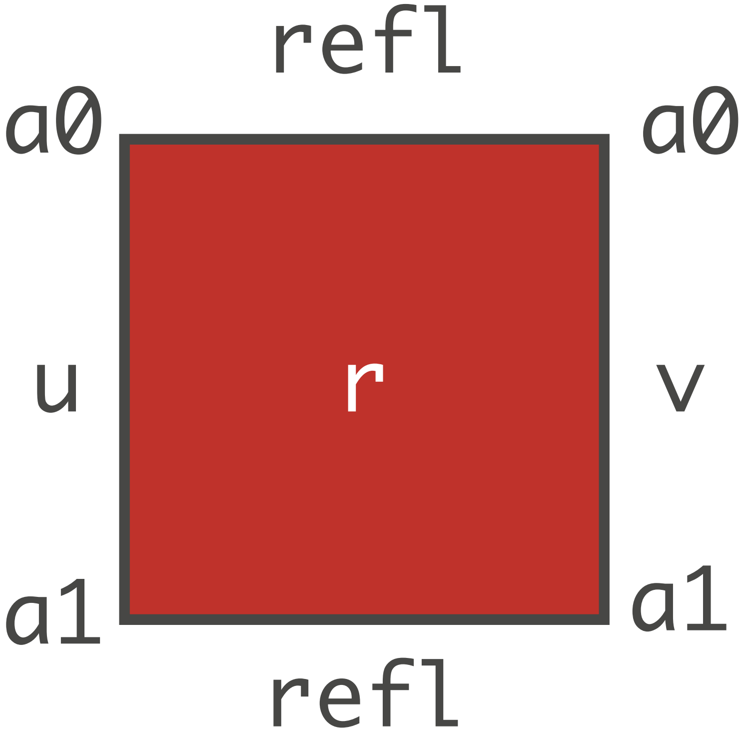
refl
a0
a0
u
r
v
a1
a1
refl

Kan condition:
any n-dimensional
open box has a lid,
and an inside

Computation

[Coquand,Huber,Bezem,Barras,
Brunerie,Licata,Harper,Shulman,
Altenkirch,Kaposi,Polonsky…]

- Bezem,Coquand,Huber, 2013 gave a constructive model of type theory in Kan cubical sets; evaluator based on this
- Today: a type theoretic paraphrase of these ideas

Choices

Parametrized judgement or internal inductive step

Boundaries-as-terms or boundaries-as-types

Cubical operations meta-operation or not

Choices

Parametrized judgement or internal inductive step

$u :^{n} A$

Boundaries-as-terms or boundaries-as-types

Cubical operations meta-operation or not

Choices

Parametrized judgement or internal inductive step

$u :^n A$ **$u \sim_A v$**

[Altenkirch&Kaposi,Polonsky]

Boundaries-as-terms or boundaries-as-types

Cubical operations meta-operation or not

Choices

Parametrized judgement or internal inductive step

$u :^n A$

$u \sim_A v$

[Altenkirch&Kaposi,Polonsky]

Boundaries-as-terms or boundaries-as-types

$u : \mathrm{Square}_A$

Cubical operations meta-operation or not

Choices

Parametrized judgement or internal inductive step

$u :^n A$

$u \sim_A v$

[Altenkirch&Kaposi,Polonsky]

Boundaries-as-terms or boundaries-as-types

$u:\mathrm{Square}_A$

$u:\mathrm{Square}_A\ l\ t\ b\ r$

Cubical operations meta-operation or not

Choices

Parametrized judgement or internal inductive step

$u :^n A$ **$u \sim_A v$**

[Altenkirch&Kaposi,Polonsky]

Boundaries-as-terms or boundaries-as-types

$u : \mathrm{Square}_A$ **$u : \mathrm{Square}_A\ l\ t\ b\ r$**

Cubical operations meta-operation or not

substitution

Choices

Parametrized judgement or internal inductive step

$u :^n A$ **$u \sim_A v$**

[Altenkirch&Kaposi,Polonsky]

Boundaries-as-terms or boundaries-as-types

$u:\mathrm{Square}_A$ **$u:\mathrm{Square}_A\ l\ t\ b\ r$**

Cubical operations meta-operation or not

substitution **explicit substitution**

A Cubical Type Theory
(∞-Dimensional, Boundaries-as-Terms, Explicit Cubical Operations)

Dimensions as names

[Coquand,Pitts]

- n-dimensional cube has n dimension names free
- α-equivalence: make {x,...}-cube into {x’,...}-cube
- Substitution of 0 or 1: faces
- Weakening: degeneracy/reflexivity
- Exchange: symmetry
- Contraction: diagonal

Properties are *cubical identities*

Dimensions as pronouns

Basic judgements

$$\Psi ::= \varnothing \mid \Psi, s\ \mathsf{dim}$$

Basic judgements

$$\Psi ::= \varnothing \mid \Psi, s\ \mathsf{dim}$$

$$\Psi; \Gamma \vdash A : \mathrm{Type}$$

Basic judgements

$$\Psi ::= \varnothing \mid \Psi, s \,\mathsf{dim}$$

$\Psi; \Gamma \vdash A : \mathsf{Type}$	A is a Ψ-cube in type
$\Psi; \Gamma \vdash u : A$	u is an Ψ-cube in A, together with its boundary

Basic judgements

dimension

$$\Psi ::= \varnothing \mid \Psi, s\ \mathsf{dim}$$

$$\Psi; \Gamma \vdash A : \mathtt{Type}$$

A is a Ψ-cube in type

$$\Psi; \Gamma \vdash u : A$$

u is an Ψ-cube in A,
together with its boundary

Basic judgements

dimension

$\Psi ::= \varnothing \mid \Psi, s\ \mathsf{dim}$

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point:

- `nat : Type`

Basic judgements

dimension

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$\Psi; \Gamma \vdash A : \mathrm{Type}$ A is a Ψ-cube in type

$\Psi; \Gamma \vdash u : A$ u is an Ψ-cube in A, together with its boundary

point: ● `nat : Type`

element of point: `0 : nat`

Basic judgements

$$\Psi ; \Gamma \vdash A : \mathtt{Type}$$

A is a Ψ-cube in type

$$\Psi ; \Gamma \vdash u : A$$

u is an Ψ-cube in A,
together with its boundary

line in type:
two equivalent types

Basic judgements

$\Psi; \Gamma \vdash A : \mathtt{Type}$

A is a Ψ-cube in type

$\Psi; \Gamma \vdash u : A$

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line in type:
two equivalent types

element of **line**:
heterogeneously
equal points

Basic judgements

$$\Psi;\Gamma \vdash A : \mathtt{Type}$$

A is a Ψ-cube in type

$$\Psi;\Gamma \vdash u : A$$

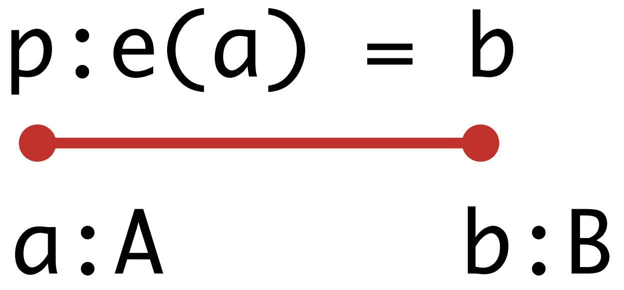
u is an Ψ-cube in A,
together with its boundary

line in type:
two equivalent types

`e:Equiv A B`

`A` `B`

element of **line**:
heterogeneously
equal points



`p:e(a) = b`

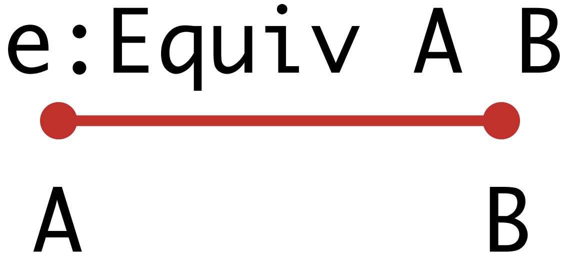
`a:A` `b:B`

Basic judgements

$\Psi ; \Gamma \vdash A : \mathtt{Type}$ A is a Ψ-cube in type

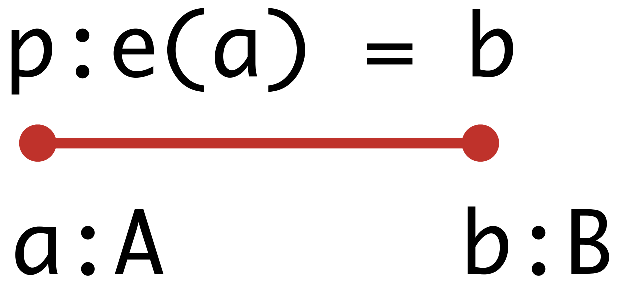
$\Psi ; \Gamma \vdash u : A$ u is an Ψ-cube in A, together with its boundary

line in type:
two equivalent types



“x dim ⊢ (A,B,e):Type”

element of **line**:
heterogeneously
equal points

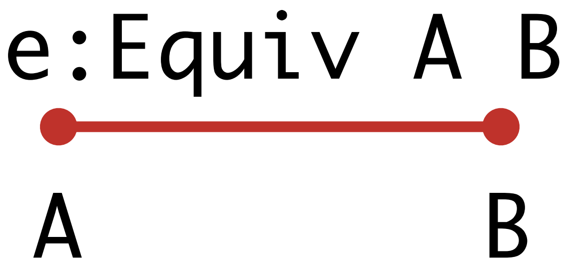


Basic judgements

$\Psi;\Gamma \vdash A : \mathtt{Type}$ A is a Ψ-cube in type

$\Psi;\Gamma \vdash u : A$ u is an Ψ-cube in A, together with its boundary

line in type:
two equivalent types

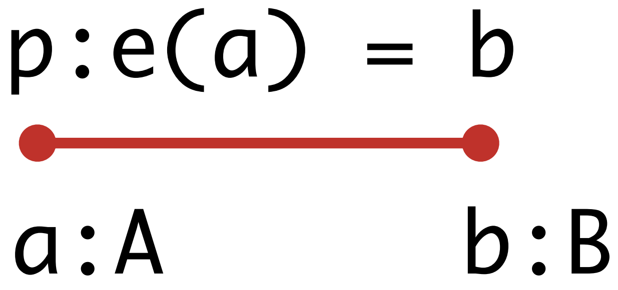


`e:Equiv A B`

`A` `B`

“`x dim ⊢ (A,B,e):Type`”

element of **line**:
heterogeneously
equal points



`p:e(a) = b`

`a:A` `b:B`

“`x dim ⊢ (a,b,p) : (A,B,e)`”

Ingredients

- Cubical operations
- Higher-dimensional substitution
- Types
- Kan filling

Ingredients

- **Cubical operations**
- Higher-dimensional substitution
- Types
- Kan filling

Degeneracy

$$\frac{\Psi, \Psi'; \Gamma \vdash u : A}{\Psi, s\ \mathsf{dim}, \Psi'; \Gamma \vdash u : A}$$

Faces/Diagonals

$$\frac{\Psi, s\ \mathsf{dim}, \Psi'; \Gamma \vdash u : A \qquad \Psi \vdash r\ \mathsf{dim}}{\Psi, \Psi'; \Gamma\langle r/s\rangle \vdash u\langle r/s\rangle : A\langle r/s\rangle}$$

$$u\langle r/s\rangle\langle r'/s'\rangle \equiv u\langle r'/s'\rangle\langle r\langle r'/s'\rangle/s\rangle$$

$$u\langle r/s\rangle \equiv u \text{ if } s\#u$$

Faces/Diagonals

r ::= s | 0 | 1

$$\frac{\Psi, s\,\mathsf{dim}, \Psi'; \Gamma \vdash u : A \qquad \Psi \vdash r\,\mathsf{dim}}{\Psi, \Psi'; \Gamma\langle r/s\rangle \vdash u\langle r/s\rangle : A\langle r/s\rangle}$$

$$u\langle r/s\rangle\langle r'/s'\rangle \equiv u\langle r'/s'\rangle\langle r\langle r'/s'\rangle/s\rangle$$

$$u\langle r/s\rangle \equiv u \text{ if } s\#u$$

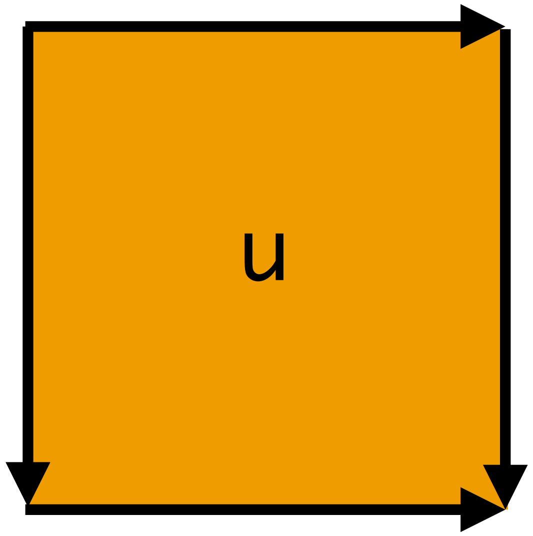
Faces

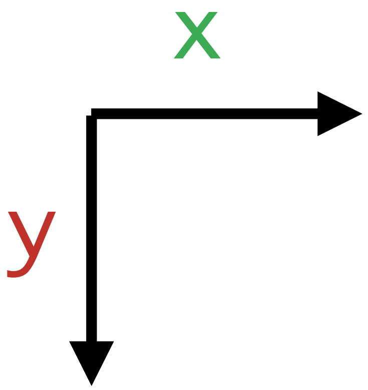
`x dim ⊢ u : A`

$$\begin{tikzcd}
{} \arrow[r, "x"] & {} & u<0/x> \arrow[r, "u"] & u<1/x>
\end{tikzcd}$$

Faces

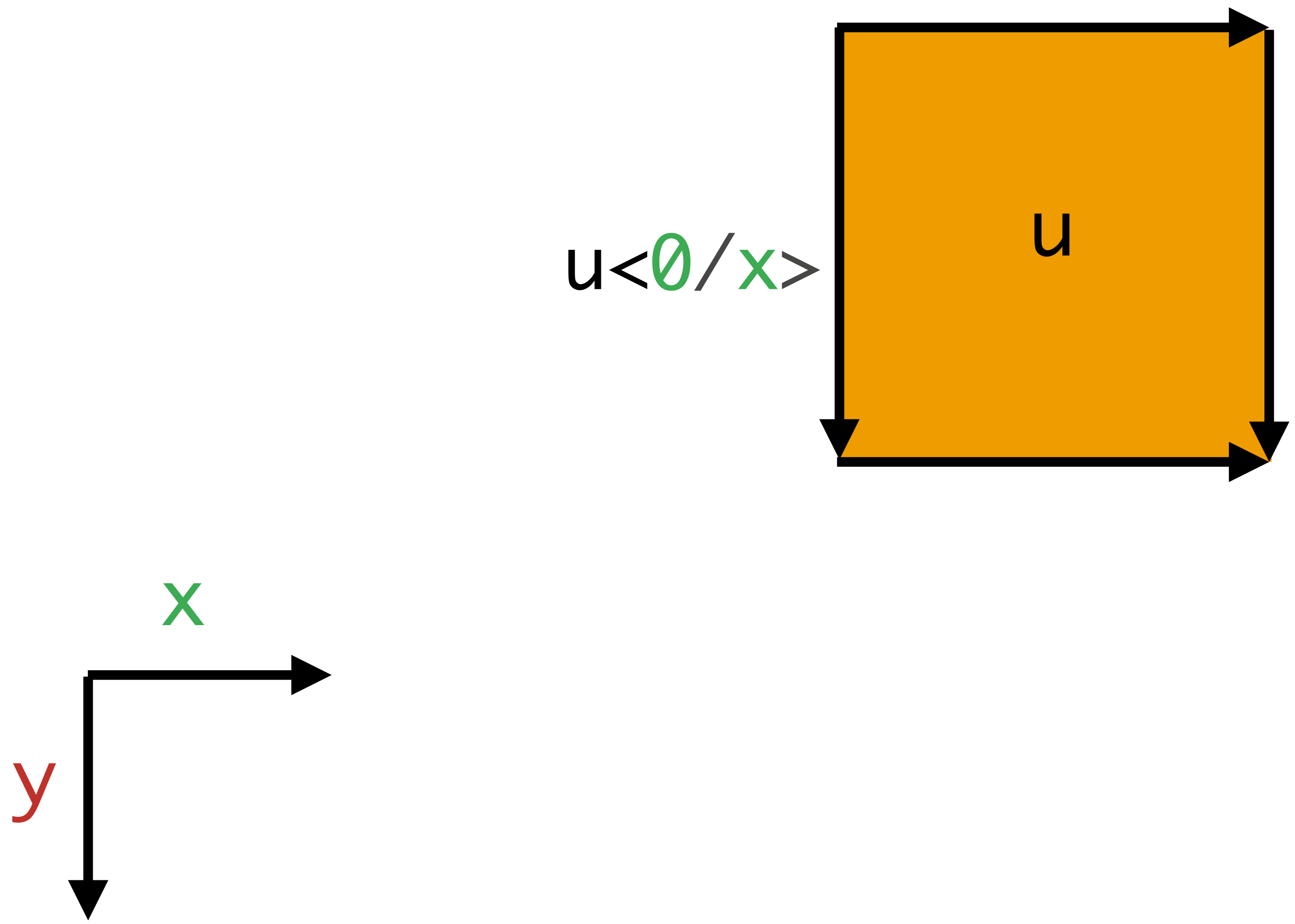
`x dim, y dim ⊢ u : A`





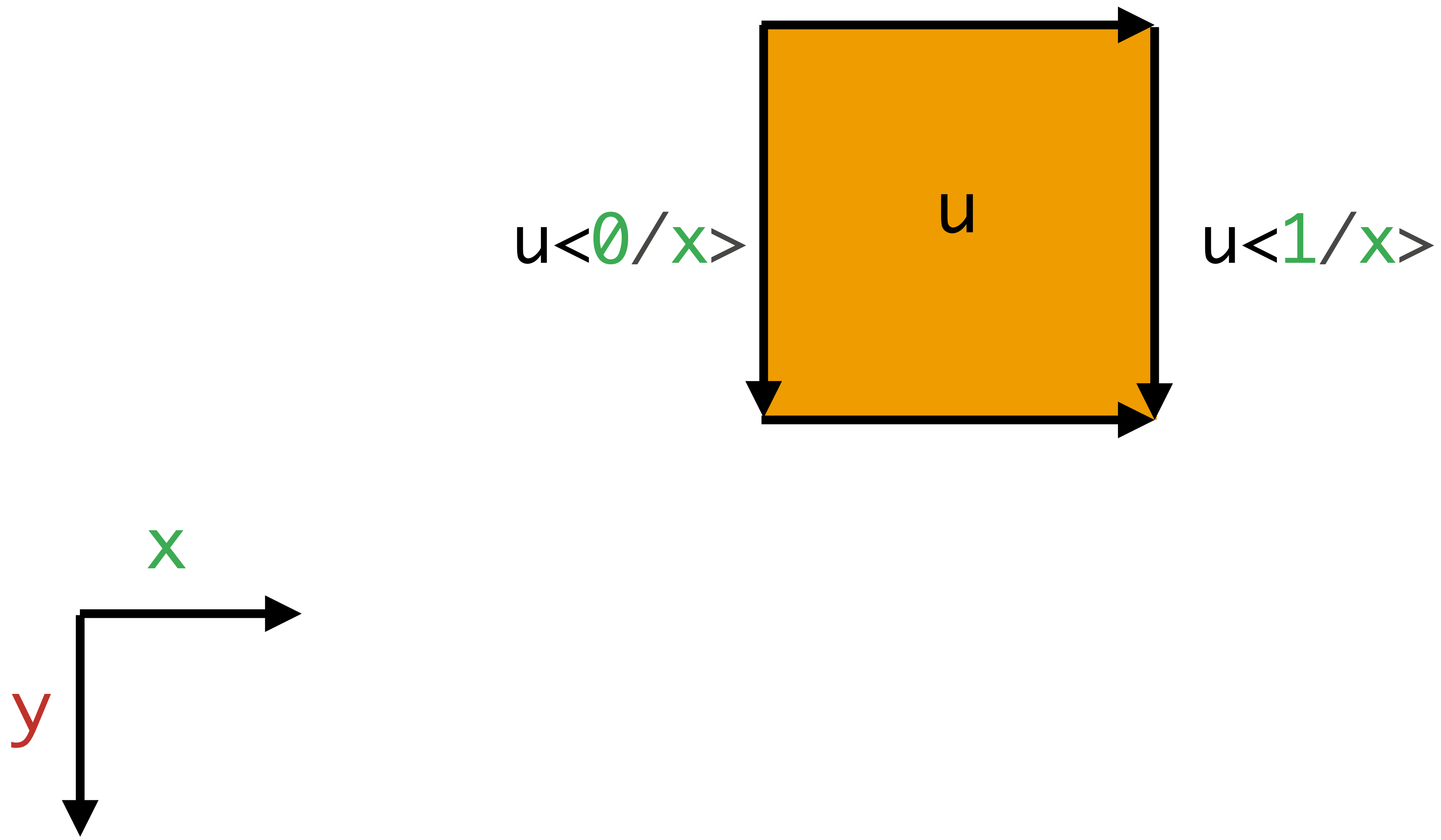
Faces

`x dim, y dim ⊢ u : A`



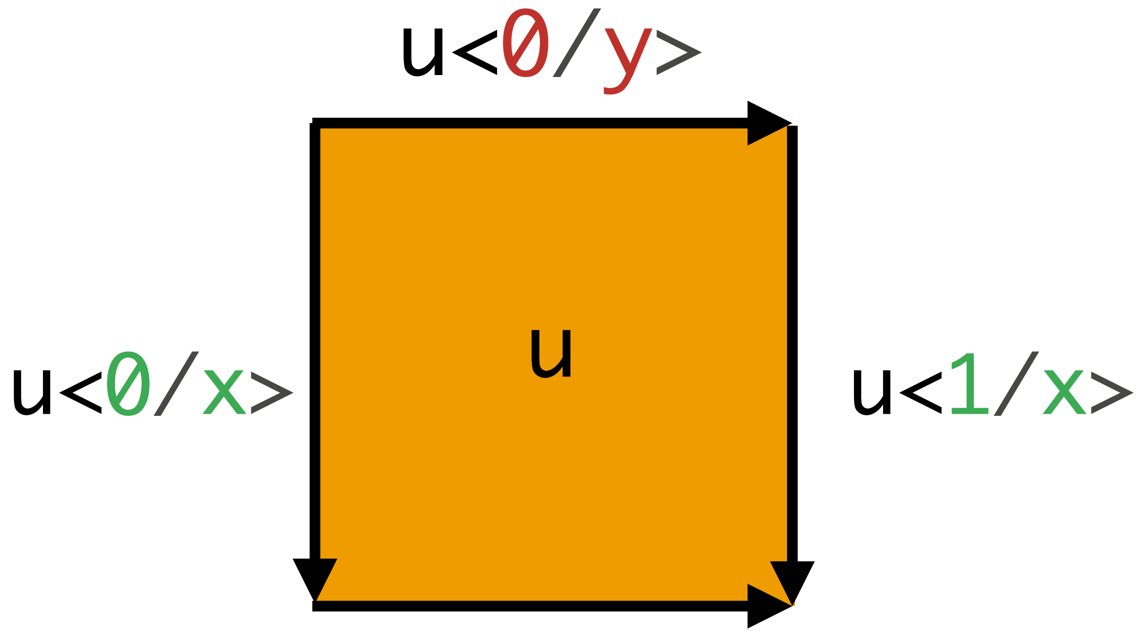
Faces

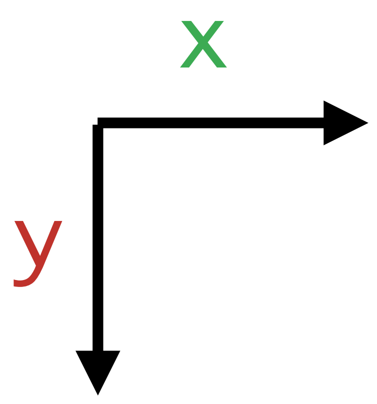
`x dim, y dim ⊢ u : A`



Faces

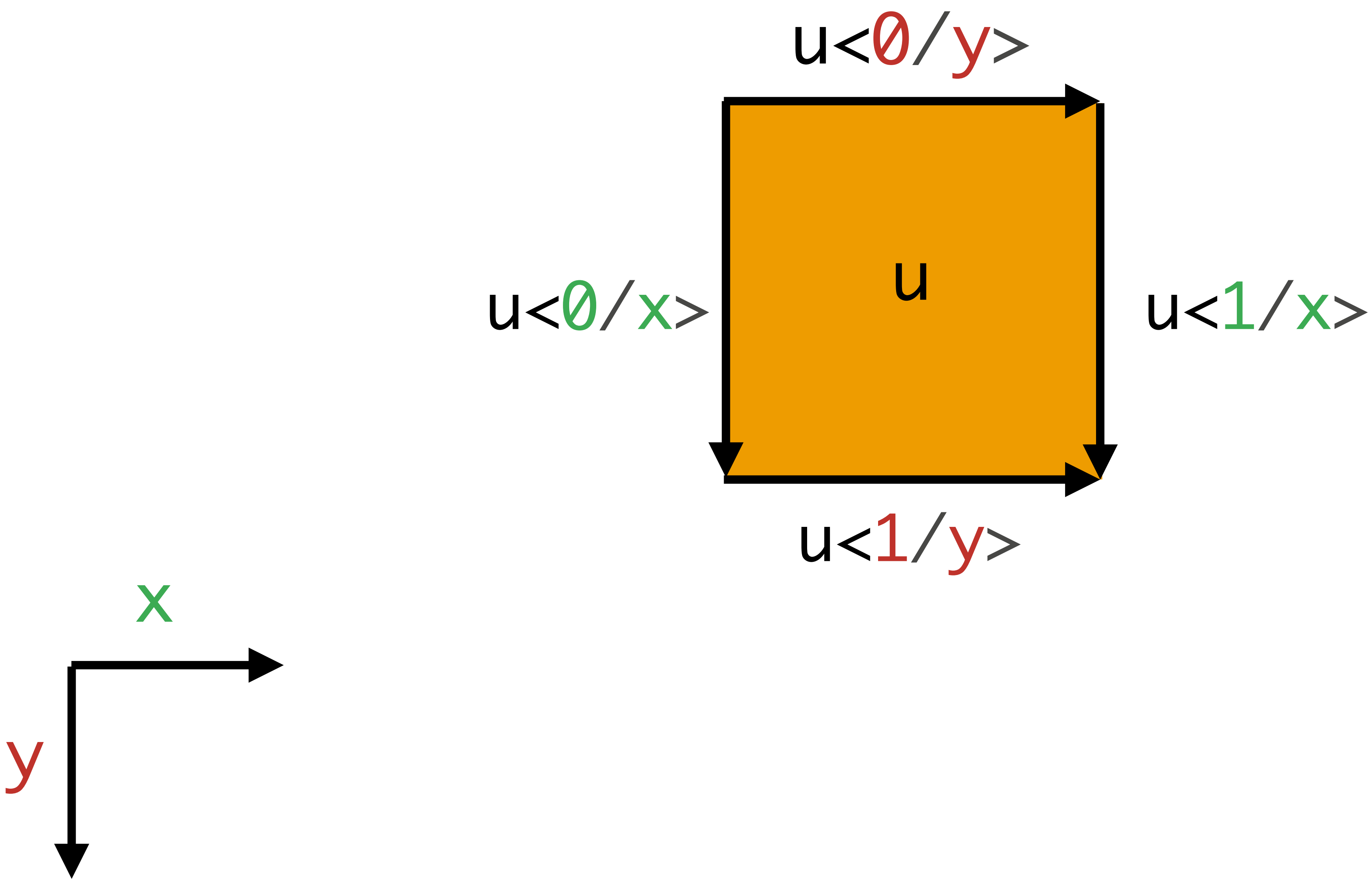
`x dim, y dim ⊢ u : A`





Faces

x dim, y dim ⊢ u : A



u<0/y>

u<0/x>

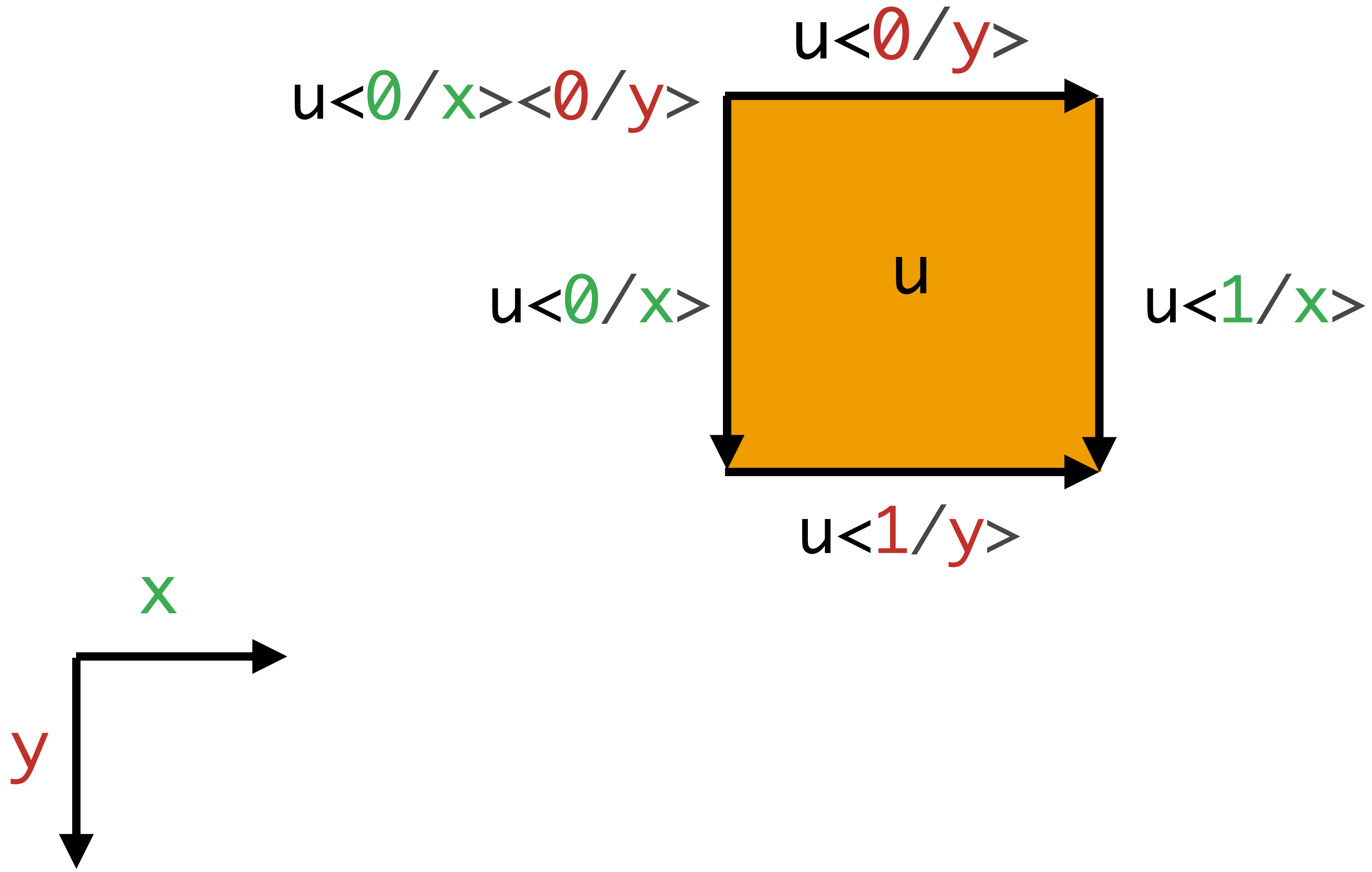
u

u<1/x>

u<1/y>

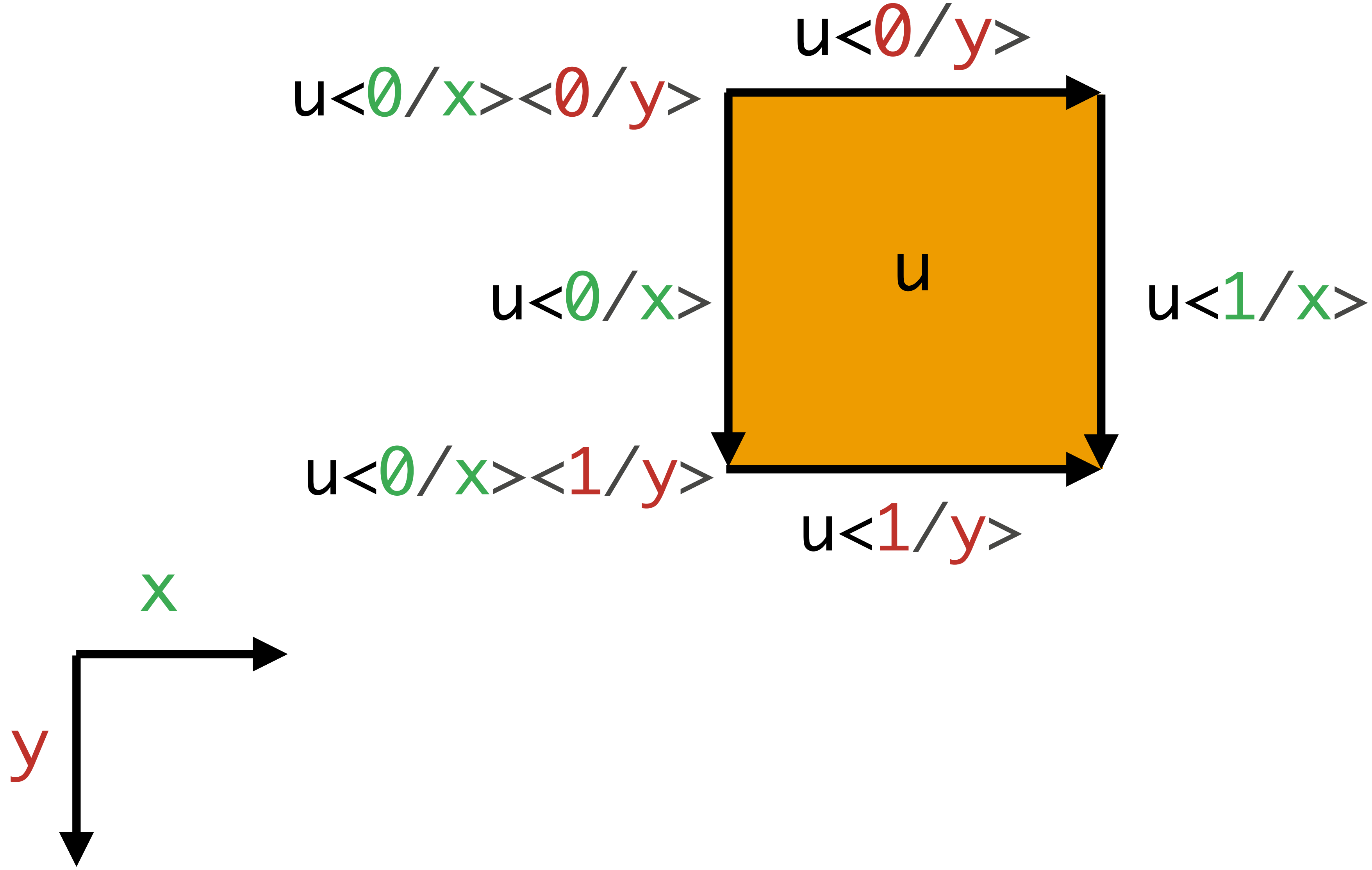
Faces

`x dim, y dim ⊢ u : A`



Faces

`x dim, y dim ⊢ u : A`



u<0/x><0/y>

u<0/y>

u<0/x>

u

u<1/x>

u<0/x><1/y>

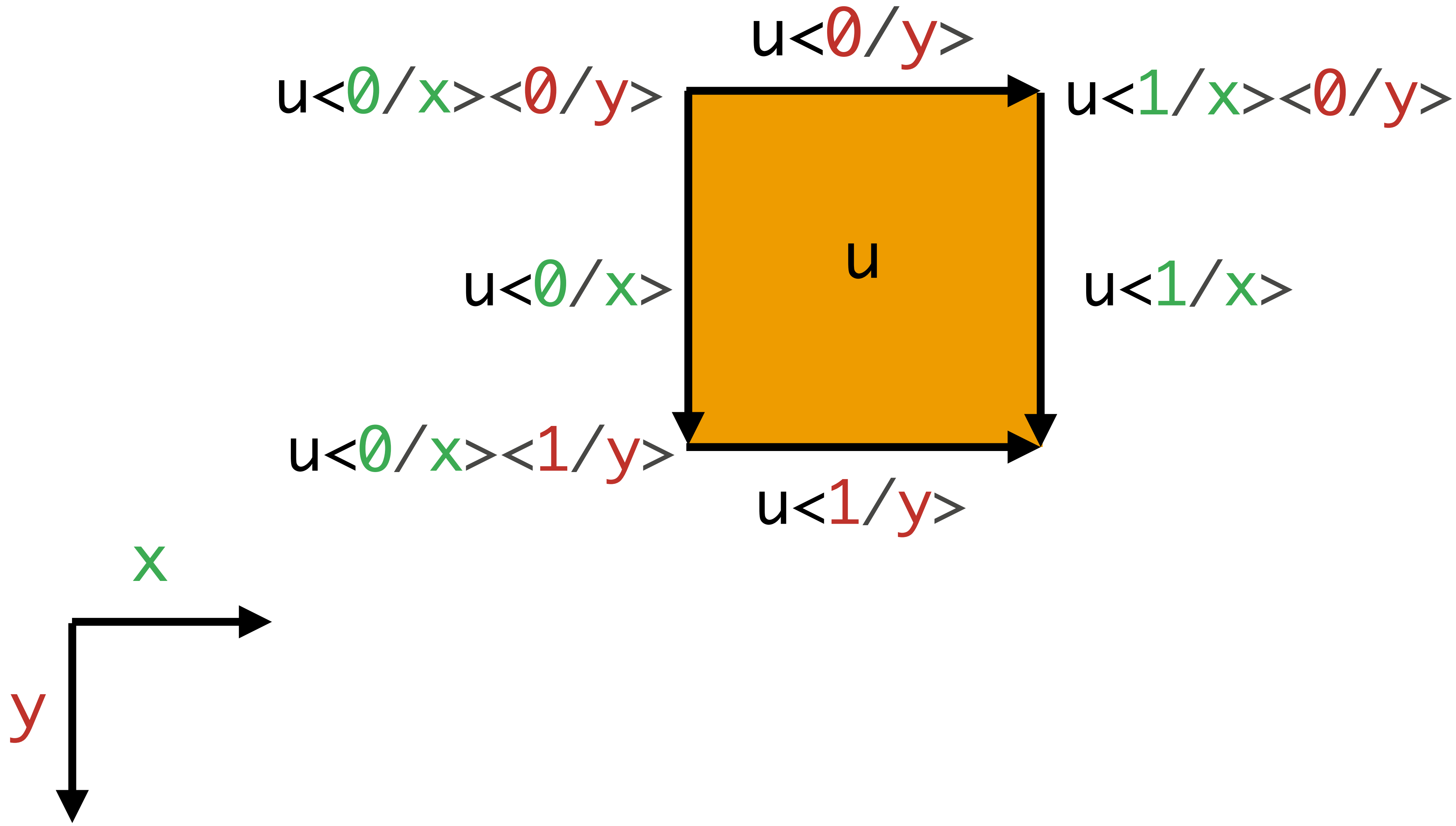
u<1/y>

x

y

Faces

`x dim, y dim ⊢ u : A`



u<0/y>

u<0/x><0/y>

u<1/x><0/y>

u<0/x>

u

u<1/x>

u<0/x><1/y>

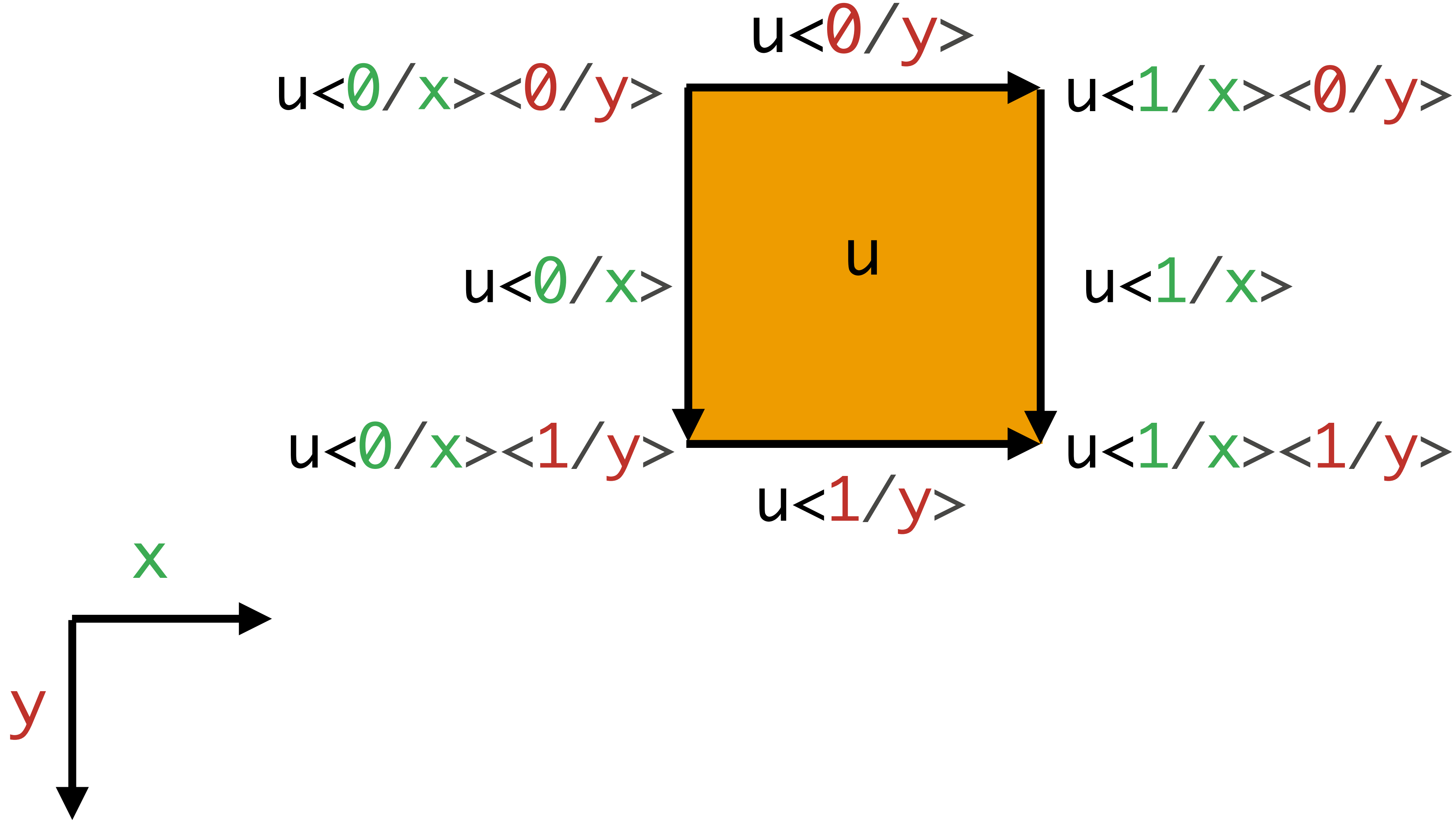
u<1/y>

x

y

Faces

`x dim, y dim ⊢ u : A`



u<0/y>

u<0/x><0/y>

u<1/x><0/y>

u<0/x>

u

u<1/x>

u<0/x><1/y>

u<1/x><1/y>

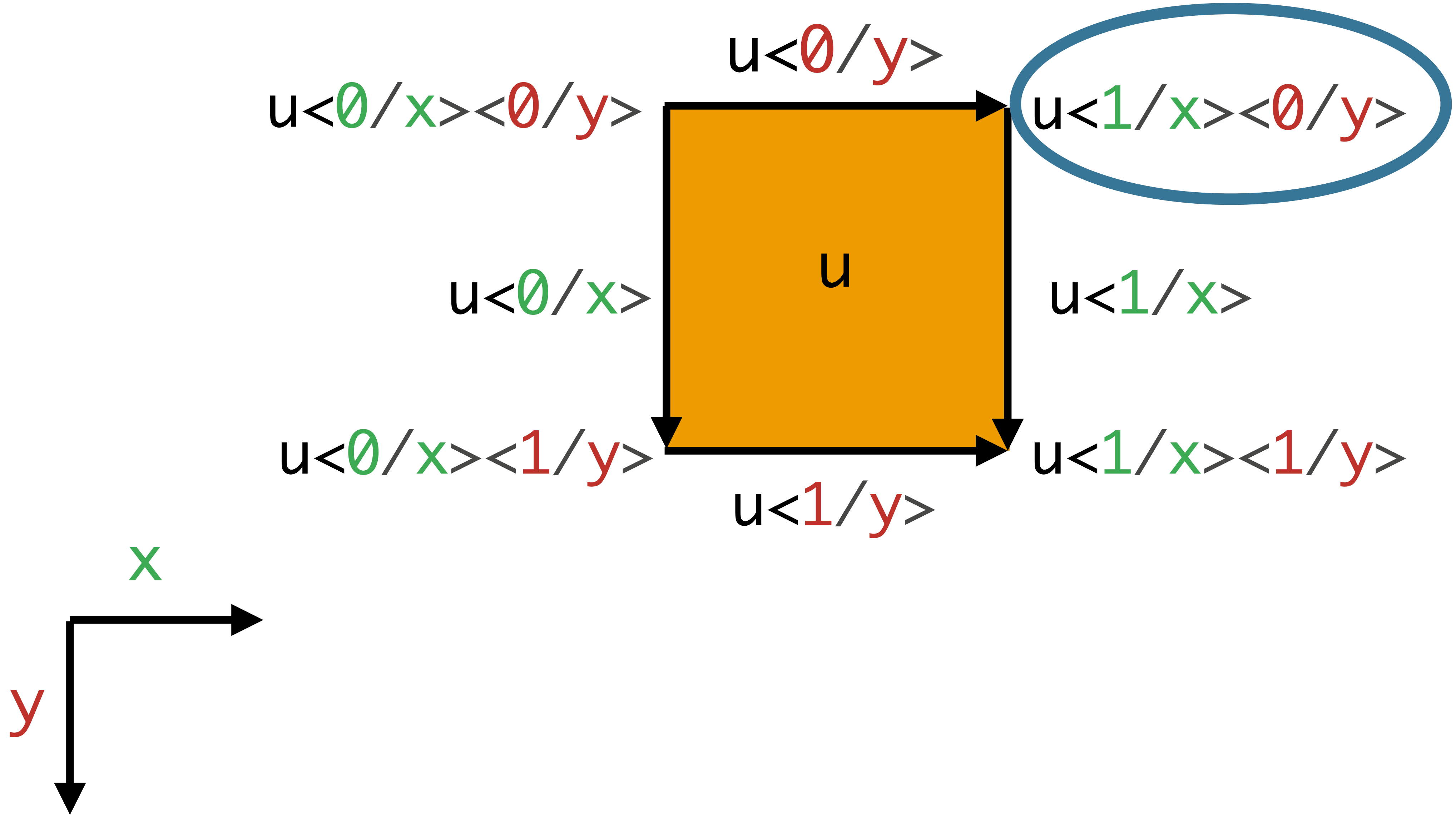
u<1/y>

x

y

Faces

`x dim, y dim ⊢ u : A`



u<0/y>

u<0/x><0/y>

u<1/x><0/y>

u<0/x>

u

u<1/x>

u<0/x><1/y>

u<1/x><1/y>

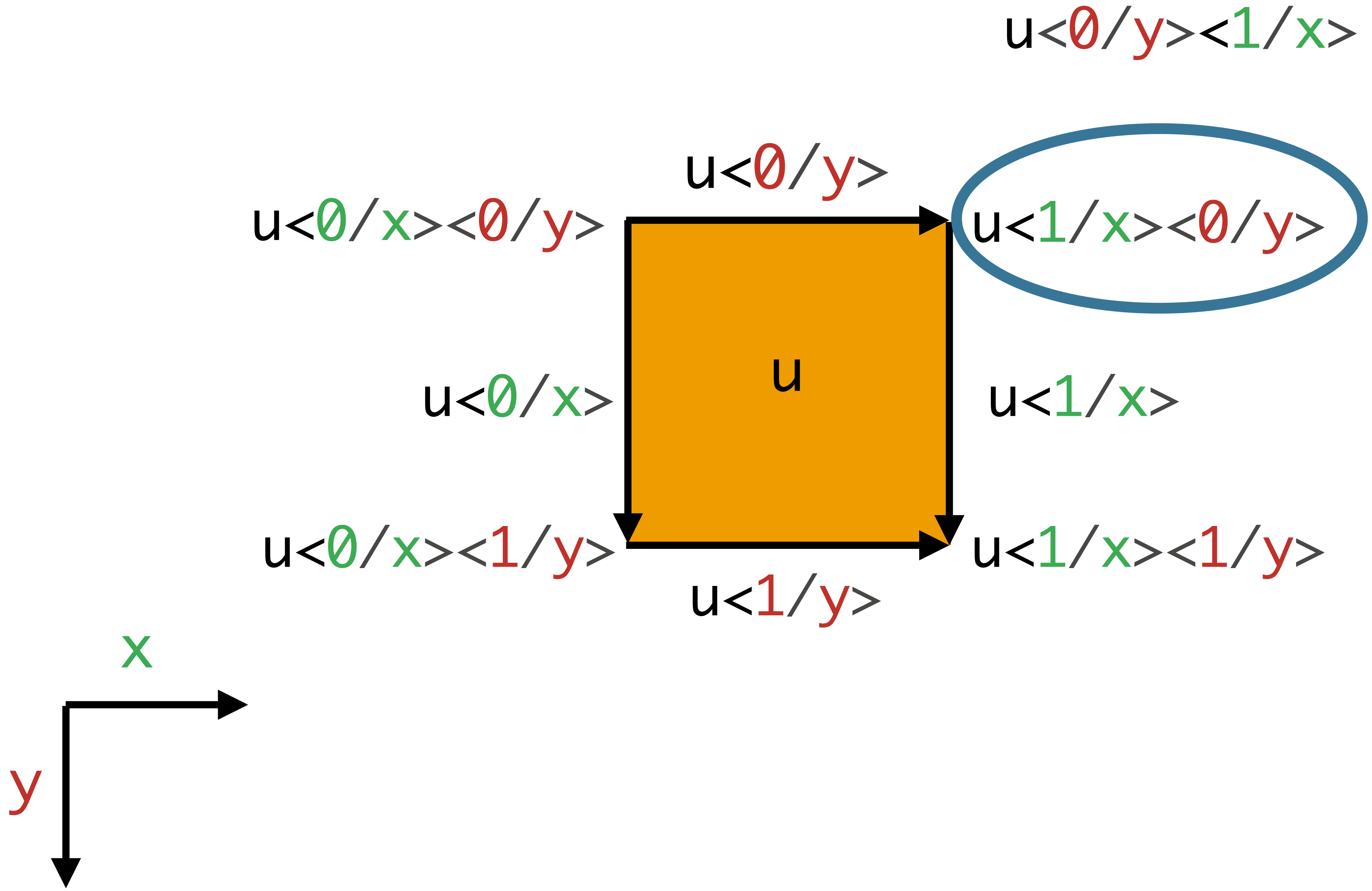
u<1/y>

x

y

Faces

x dim, y dim ⊢ u : A



u<0/y><1/x>

u<0/y>

u<0/x><0/y>

u<1/x><0/y>

u<0/x>

u

u<1/x>

u<0/x><1/y>

u<1/x><1/y>

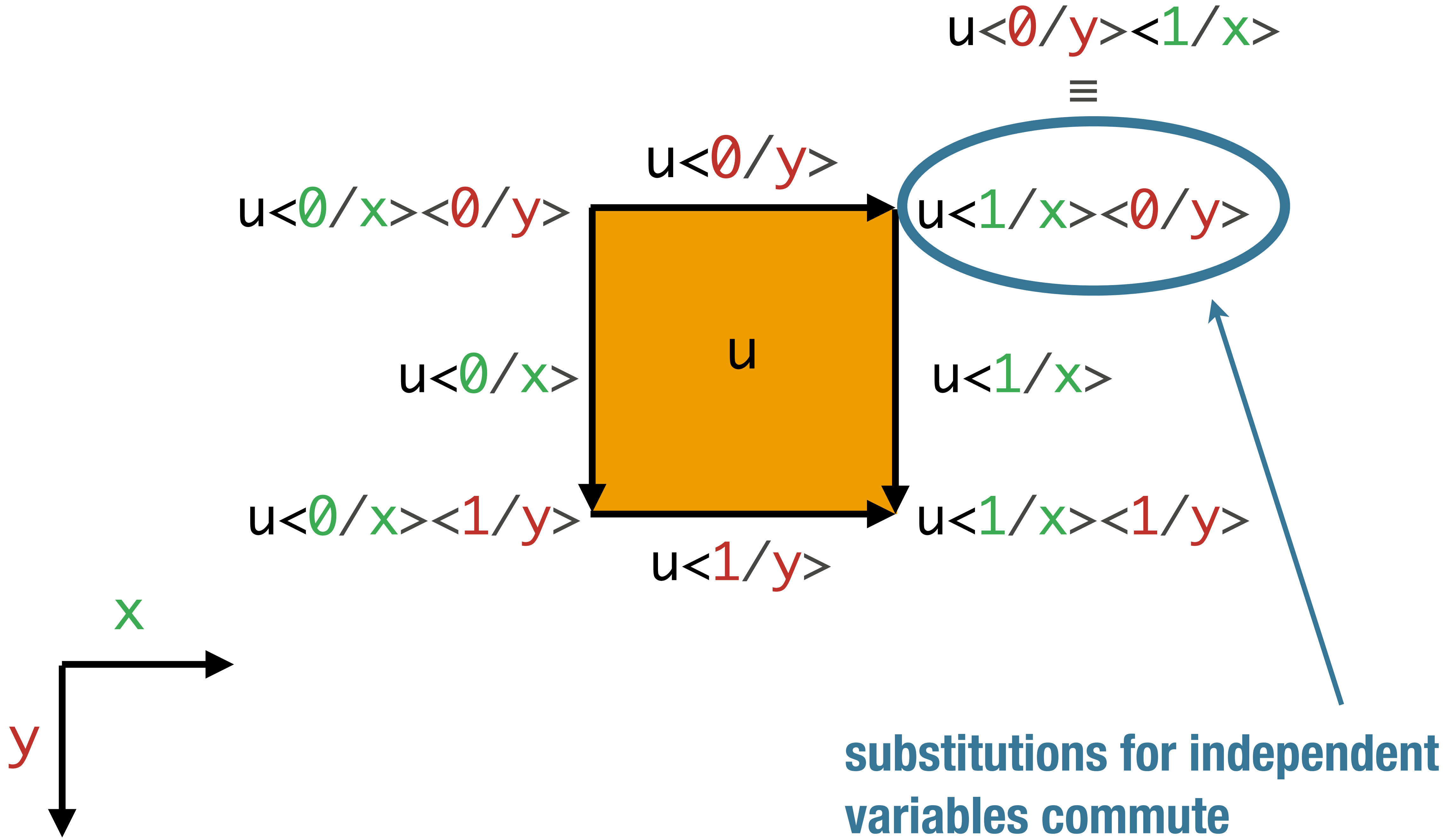
u<1/y>

x

y

Faces

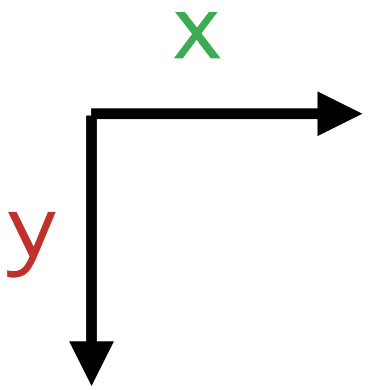
`x dim, y dim ⊢ u : A`



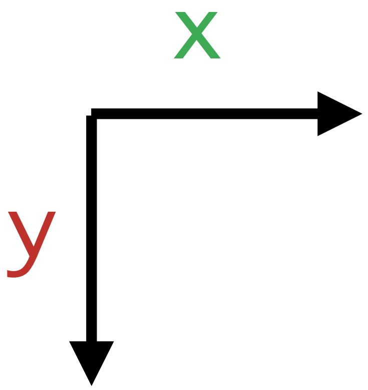
substitutions for independent variables commute

Degeneracies

a



Degeneracies

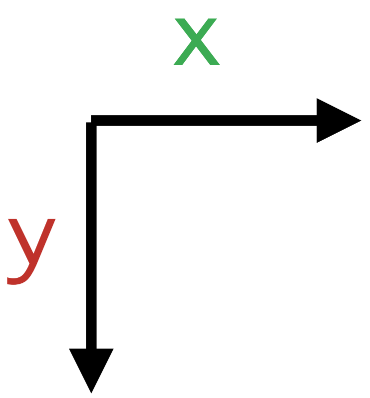


Degeneracies

a

$$\begin{tikzcd} a \arrow[r, "a"] & a \end{tikzcd}$$

$$a \langle 0/x \rangle \equiv a$$



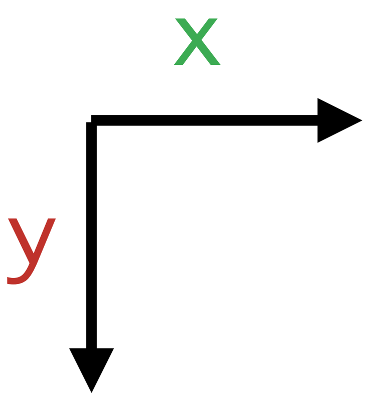
Degeneracies

a

$$\begin{tikzcd} a \arrow[r, "a"] & a \end{tikzcd}$$

$$a <0/x> \equiv a$$

$$a <1/x> \equiv a$$



Degeneracies

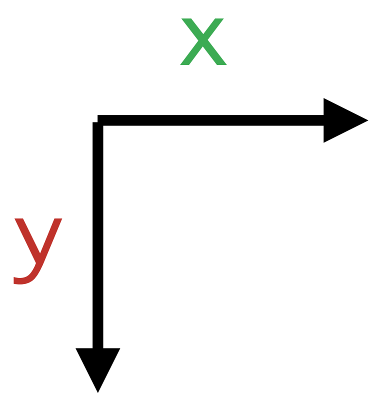
a

$$a \xrightarrow{a} a$$

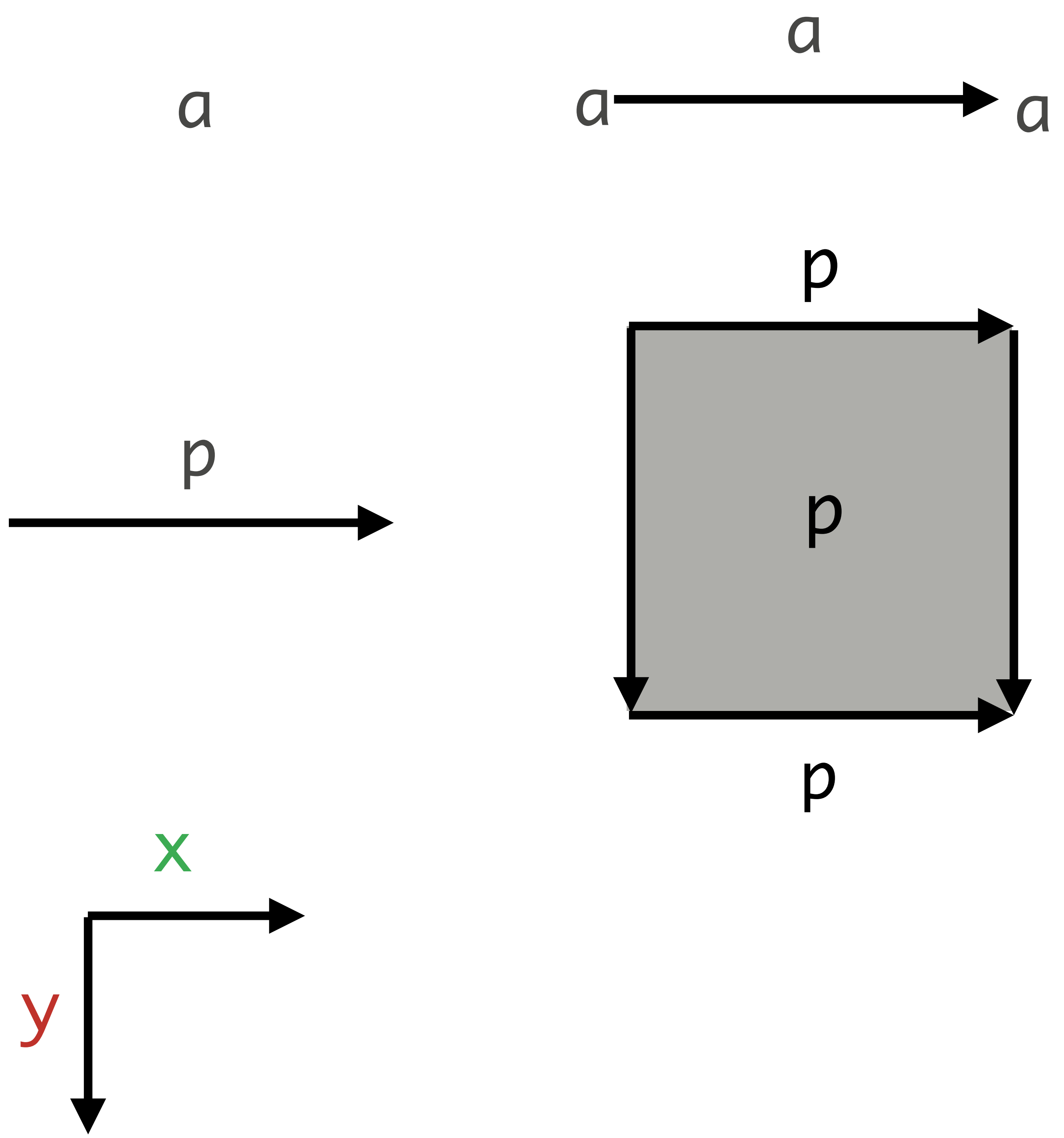
a <0/x> ≡ a

a <1/x> ≡ a

$$\xrightarrow{p}$$



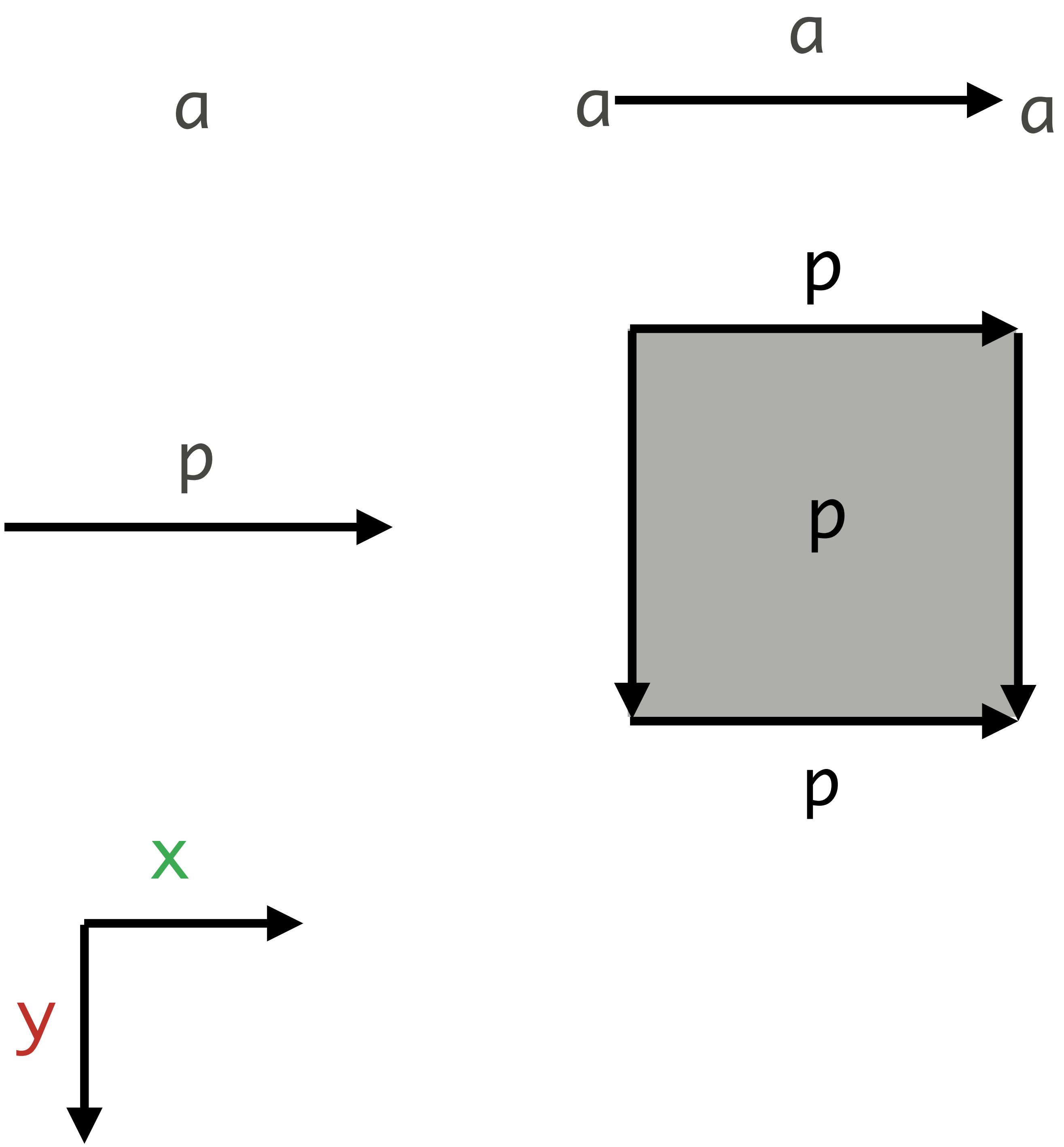
Degeneracies



`a <0/x> ≡ a`

`a <1/x> ≡ a`

Degeneracies



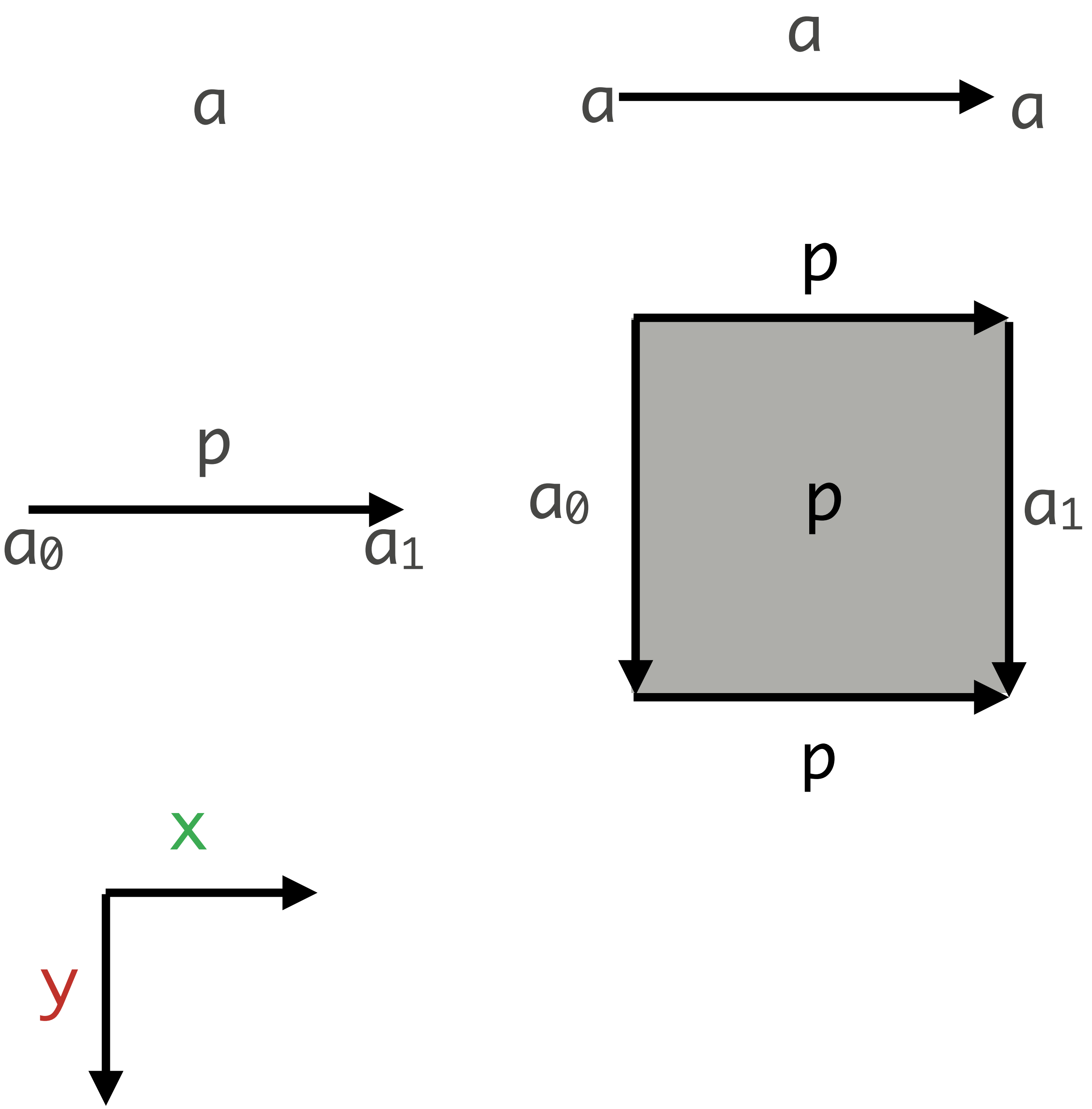
a <0/x> ≡ a

a <1/x> ≡ a

p<0/y> ≡ p

p<1/y> ≡ p

Degeneracies



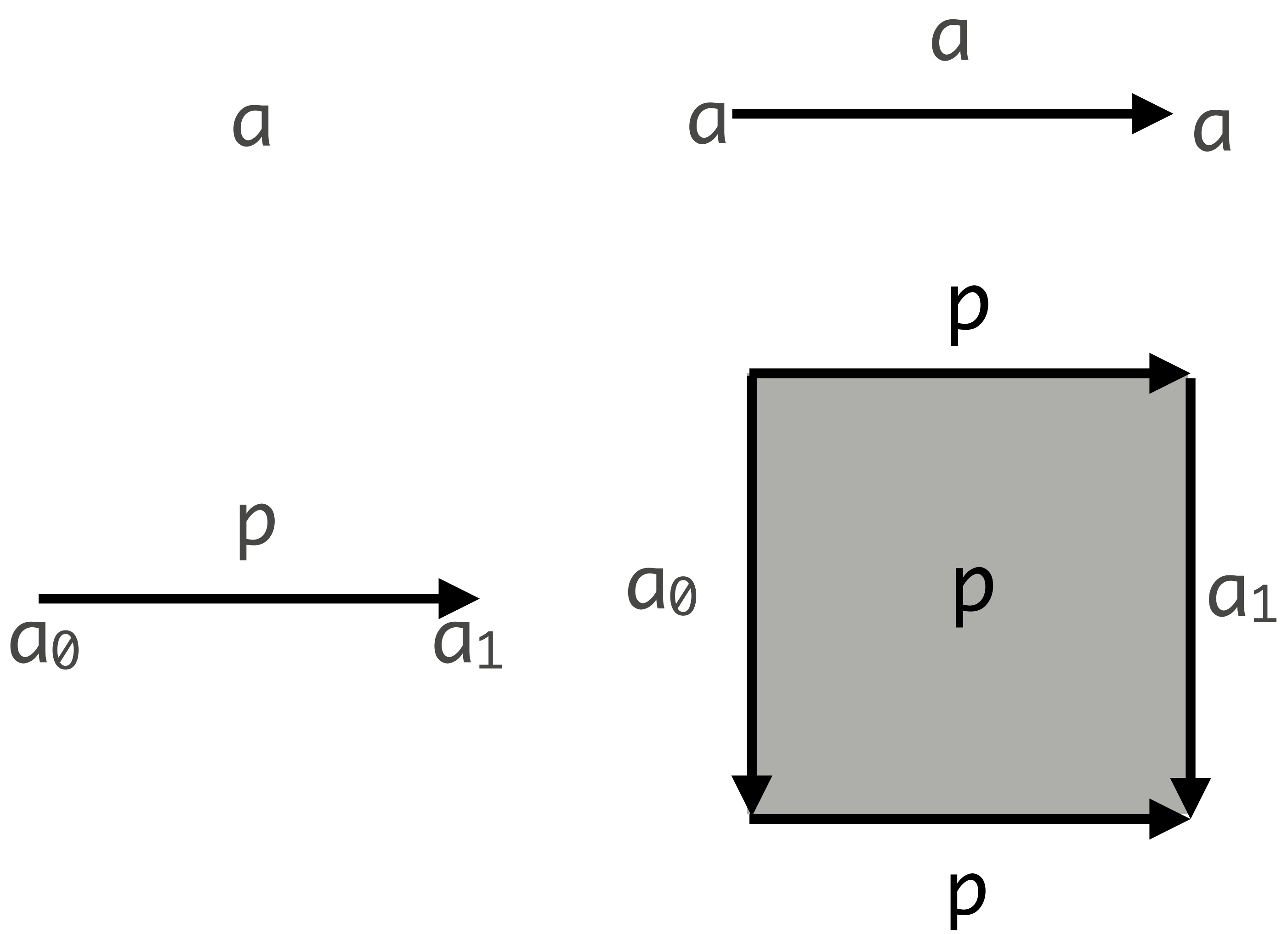
`a <0/x> ≡ a`

`a <1/x> ≡ a`

`p<0/y> ≡ p`

`p<1/y> ≡ p`

Degeneracies

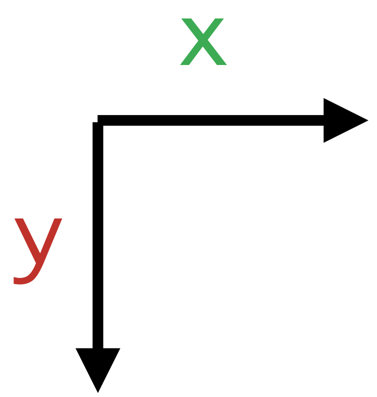


`a <0/x> ≡ a`

`a <1/x> ≡ a`

`p<0/y> ≡ p`

`p<1/y> ≡ p`



substitution after weakening is identity, otherwise pushes inside

Diagonals

$$\begin{tikzcd}
\texttt{u<0/x><0/y>} \arrow[r, "\texttt{u<0/y>}"] \arrow[d, "\texttt{u<0/x>}"'] & \texttt{u<1/x><0/y>} \arrow[d, "\texttt{u<1/x>}"] \\
\texttt{u<0/x><1/y>} \arrow[r, "\texttt{u<1/y>}"'] & \texttt{u<1/x><1/y>}
\end{tikzcd}$$

Diagonals

$$\begin{tikzcd}
\texttt{u<0/x><0/y>} \arrow[r, "\texttt{u<0/y>}"] \arrow[d, "\texttt{u<0/x>}"'] & \texttt{u<1/x><0/y>} \arrow[d, "\texttt{u<1/x>}"] \\
\texttt{u<0/x><1/y>} \arrow[r, "\texttt{u<1/y>}"'] & \texttt{u<1/x><1/y>}
\end{tikzcd}$$

`u<x/y>` is a line ({x}-cube)

Diagonals

$$\begin{tikzcd}
\mathtt{u<0/x><0/y>} \arrow[r, "\mathtt{u<0/y>}"] \arrow[d, "\mathtt{u<0/x>}"'] & \mathtt{u<1/x><0/y>} \arrow[d, "\mathtt{u<1/x>}"] \\
\mathtt{u<0/x><1/y>} \arrow[r, "\mathtt{u<1/y>}"'] & \mathtt{u<1/x><1/y>}
\end{tikzcd}$$

`u<x/y>` is a line ({x}-cube)

`u<x/y><0/x> ≡ u<0/x><0/y>`

Diagonals

$$\begin{tikzcd}
u<0/x><0/y> \arrow[r, "u<0/y>"] \arrow[d, "u<0/x>"'] & u<1/x><0/y> \arrow[d, "u<1/x>"] \\
u<0/x><1/y> \arrow[r, "u<1/y>"'] & u<1/x><1/y>
\end{tikzcd}$$

u<x/y> is a line ({x}-cube)

u<x/y><0/x> ≡ u<0/x><0/y>

Diagonals

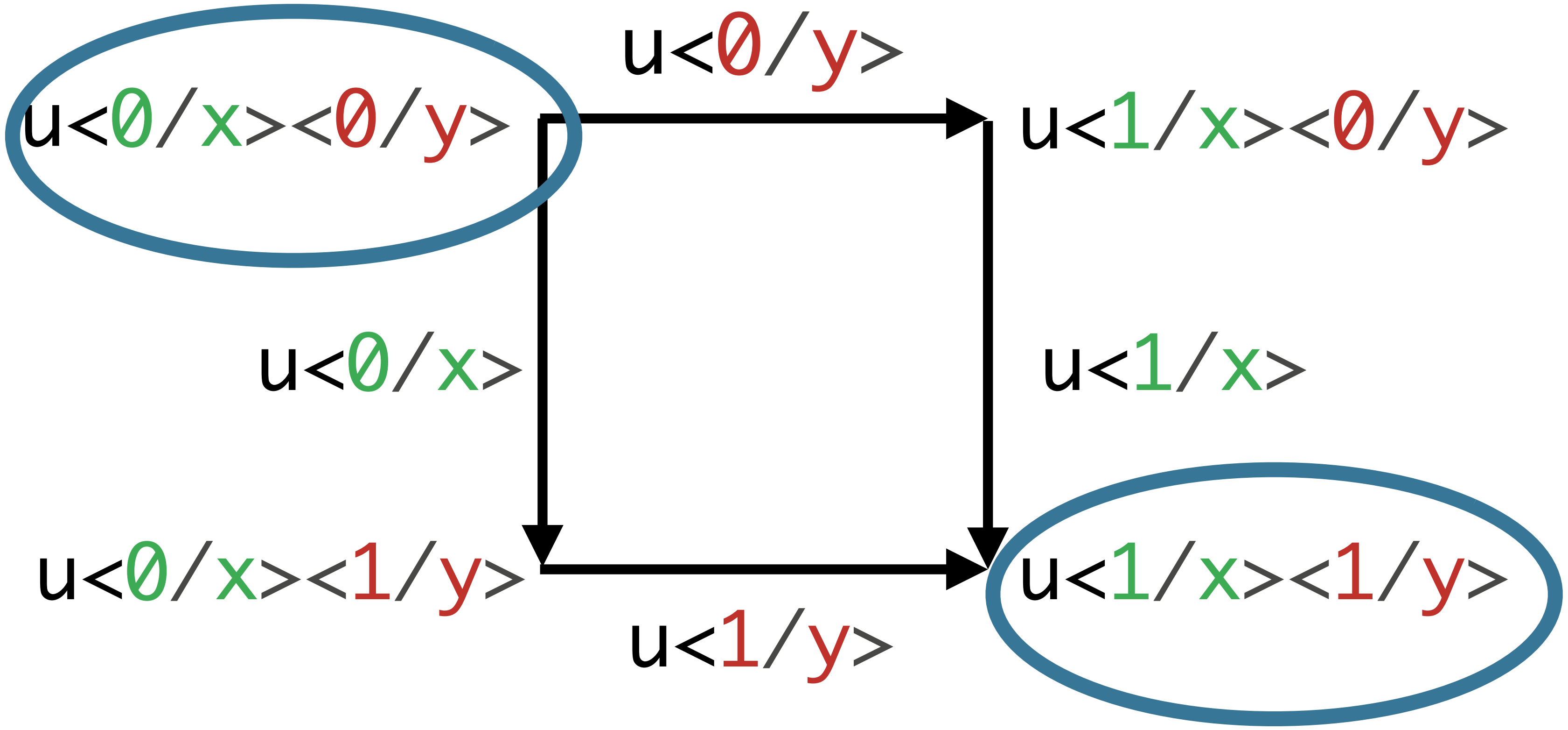
$$\begin{tikzcd}
\text{u<0/x><0/y>} \arrow[r, "\text{u<0/y>}"] \arrow[d, "\text{u<0/x>}"'] & \text{u<1/x><0/y>} \arrow[d, "\text{u<1/x>}"] \\
\text{u<0/x><1/y>} \arrow[r, "\text{u<1/y>}"'] & \text{u<1/x><1/y>}
\end{tikzcd}$$

`u<x/y>` is a line ({x}-cube)

`u<x/y><0/x> ≡ u<0/x><0/y>`

`u<x/y><1/x> ≡ u<1/x><1/y>`

Diagonals



$$\begin{tikzcd}
u\langle 0/x\rangle\langle 0/y\rangle \arrow[r, "u\langle 0/y\rangle"] \arrow[d, "u\langle 0/x\rangle"'] & u\langle 1/x\rangle\langle 0/y\rangle \arrow[d, "u\langle 1/x\rangle"] \\
u\langle 0/x\rangle\langle 1/y\rangle \arrow[r, "u\langle 1/y\rangle"'] & u\langle 1/x\rangle\langle 1/y\rangle
\end{tikzcd}$$

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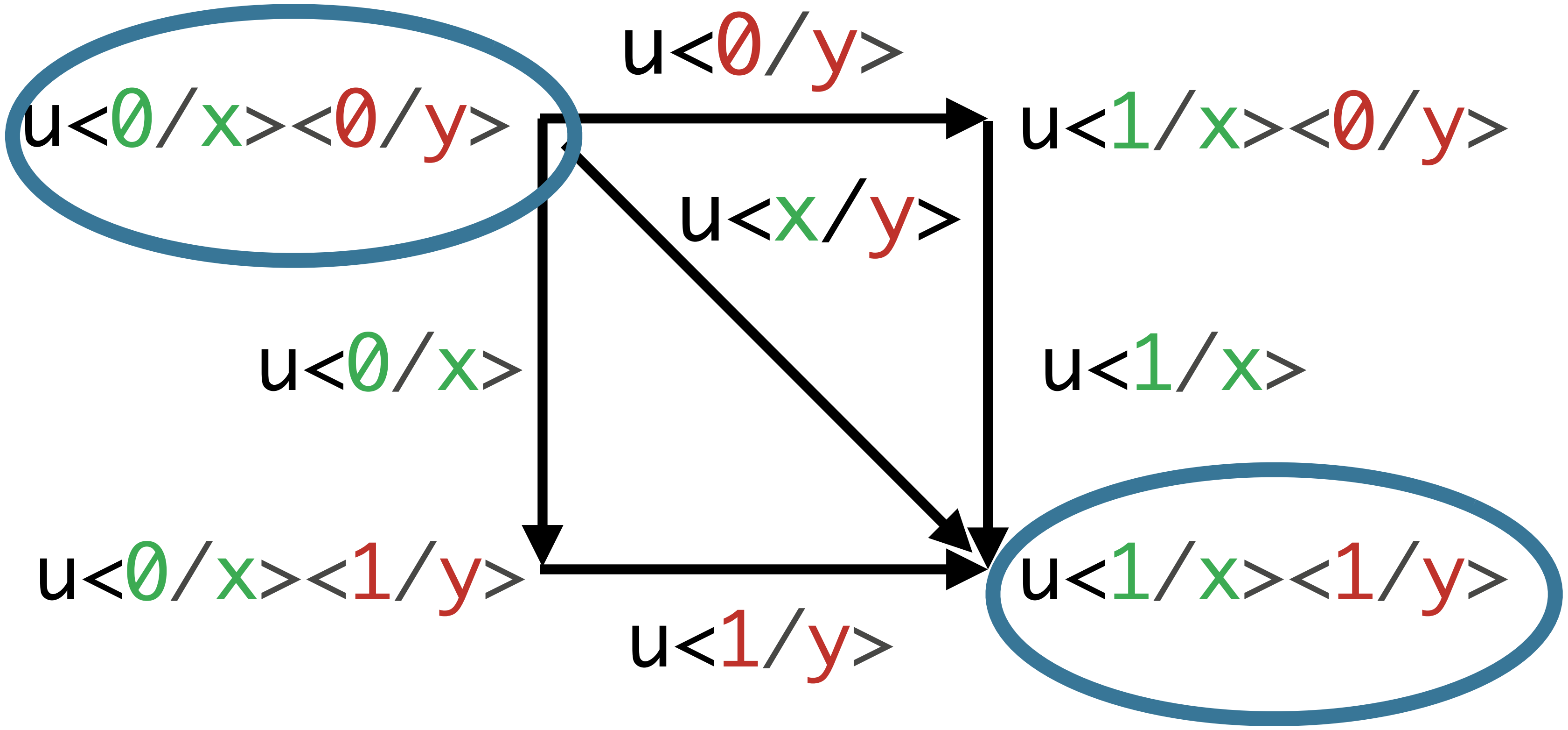
Diagonals



$$\begin{tikzcd}
u<0/x><0/y> \arrow[r, "u<0/y>"] \arrow[d, "u<0/x>"'] \arrow[dr, "u<x/y>"] & u<1/x><0/y> \arrow[d, "u<1/x>"] \\
u<0/x><1/y> \arrow[r, "u<1/y>"'] & u<1/x><1/y>
\end{tikzcd}$$

u<x/y> is a line ({x}-cube)

u<x/y><0/x> ≡ u<0/x><0/y>

u<x/y><1/x> ≡ u<1/x><1/y>

Ingredients

- Cubical operations
- **Higher-dimensional substitution**
- Types
- Kan filling

Higher-dimensional subst.

Higher-dimensional subst.

$$\mathtt{ap} : (f : A \to B) \to a_0 =_A a_1 \to f(a_0) =_A f(a_1)$$

Function f : A → B is a “∞-functor”:
takes points to points, lines to lines,
squares to squares, cubes to cubes, …

Higher-dimensional subst.

$$\mathtt{ap} : (f : A \to B) \to a_0 =_A a_1 \to f(a_0) =_A f(a_1)$$

Function f : A → B is a “∞-functor”:
takes points to points, lines to lines,
squares to squares, cubes to cubes, …

Implement this by a fancy substitution operation on open terms $x{:}A \vdash u : B$

Higher-dimensional subst.

$$\frac{\Gamma, x : A, \Gamma' \vdash u : B \qquad \Gamma \vdash a : A}{\Gamma, \Gamma'[a/x] \vdash u[a/x] : B[a/x]}$$

Higher-dimensional subst.

$$\frac{\Psi; \Gamma, x : A, \Gamma' \vdash u : B \qquad \Psi; \Gamma \vdash a : A}{\Psi; \Gamma, \Gamma'[a/x] \vdash u[a/x] : B[a/x]}$$

Higher-dimensional subst.

$$\frac{\Psi;\Gamma, x : A, \Gamma' \vdash u : B \qquad \Psi;\Gamma \vdash a : A}{\Psi;\Gamma, \Gamma'[a/x] \vdash u[a/x] : B[a/x]}$$

$$u[a/x] \equiv u$$

Higher-dimensional subst.

$$\frac{\Psi; \Gamma, x : A, \Gamma' \vdash u : B \qquad \Psi; \Gamma \vdash a : A}{\Psi; \Gamma, \Gamma'[a/x] \vdash u[a/x] : B[a/x]}$$

$$u[a/x] \equiv u$$

$$u[a/x][b/y] \equiv u[b/y][a[b/y]/x]$$

Higher-dimensional subst.

$$\frac{\Psi; \Gamma, x : A, \Gamma' \vdash u : B \qquad \Psi; \Gamma \vdash a : A}{\Psi; \Gamma, \Gamma'[a/x] \vdash u[a/x] : B[a/x]}$$

$$u[a/x] \equiv u$$

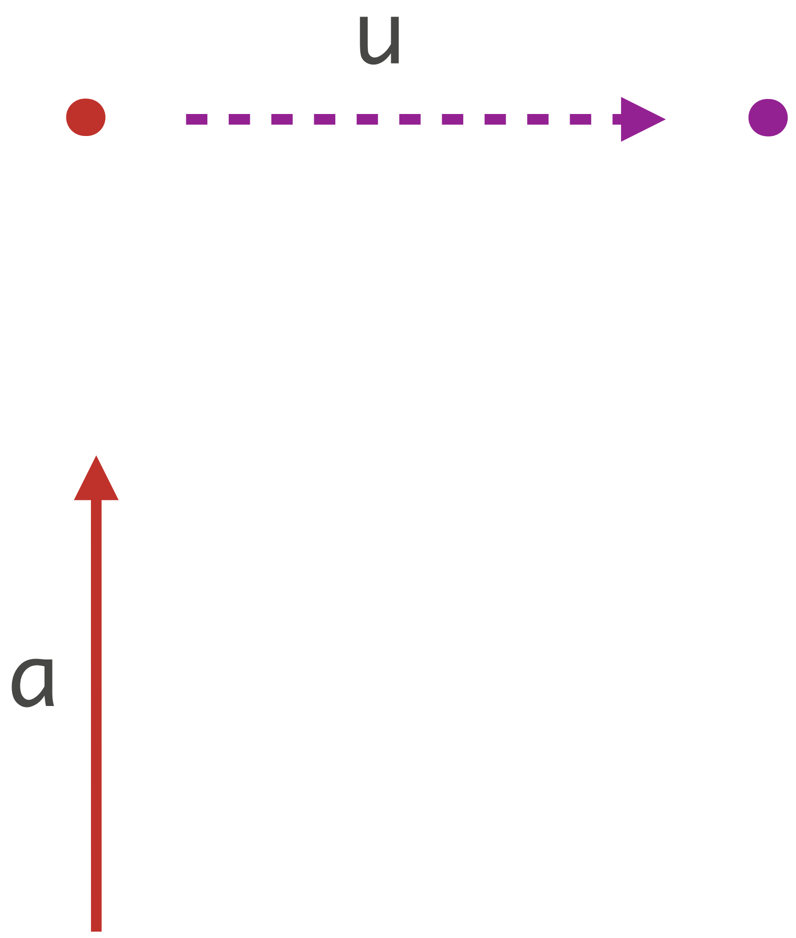
$$u[a/x][b/y] \equiv u[b/y][a[b/y]/x]$$

$$u[a/x]\langle r/s \rangle \equiv u\langle r/s \rangle[a\langle r/s \rangle/x]$$

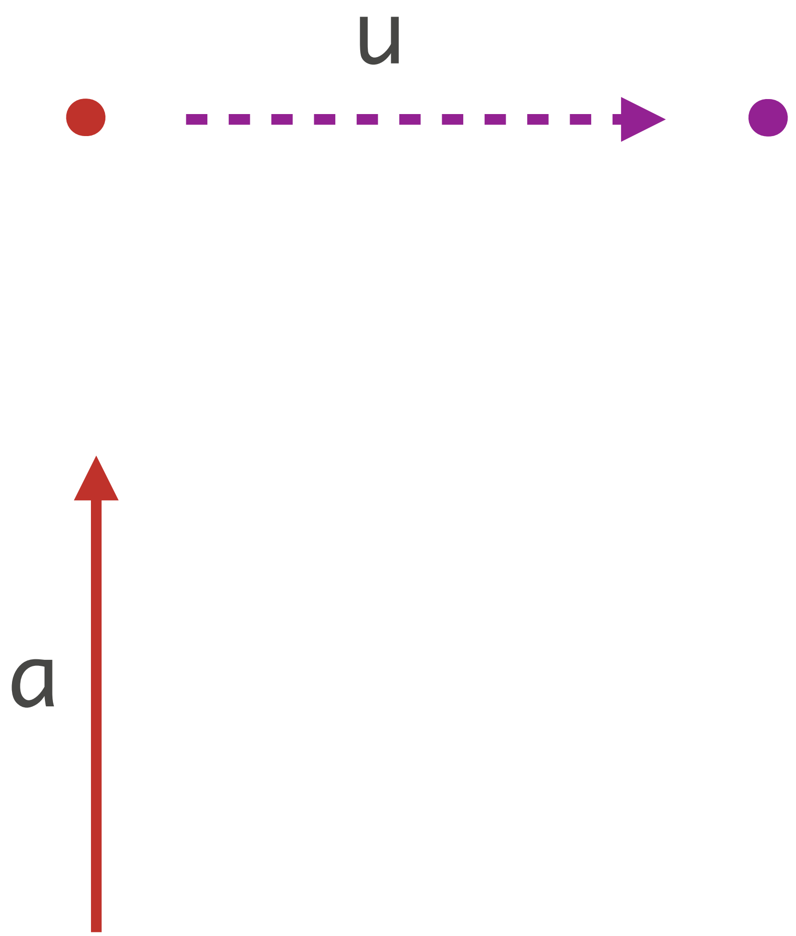
Higher-dimensional subst.

$\emptyset; w : A \vdash u : B$

$y \text{ dim} \vdash a$



Higher-dimensional subst.



$\varnothing; \mathrm{w} : \mathrm{A} \vdash \mathrm{u} : \mathrm{B}$

$\mathrm{y}\ \mathrm{dim}; \mathrm{w} : \mathrm{A} \vdash \mathrm{u} : \mathrm{B}$

$\mathrm{y}\ \mathrm{dim} \vdash \mathrm{a}$

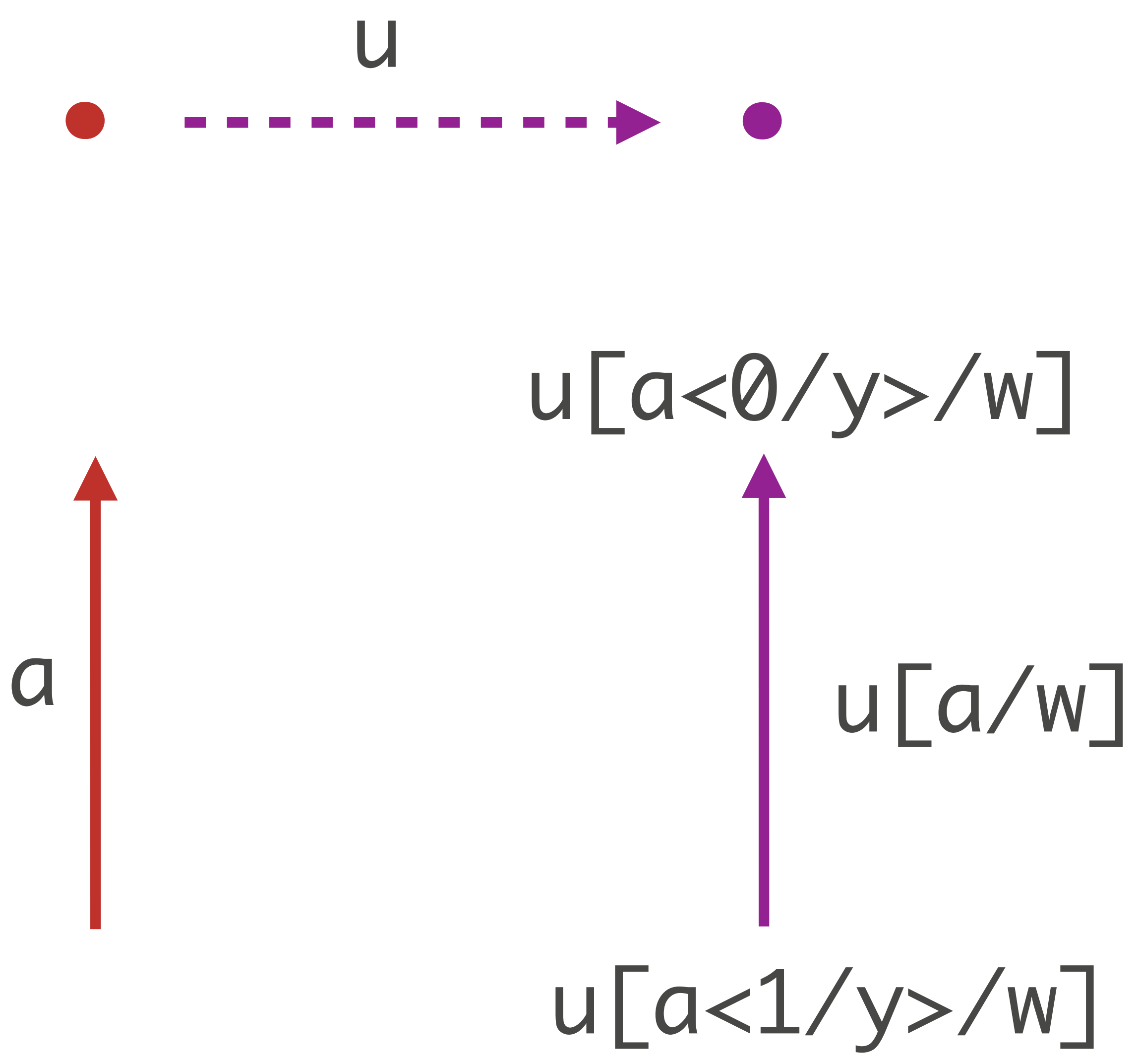
Higher-dimensional subst.

$\emptyset; w : A \vdash u : B$

$y \text{ dim}; w : A \vdash u : B$

$y \text{ dim} \vdash a$

$y \text{ dim} \vdash u[a/w]$



Ingredients

- Cubical operations
- Higher-dimensional substitution
- **Types**
- Kan filling

Π types

$$\frac{\Psi;\Gamma \vdash A : \mathtt{Type} \qquad \Psi;\Gamma, x : A \vdash B : \mathtt{Type}}{\Psi;\Gamma \vdash \Pi\, x{:}A.\, B : \mathtt{Type}}$$

$$\frac{\Psi;\Gamma \vdash f : \Pi\, x{:}A.\, B \qquad \Psi;\Gamma \vdash a : A}{\Psi;\Gamma \vdash \mathtt{app}(f, a) : B[a/x]}$$

$$\frac{\Psi;\Gamma, x : A \vdash u : B}{\Psi;\Gamma \vdash \lambda x.u : \Pi\, x{:}A.\, B}$$

$$f \equiv \lambda\, x.\, \mathtt{app}(f, x)$$

$$\mathtt{app}(\lambda x.u, a) \equiv u[a/x] : B[a/x]$$

Π types

$$\frac{\Psi;\Gamma \vdash A : \mathtt{Type} \qquad \Psi;\Gamma, x : A \vdash B : \mathtt{Type}}{\Psi;\Gamma \vdash \Pi\, x{:}A.\, B : \mathtt{Type}}$$

$$\frac{\Psi;\Gamma \vdash f : \Pi\, x{:}A.\, B \qquad \Psi;\Gamma \vdash a : A}{\Psi;\Gamma \vdash \mathtt{app}(f, a) : B[a/x]}$$

$$\frac{\Psi;\Gamma, x : A \vdash u : B}{\Psi;\Gamma \vdash \lambda x.u : \Pi\, x{:}A.\, B}$$

$$f \equiv \lambda\, x.\, \mathtt{app}(f, x)$$

$$\mathtt{app}(\lambda x.u, a) \equiv u[a/x] : B[a/x]$$

Π types

$$(\Pi\, x{:}A.\, B)\langle r/s\rangle \equiv \Pi\, x{:}A\langle r/s\rangle.\, B\langle r/s\rangle$$

$$(\lambda x.u)\langle r/s\rangle \equiv \lambda x.u\langle r/s\rangle$$

$$\mathtt{app}(f, a)\langle r/s\rangle \equiv \mathtt{app}(f\langle r/s\rangle, a\langle r/s\rangle)$$

Π types

$$(\Pi\, x{:}A.\, B)\langle r/s\rangle \equiv \Pi\, x{:}A\langle r/s\rangle.\, B\langle r/s\rangle$$

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Π types

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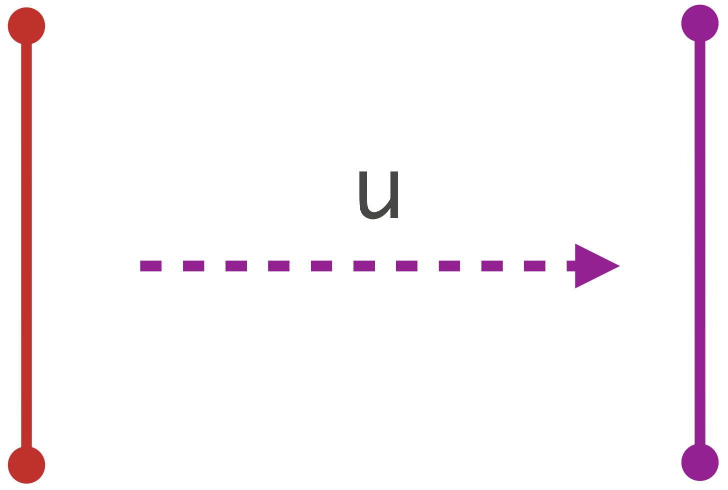
$$\mathsf{app}(f, a)\langle r/s\rangle \equiv \mathsf{app}(f\langle r/s\rangle, a\langle r/s\rangle)$$

dimension subst applies “horizontally”

$$\frac{\Psi, s\ \mathsf{dim}; \Gamma \vdash u : A \qquad \Psi \vdash r\ \mathsf{dim}}{\Psi; \Gamma\langle r/s\rangle \vdash u\langle r/s\rangle : A\langle r/s\rangle}$$

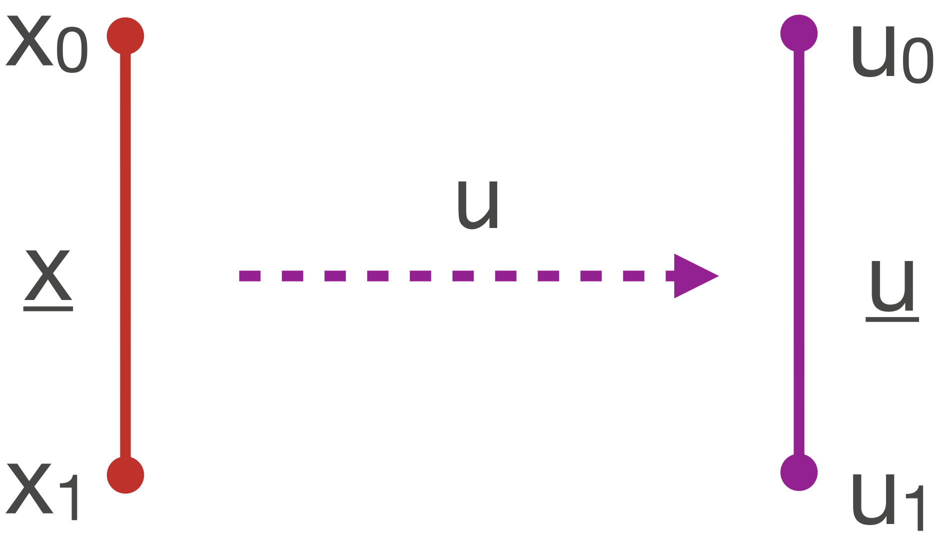
Dimension hypothetical

$s \text{ dim}; x : A \vdash u : B$



Dimension hypothetical

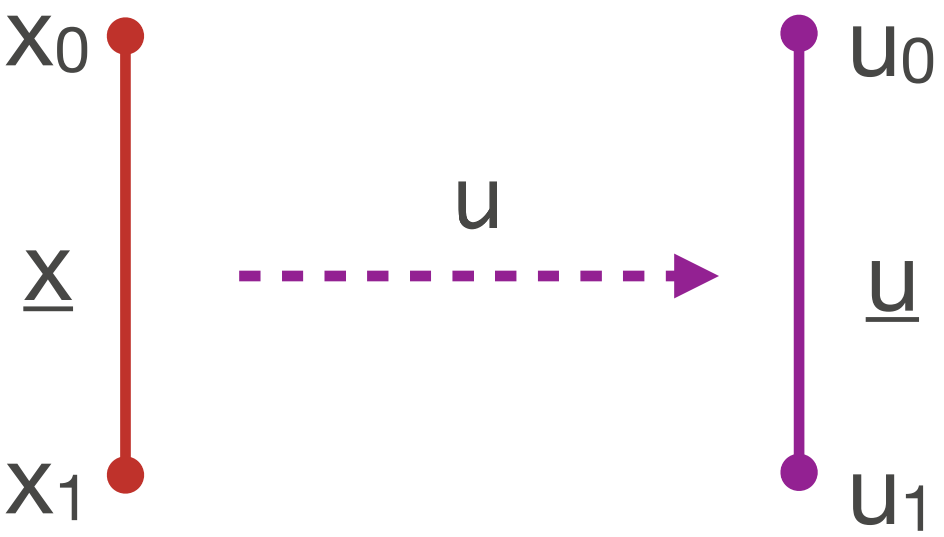
$s \text{ dim}; x : A \vdash u : B$



“$(x_0,x_1,\underline{x}) : A \vdash (u_0,u_1,\underline{u}) : B$”

Dimension hypothetical

$s\ \mathrm{dim};\ x : A \vdash u : B$



“$(x_0, x_1, \underline{x}) : A \vdash (u_0, u_1, \underline{u}) : B$”

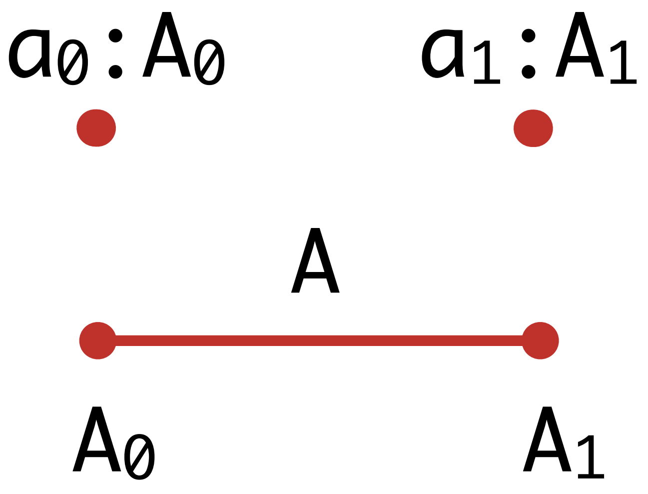
“$(x_0, x_1, \underline{x}) : A \vdash (u_0[x_0], u_1[x_1], \underline{u}[x_0, x_1, \underline{x}]) : B$”

Identity type

Heterogeneous identity type as primitive:

$$\frac{\Psi, s\ \mathsf{dim}; \Gamma \vdash A : \mathtt{Type} \qquad \Psi; \Gamma \vdash a_0 : A\langle 0/s \rangle \qquad \Psi; \Gamma \vdash a_1 : A\langle 1/s \rangle}{\Psi; \Gamma \vdash \mathtt{Id}_A^{[s]}(a_0, a_1) : \mathtt{Type}}$$

$\mathrm{Id}^{[x]}_{A}(a_0,a_1)$

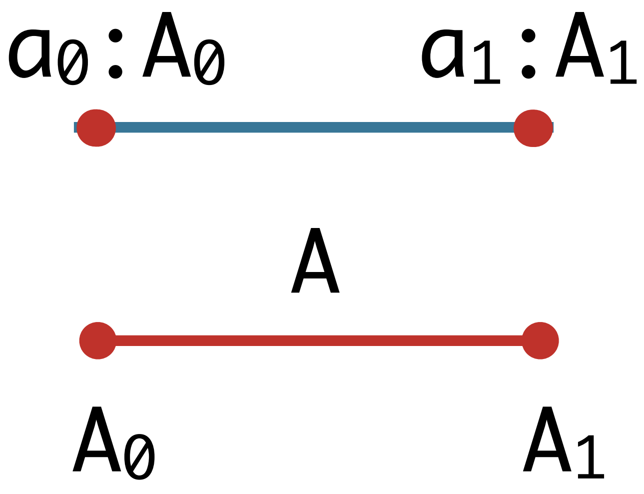


Identity type

Heterogeneous identity type as primitive:

$$\frac{\Psi, s\ \mathsf{dim}; \Gamma \vdash A : \mathtt{Type} \qquad \Psi; \Gamma \vdash a_0 : A\langle 0/s \rangle \qquad \Psi; \Gamma \vdash a_1 : A\langle 1/s \rangle}{\Psi; \Gamma \vdash \mathtt{Id}^{[s]}_{A}(a_0, a_1) : \mathtt{Type}}$$

u : $\mathrm{Id}^{[x]}{}_{A}(a_0, a_1)$



Identity type intro

$$\frac{\Psi, s\ \mathsf{dim}; \Gamma \vdash u : A \quad u\langle 0/s\rangle \equiv a_0 \quad u\langle 1/s\rangle \equiv a_1}{\Psi; \Gamma \vdash \langle s\rangle u : \mathtt{Id}_A^{[s]}(a_0, a_1)}$$

$$(\langle s\rangle u)\langle r/s'\rangle \equiv \langle s\rangle(u\langle r/s'\rangle)$$

Identity type elim

$$\frac{\Psi; \Gamma \vdash u : \mathtt{Id}_{A}^{[s]}(a, b) \qquad \Psi \vdash r\ \mathsf{dim}}{\Psi; \Gamma \vdash u@r : A\langle r/s\rangle}$$

Identity type elim

$$\frac{\Psi;\Gamma \vdash u : \mathtt{Id}_{A}^{[s]}(a,b) \qquad \Psi \vdash r\ \mathsf{dim}}{\Psi;\Gamma \vdash u@r : A\,\langle r/s\rangle}$$

$$(u@r')\langle r/s\rangle \equiv u\langle r/s\rangle@r'\langle r/s\rangle$$

Identity type elim

$$\frac{\Psi; \Gamma \vdash u : \mathtt{Id}_{A}^{[s]}(a, b) \qquad \Psi \vdash r\ \mathsf{dim}}{\Psi; \Gamma \vdash u@r : A\langle r/s\rangle}$$

$$(u@r')\langle r/s\rangle \equiv u\langle r/s\rangle@r'\langle r/s\rangle$$

$$(\langle s\rangle u)@r \equiv u\langle r/s\rangle$$

Identity type elim

$$\frac{\Psi ; \Gamma \vdash u : \mathtt{Id}_A^{[s]}(a, b) \quad \Psi \vdash r \; \mathsf{dim}}{\Psi ; \Gamma \vdash u @ r : A\langle r/s\rangle}$$

$$(u @ r')\langle r/s\rangle \equiv u\langle r/s\rangle @ r'\langle r/s\rangle$$

$$(\langle s\rangle u) @ r \equiv u\langle r/s\rangle$$

$$u \equiv \langle s\rangle (u @ s)$$

Identity type elim

[c.f. Stone-Harper singleton calculus]

$$\frac{\Psi;\Gamma \vdash u : \mathtt{Id}_{A}^{[s']}(a_0, a_1)}{\Psi;\Gamma \vdash u@0 \equiv a_0 \qquad\qquad : A\langle 0/s'\rangle}$$

$$\frac{\Psi;\Gamma \vdash u : \mathtt{Id}_{A}^{[s']}(a_0, a_1)}{\Psi;\Gamma \vdash u@1 \equiv a_1 \qquad\qquad : A\langle 1/s'\rangle}$$

Example: Funext

$h : \Pi x{:}A.\mathrm{Id}_B(f\ x, g\ x) \vdash\ ?\ : \mathrm{Id}_{\Pi:A.B}(f,g)$

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$h : \Pi x{:}A.\mathrm{Id}_B(f\ x, g\ x) \vdash ?\ : \mathrm{Id}_{\Pi:A.B}(f,g)$

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$?\langle 0/s \rangle \equiv f$

$?\langle 1/s \rangle \equiv g$

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$h : \Pi x{:}A.\mathrm{Id}_B(f\ x,\ g\ x) \vdash \langle s\rangle\ ?\ \ : \mathrm{Id}_{\Pi:A.B}(f,g)$

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$$?\langle 0/s\rangle \equiv f$$

$$?\langle 1/s\rangle \equiv g$$

$$s\ \mathrm{dim}; h : \Pi x{:}A.\mathrm{Id}_B(f\ x, g\ x), x{:}A \vdash\ ?\ : B$$

Example: Funext

$h : \Pi x{:}A.\mathrm{Id}_B(f\ x, g\ x) \vdash \langle s\rangle\backslash x.?\ : \mathrm{Id}_{\Pi:A.B}(f,g)$

$s\ \mathrm{dim}; h : \Pi x{:}A.\mathrm{Id}_B(f\ x, g\ x) \vdash \backslash x.?\ : \Pi{:}A.B$

$\backslash x.?\langle 0/s\rangle \equiv f$

$\backslash x.?\langle 1/s\rangle \equiv g$

$s\ \mathrm{dim}; h : \Pi x{:}A.\mathrm{Id}_B(f\ x, g\ x), x{:}A \vdash ?\ : B$

Example: Funext

$h : \Pi x{:}A.\mathrm{Id}_B(f\ x,\ g\ x) \vdash \langle s\rangle \backslash x.?\ \ : \mathrm{Id}_{\Pi:A.B}(f,g)$

$s\ \mathrm{dim};\ h : \Pi x{:}A.\mathrm{Id}_B(f\ x,\ g\ x) \vdash \backslash x.?\ \ : \Pi{:}A.B$

$\backslash x.?\langle 0/s\rangle \equiv f$

$\backslash x.?\langle 1/s\rangle \equiv g$

$s\ \mathrm{dim};\ h : \Pi x{:}A.\mathrm{Id}_B(f\ x,\ g\ x), x{:}A \vdash (h\ x)@s\ \ : B$

Example: Funext

$$h : \Pi x{:}A.\mathrm{Id}_B(f\ x,\ g\ x) \vdash <s>\backslash x.?\ \ : \mathrm{Id}_{\Pi{:}A.B}(f,g)$$

$$s\ \mathrm{dim};\ h : \Pi x{:}A.\mathrm{Id}_B(f\ x,\ g\ x) \vdash \backslash x.?\ \ : \Pi{:}A.B$$

$$\backslash x.?<0/s> \equiv f$$

$$\backslash x.?<1/s> \equiv g$$

$$s\ \mathrm{dim};\ h : \Pi x{:}A.\mathrm{Id}_B(f\ x,\ g\ x), x{:}A \vdash (h\ x)@s\ \ : B$$

$$((h\ x)@s)<0/s> \equiv f(x)$$

$$((h\ x)@s)<1/s> \equiv g(x)$$

Example: Funext

$h : \Pi x{:}A.\mathrm{Id}_B(f\ x, g\ x) \vdash \texttt{<s>}\backslash x.h(x) \ \ : \mathrm{Id}_{\Pi:A.B}(f,g)$

$s\ \mathrm{dim}; h : \Pi x{:}A.\mathrm{Id}_B(f\ x, g\ x) \vdash \backslash x.h(x)@s \ \ : \Pi{:}A.B$

$(\backslash x.h(x)@s)\langle 0/s\rangle \equiv f$

$(\backslash x.h(x)@s)\langle 1/s\rangle \equiv g$

$s\ \mathrm{dim}; h : \Pi x{:}A.\mathrm{Id}_B(f\ x, g\ x), x{:}A \vdash (h\ x)@s \ \ : B$

$((h\ x)@s)\langle 0/s\rangle \equiv f(x)$

$((h\ x)@s)\langle 1/s\rangle \equiv g(x)$

Circle

$$\frac{}{\Psi; \Gamma \vdash \mathtt{base} : \mathbb{S}^1}$$

$$\frac{\Psi \vdash r\ \mathsf{dim}}{\Psi; \Gamma \vdash \mathtt{loop}_r : \mathbb{S}^1}$$

Circle

$$\frac{}{\Psi; \Gamma \vdash \mathsf{base} : \mathbb{S}^1}$$

$$\frac{\Psi \vdash r\ \mathsf{dim}}{\Psi; \Gamma \vdash \mathtt{loop}_r : \mathbb{S}^1}$$

$$\mathsf{base}\langle r/s \rangle \equiv \mathsf{base}$$

Circle

$$\frac{}{\Psi; \Gamma \vdash \texttt{base} : \mathbb{S}^1}$$

$$\frac{\Psi \vdash r \ \mathsf{dim}}{\Psi; \Gamma \vdash \texttt{loop}_r : \mathbb{S}^1}$$

$$\texttt{base}\langle r/s \rangle \equiv \texttt{base}$$

$$\texttt{loop}_{r'}\langle r/s \rangle \equiv \texttt{loop}_{r'\langle r/s \rangle}$$

Circle

$$\frac{}{\Psi; \Gamma \vdash \mathsf{base} : \mathbb{S}^1}$$

$$\frac{\Psi \vdash r\ \mathsf{dim}}{\Psi; \Gamma \vdash \mathtt{loop}_r : \mathbb{S}^1}$$

$$\mathtt{loop}_0 \equiv \mathsf{base}$$

$$\mathtt{loop}_1 \equiv \mathsf{base}$$

$$\mathsf{base}\langle r/s \rangle \equiv \mathsf{base}$$

$$\mathtt{loop}_{r'}\langle r/s \rangle \equiv \mathtt{loop}_{r'\langle r/s \rangle}$$

Circle

$$\frac{\begin{array}{l}\Psi;\Gamma, x:\mathbb{S}^1 \vdash A : \texttt{Type} \\ \Psi;\Gamma \vdash b : A[\texttt{base}/x] \\ \Psi, s\ \mathsf{dim};\Gamma \vdash l : A[\texttt{loop}_s/x] \\ \Psi;\Gamma \vdash l\langle 0/s\rangle \equiv b : A[\texttt{base}/x] \\ \Psi;\Gamma \vdash l\langle 1/s\rangle \equiv b : A[\texttt{base}/x] \\ \Psi;\Gamma \vdash u : \mathbb{S}^1\end{array}}{\Psi;\Gamma \vdash \mathbb{S}^{x.A}_{\mathrm{elim}}(b, s.l; u) : A[u/x]}$$

Circle

$$\frac{\begin{array}{l}\Psi;\Gamma,x:\mathbb{S}^1\vdash A:\mathtt{Type}\\ \Psi;\Gamma\vdash b:A[\mathtt{base}/x]\\ \Psi,s\ \mathsf{dim};\Gamma\vdash l:A[\mathtt{loop}_s/x]\\ \Psi;\Gamma\vdash l\langle 0/s\rangle\equiv b:A[\mathtt{base}/x]\\ \Psi;\Gamma\vdash l\langle 1/s\rangle\equiv b:A[\mathtt{base}/x]\\ \Psi;\Gamma\vdash u:\mathbb{S}^1\end{array}}{\Psi;\Gamma\vdash\mathbb{S}^{x.A}_{\mathrm{elim}}(b,s.l;u):A[u/x]}$$

$$\mathbb{S}^{x.A}_{\mathrm{elim}}(b,s.l;\mathtt{base})\equiv b$$

Circle

$$\frac{\begin{array}{l}\Psi;\Gamma, x:\mathbb{S}^1 \vdash A : \texttt{Type}\\ \Psi;\Gamma \vdash b : A[\texttt{base}/x]\\ \Psi, s\ \mathsf{dim};\Gamma \vdash l : A[\texttt{loop}_s/x]\\ \Psi;\Gamma \vdash l\langle 0/s\rangle \equiv b : A[\texttt{base}/x]\\ \Psi;\Gamma \vdash l\langle 1/s\rangle \equiv b : A[\texttt{base}/x]\\ \Psi;\Gamma \vdash u : \mathbb{S}^1\end{array}}{\Psi;\Gamma \vdash \mathbb{S}^{x.A}_{\mathrm{elim}}(b, s.l; u) : A[u/x]}$$

$$\mathbb{S}^{x.A}_{\mathrm{elim}}(b, s.l; \texttt{base}) \equiv b$$

$$\mathbb{S}^{x.A}_{\mathrm{elim}}(b, s.l; \texttt{loop}_{s'}) \equiv l\langle s'/s\rangle$$

Ingredients

- Cubical operations
- Higher-dimensional substitution
- Types
- **Kan filling**

Kan filling

$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \ \textsf{dim} \text{ and are distinct} \\ \Psi \vdash r \ \textsf{dim} \text{ and } r \neq s \\ \Psi; \Gamma \vdash A : \textsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

Kan filling

$$\dfrac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j \mid i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

tube dimensions

Kan filling

$$\frac{\begin{array}{l} \Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^\varepsilon : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle \end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r \to \bullet}(b) : A}$$

tube dimensions

filling dimension

Kan filling

$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi;\Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i;\Gamma\langle\varepsilon/s_i\rangle \vdash t_i^{\varepsilon} : A\langle\varepsilon/s_i\rangle \\ \Psi - s;\Gamma\langle\varepsilon/r\rangle \vdash b : A\langle r/s\rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle\varepsilon/s_j\rangle \equiv t_j^{\varepsilon'}\langle\varepsilon/s_i\rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle\varepsilon/s_i\rangle/s\rangle \equiv b\langle\varepsilon/s_i\rangle\end{array}}{\Psi;\Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

tube dimensions

transverse face you have

filling dimension

Kan filling

sides of of tube

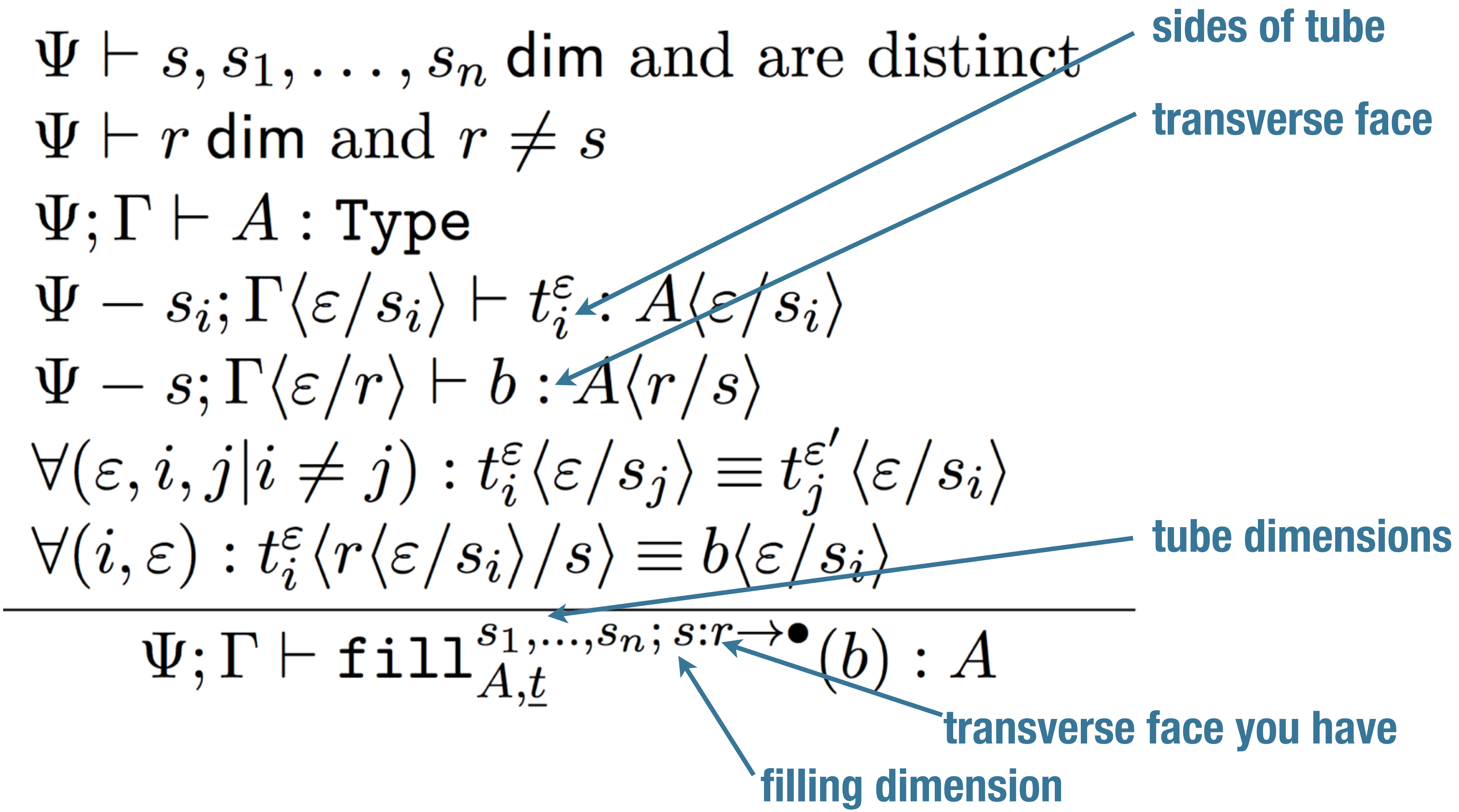
$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r \to \bullet}(b) : A}$$

tube dimensions

transverse face you have

filling dimension

Kan filling



$$\frac{\begin{array}{l} \Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \text{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle \end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r \to \bullet}(b) : A}$$

sides of tube

transverse face

tube dimensions

transverse face you have

filling dimension

Kan filling

$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \ \mathsf{dim\ and\ are\ distinct} \\ \Psi \vdash r \ \mathsf{dim\ and}\ r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle\varepsilon/s_i\rangle \vdash t_i^{\varepsilon} : A\langle\varepsilon/s_i\rangle \\ \Psi - s; \Gamma\langle\varepsilon/r\rangle \vdash b : A\langle r/s\rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle\varepsilon/s_j\rangle \equiv t_j^{\varepsilon'}\langle\varepsilon/s_i\rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle\varepsilon/s_i\rangle/s\rangle \equiv b\langle\varepsilon/s_i\rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

sides of tube

transverse face

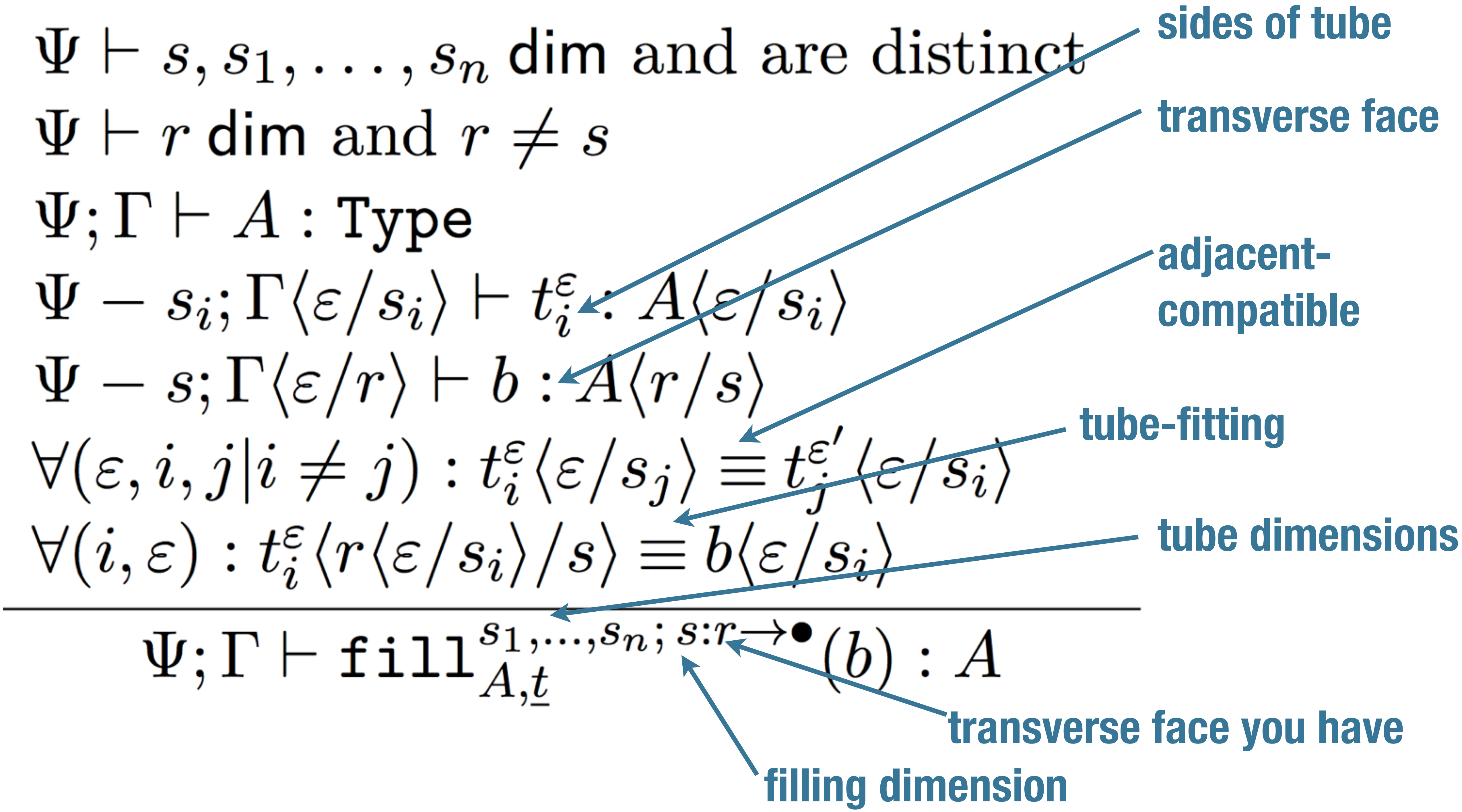
adjacent-compatible

tube dimensions

transverse face you have

filling dimension

Kan filling



$$\frac{\begin{array}{l} \Psi \vdash s, s_1, \ldots, s_n \ \mathsf{dim}\ \text{and are distinct} \\ \Psi \vdash r\ \mathsf{dim}\ \text{and}\ r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle \end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

sides of tube

transverse face

adjacent-compatible

tube-fitting

tube dimensions

transverse face you have

filling dimension

Kan filling

$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathtt{Type} \\ \Psi - s_i; \Gamma\langle\varepsilon/s_i\rangle \vdash t_i^\varepsilon : A\langle\varepsilon/s_i\rangle \\ \Psi - s; \Gamma\langle\varepsilon/r\rangle \vdash b : A\langle r/s\rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon\langle\varepsilon/s_j\rangle \equiv t_j^{\varepsilon'}\langle\varepsilon/s_i\rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle\varepsilon/s_i\rangle/s\rangle \equiv b\langle\varepsilon/s_i\rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r \to \bullet}(b) : A}$$

b

tube: empty
filling dimension: x
transverse side: r = 0

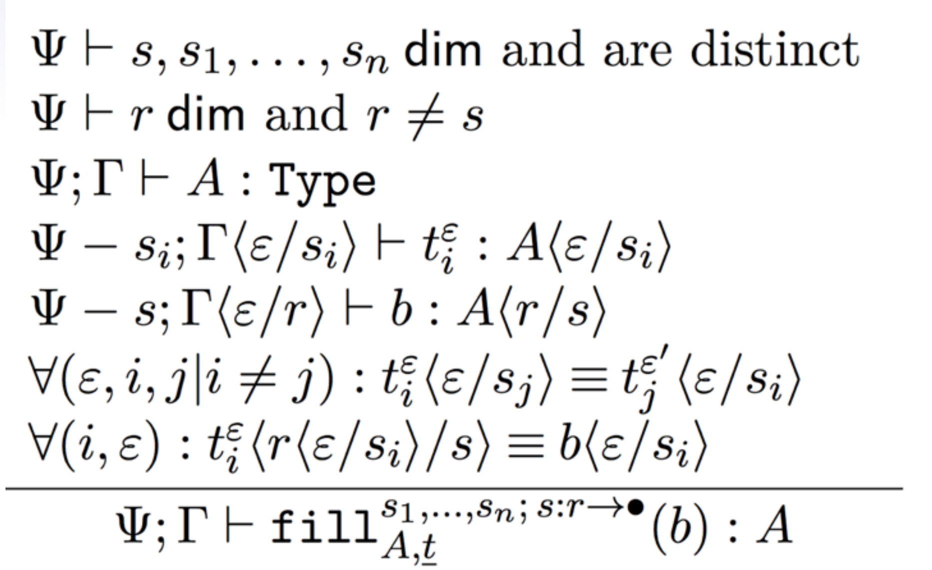
Kan filling

$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathtt{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

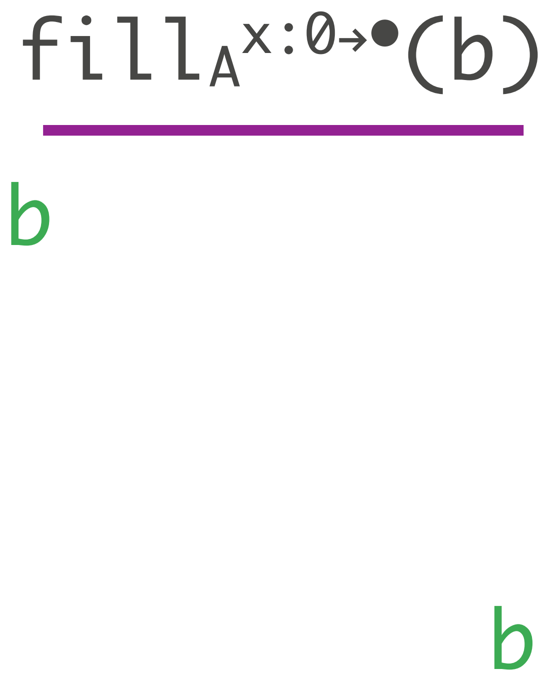
$$\frac{\mathtt{fill}_A^{x:0\to\bullet}(b)}{b}$$

tube: empty
filling dimension: x
transverse side: r = 0

Kan filling



$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \texttt{Type} \\ \Psi - s_i; \Gamma\langle\varepsilon/s_i\rangle \vdash t_i^\varepsilon : A\langle\varepsilon/s_i\rangle \\ \Psi - s; \Gamma\langle\varepsilon/r\rangle \vdash b : A\langle r/s\rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon\langle\varepsilon/s_j\rangle \equiv t_j^{\varepsilon'}\langle\varepsilon/s_i\rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle\varepsilon/s_i\rangle/s\rangle \equiv b\langle\varepsilon/s_i\rangle\end{array}}{\Psi; \Gamma \vdash \texttt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$



$\texttt{fill}_A^{x:0\to\bullet}(b)$

tube: empty
filling dimension: x
transverse side: r = 0

tube: empty
filling dimension: x
transverse side: r = 1

Kan filling

$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle \end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A, \underline{t}}^{s_1, \ldots, s_n; \, s : r \to \bullet}(b) : A}$$

$\mathtt{fill}_A^{x:0\to\bullet}(\mathtt{b})$

tube: empty
filling dimension: x
transverse side: r = 0

$\mathtt{fill}_A^{x:1\to\bullet}(\mathtt{b})$

tube: empty
filling dimension: x
transverse side: r = 1

Kan filling

$$\frac{\begin{array}{l} \Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle \end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

tube dimension: y
filling dimension: x
transverse side: 0

$\mathtt{fill}_{\mathtt{A;ty1,ty0}}^{\mathtt{y;x:0\to\bullet}}(\mathtt{b})$

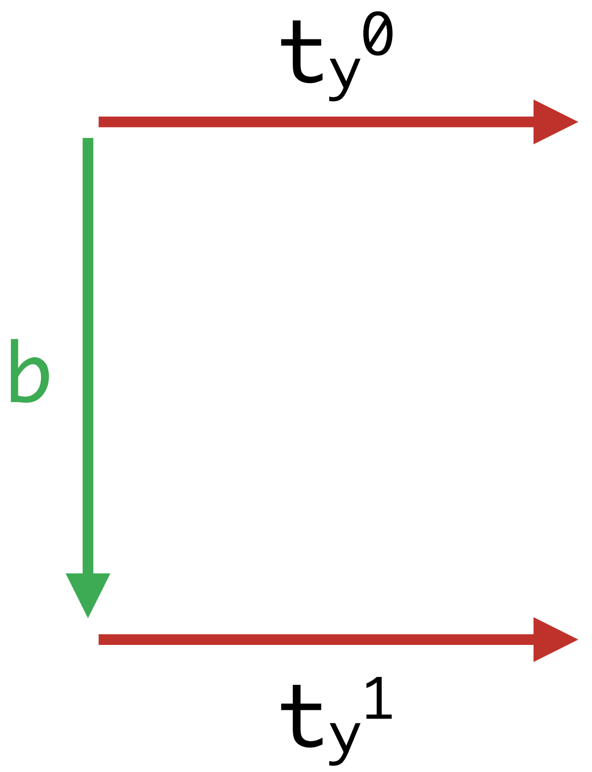
Kan filling

$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi ; \Gamma \vdash A : \mathsf{Type} \\ \boxed{\Psi - s_i ; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^\varepsilon : A\langle \varepsilon / s_i \rangle} \\ \Psi - s ; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon \langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'} \langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon \langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle \end{array}}{\Psi ; \Gamma \vdash \mathtt{fill}_{A, \underline{t}}^{s_1, \ldots, s_n ;\, s : r \to \bullet}(b) : A}$$

tube dimension: y
filling dimension: x
transverse side: 0

$\mathtt{fill}_{\mathtt{A;ty1,ty0}}^{\mathtt{y;x:0\to\bullet}}(\mathtt{b})$

Kan filling

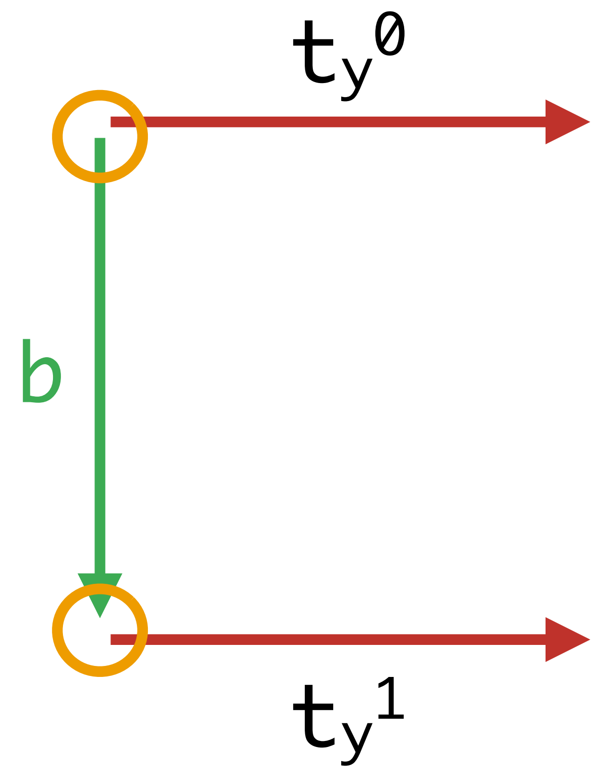


$$
\frac{
\begin{array}{l}
\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\
\Psi \vdash r \text{ dim and } r \neq s \\
\Psi; \Gamma \vdash A : \mathsf{Type} \\
\Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^\varepsilon : A\langle \varepsilon / s_i \rangle \\
\Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\
\forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\
\forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle
\end{array}
}{
\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r \to \bullet}(b) : A
}
$$

tube dimension: y
filling dimension: x
transverse side: 0

$\mathtt{fill}_{\mathtt{A;ty1,ty0}}^{\mathtt{y;x:0} \to \bullet}(\mathtt{b})$

Kan filling

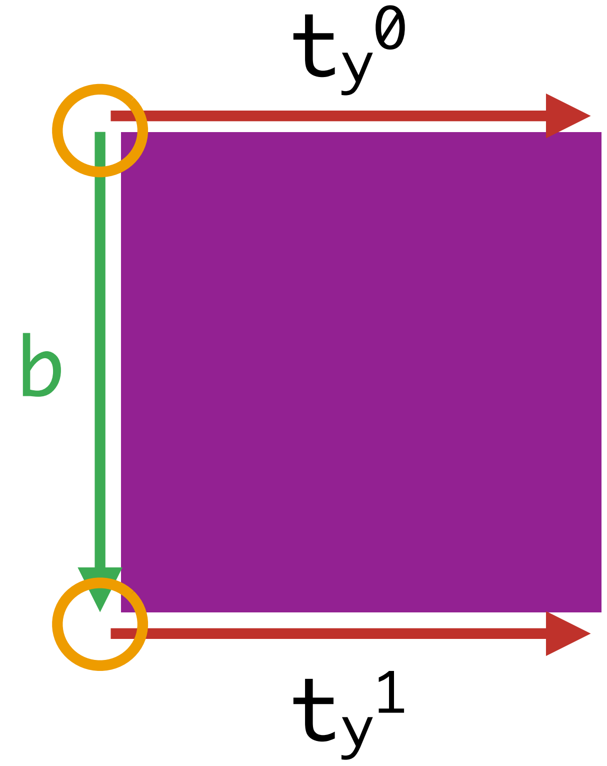


$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \ \mathsf{dim} \text{ and are distinct} \\ \Psi \vdash r \ \mathsf{dim} \text{ and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle\varepsilon/s_i\rangle \vdash t_i^\varepsilon : A\langle\varepsilon/s_i\rangle \\ \Psi - s; \Gamma\langle\varepsilon/r\rangle \vdash b : A\langle r/s\rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon\langle\varepsilon/s_j\rangle \equiv t_j^{\varepsilon'}\langle\varepsilon/s_i\rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle\varepsilon/s_i\rangle/s\rangle \equiv b\langle\varepsilon/s_i\rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

tube dimension: y
filling dimension: x
transverse side: 0

$\mathtt{fill}_{A;ty1,ty0}^{y;x:0\to\bullet}(b)$

Kan filling

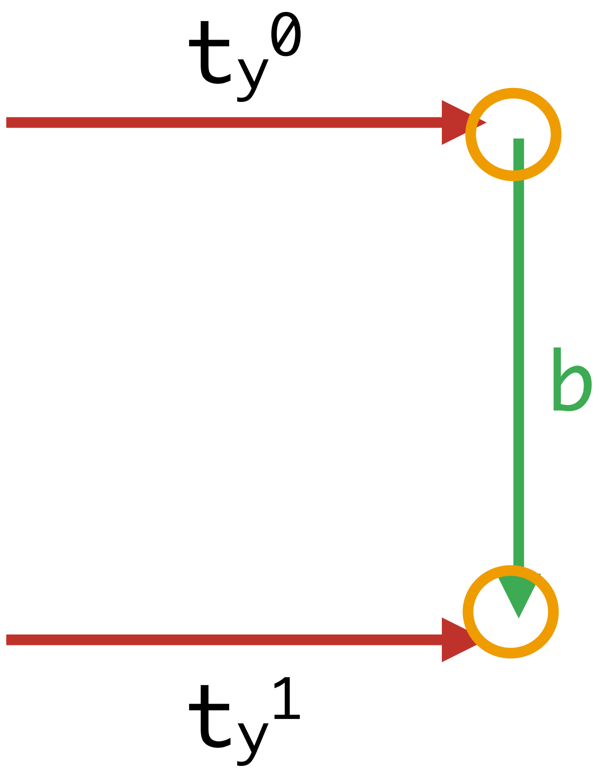


$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \textsf{ dim and are distinct} \\ \Psi \vdash r \textsf{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \textsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle \end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A, \underline{t}}^{s_1, \ldots, s_n;\, s : r \to \bullet}(b) : A}$$

tube dimension: y
filling dimension: x
transverse side: 0

$\mathtt{fill}_{A;ty1,ty0}^{y;x:0\to\bullet}(b)$

Kan filling

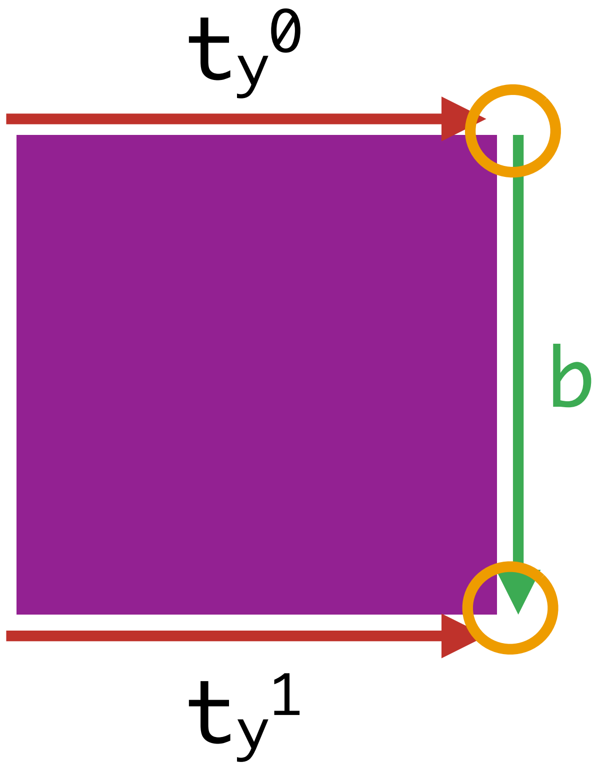


$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^\varepsilon : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

tube dimension: y
filling dimension: x
transverse side: 1

$\mathtt{fill}_{\mathtt{A;ty1,ty0}}{}^{\mathtt{y;x:1}\to\bullet}(\mathtt{b})$

Kan filling

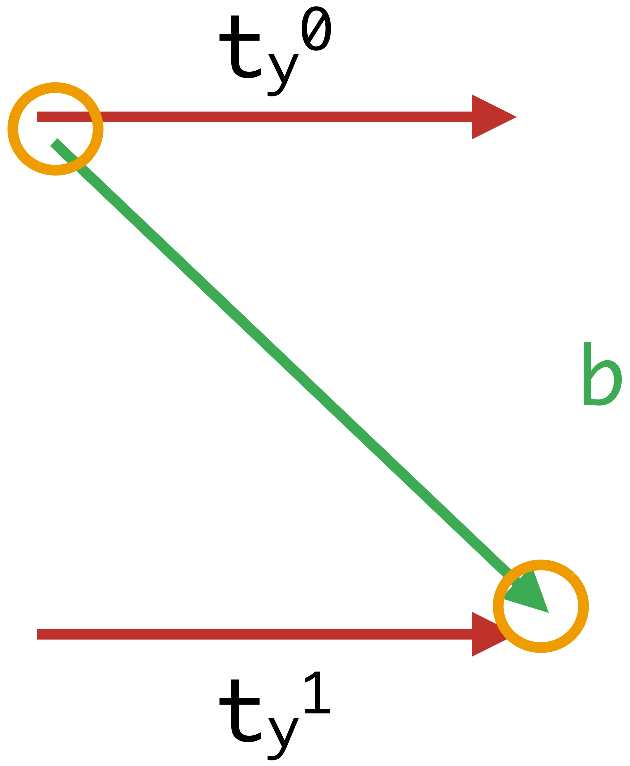


$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^\varepsilon : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

tube dimension: y
filling dimension: x
transverse side: 1

$\mathtt{fill}_{A;ty1,ty0}^{y;x:1\to\bullet}(b)$

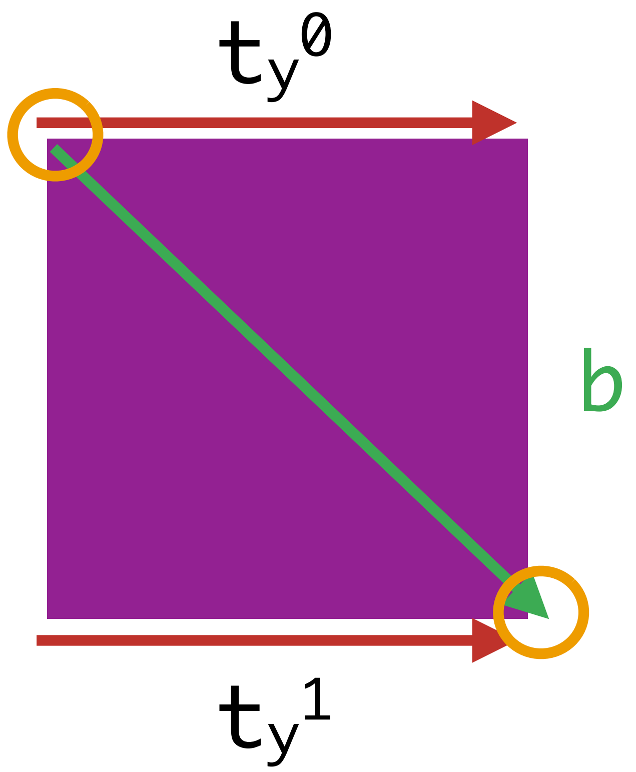
Kan filling



$$\frac{\begin{array}{l} \Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi;\Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle\varepsilon/s_i\rangle \vdash t_i^\varepsilon : A\langle\varepsilon/s_i\rangle \\ \Psi - s; \Gamma\langle\varepsilon/r\rangle \vdash b : A\langle r/s\rangle \\ \forall(\varepsilon, i, j|i \neq j) : t_i^\varepsilon\langle\varepsilon/s_j\rangle \equiv t_j^{\varepsilon'}\langle\varepsilon/s_i\rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle\varepsilon/s_i\rangle/s\rangle \equiv b\langle\varepsilon/s_i\rangle \end{array}}{\Psi;\Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

tube dimension: y
filling dimension: x
transverse side: y

Kan filling



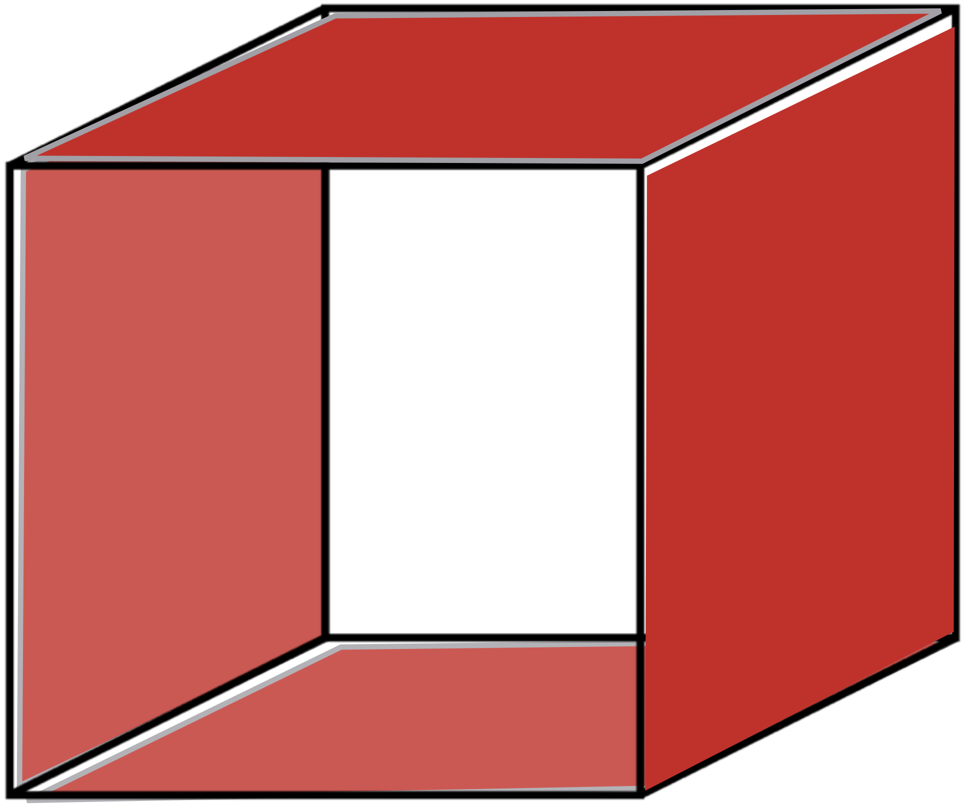
$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle\varepsilon/s_i\rangle \vdash t_i^\varepsilon : A\langle\varepsilon/s_i\rangle \\ \Psi - s; \Gamma\langle\varepsilon/r\rangle \vdash b : A\langle r/s\rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon\langle\varepsilon/s_j\rangle \equiv t_j^{\varepsilon'}\langle\varepsilon/s_i\rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle\varepsilon/s_i\rangle/s\rangle \equiv b\langle\varepsilon/s_i\rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

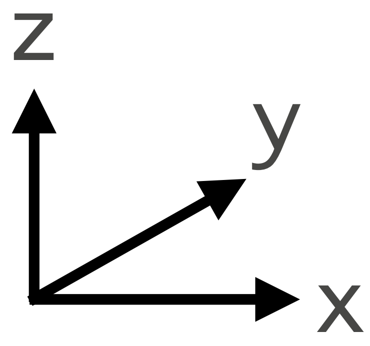
tube dimension: y
filling dimension: x
transverse side: y

$\mathtt{fill}_{A;ty1,ty0}^{y;x:y\to\bullet}(b)$

Kan filling

$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \ \textsf{dim} \text{ and are distinct} \\ \Psi \vdash r \ \textsf{dim} \text{ and } r \neq s \\ \Psi; \Gamma \vdash A : \texttt{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle\end{array}}{\Psi; \Gamma \vdash \texttt{fill}_{A,\underline{t}}^{s_1, \ldots, s_n;\, s:r \to \bullet}(b) : A}$$

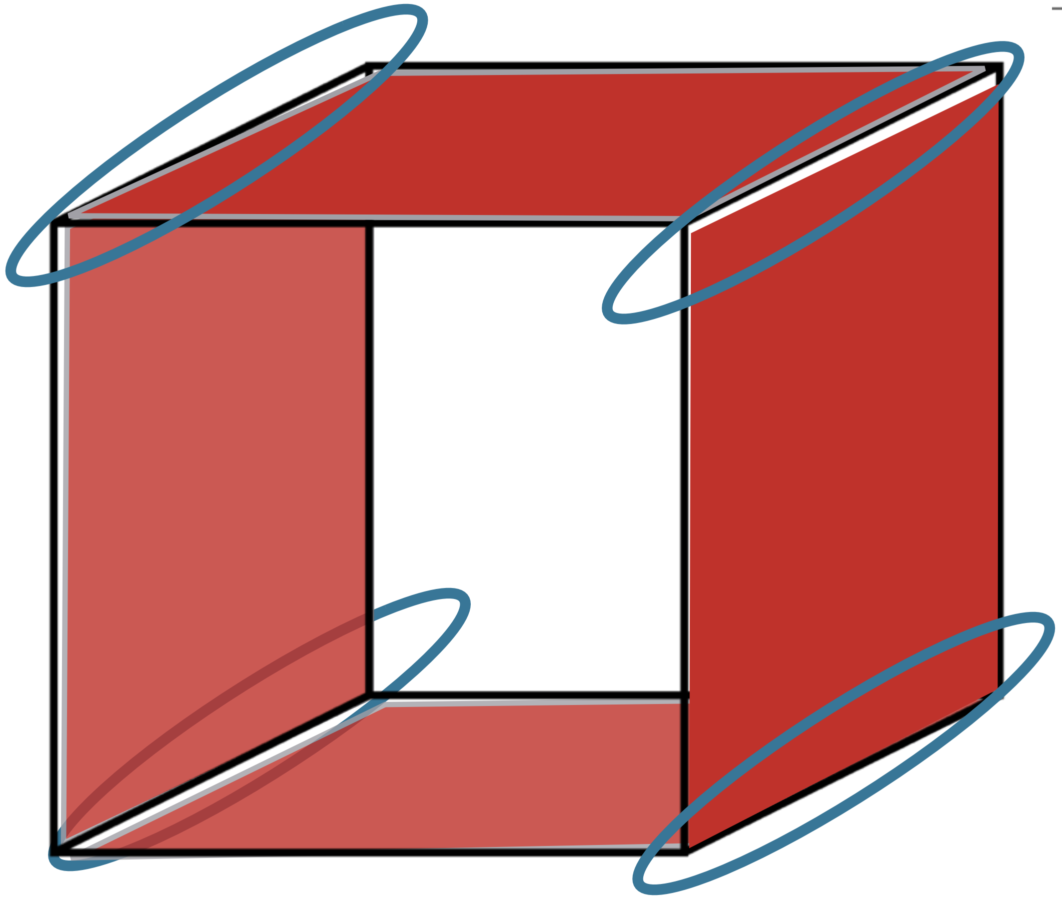


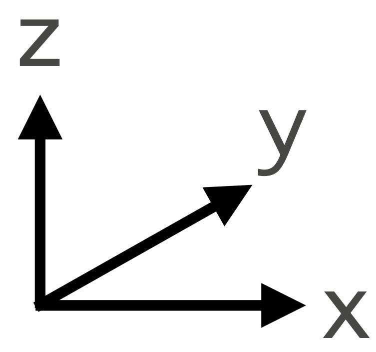


tube dimensions: x,z
filling dimension: y
transverse side: 1

Kan filling

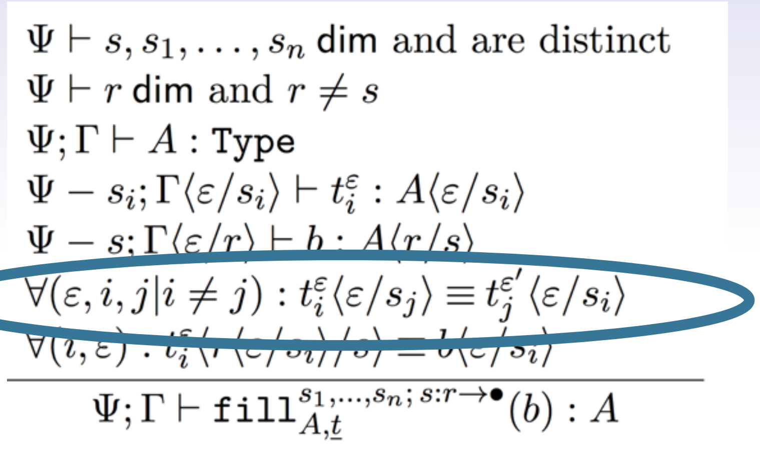
$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^{\varepsilon} : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^{\varepsilon}\langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'}\langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^{\varepsilon}\langle r\langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A, \underline{t}}^{s_1, \ldots, s_n;\, s : r \to \bullet}(b) : A}$$



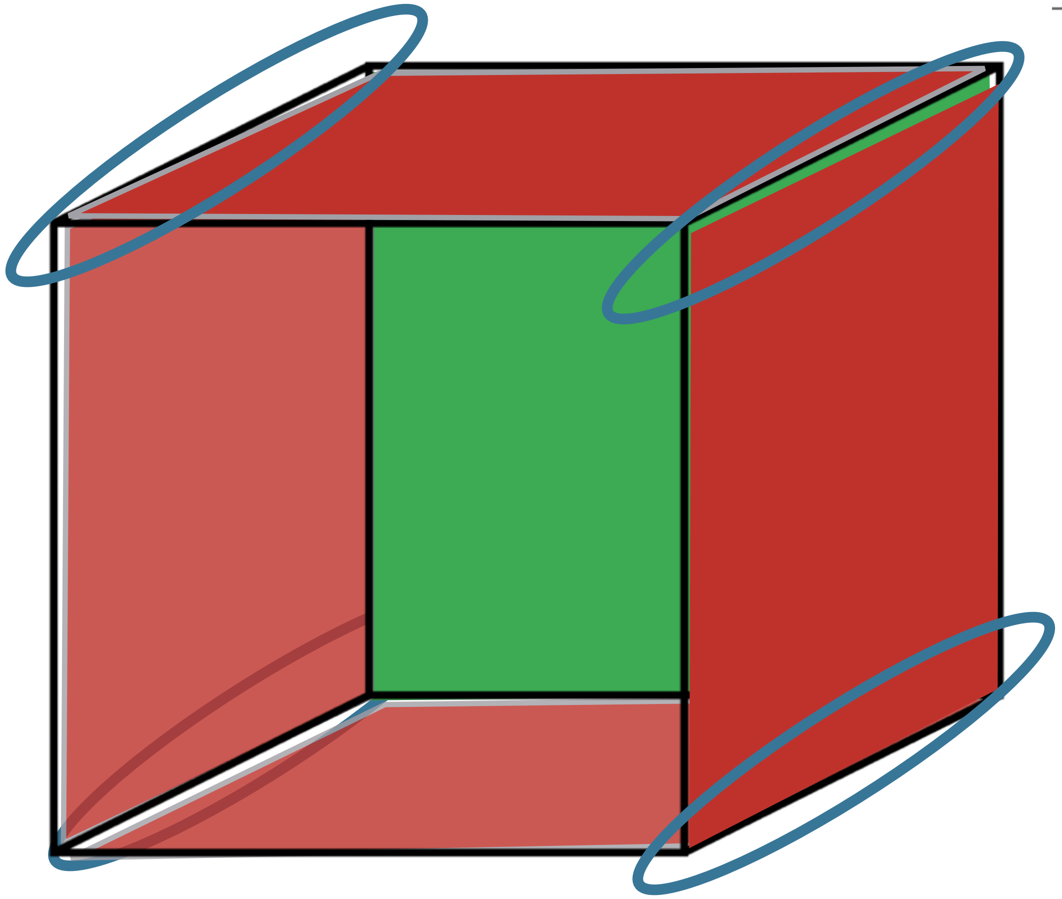


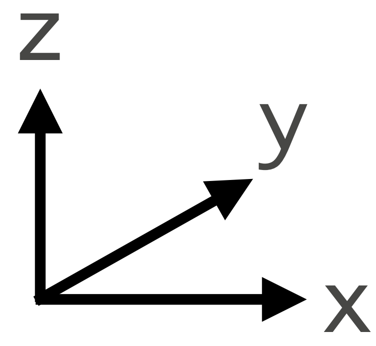
tube dimensions: x,z
filling dimension: y
transverse side: 1

Kan filling



$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathsf{Type} \\ \Psi - s_i; \Gamma\langle \varepsilon / s_i \rangle \vdash t_i^\varepsilon : A\langle \varepsilon / s_i \rangle \\ \Psi - s; \Gamma\langle \varepsilon / r \rangle \vdash b : A\langle r / s \rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon \langle \varepsilon / s_j \rangle \equiv t_j^{\varepsilon'} \langle \varepsilon / s_i \rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon \langle r \langle \varepsilon / s_i \rangle / s \rangle \equiv b\langle \varepsilon / s_i \rangle \end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$

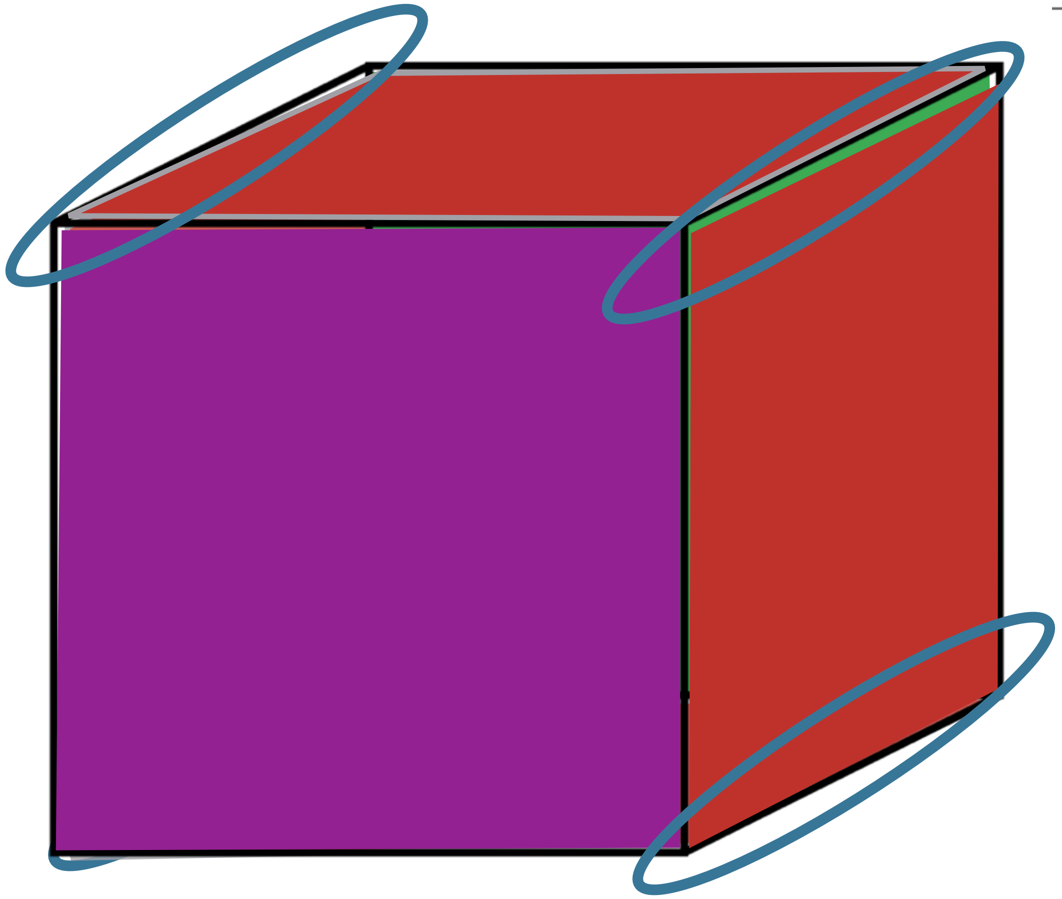


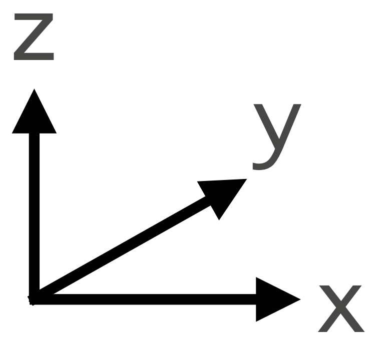


tube dimensions: x,z
filling dimension: y
transverse side: 1

Kan filling

$$\frac{\begin{array}{l}\Psi \vdash s, s_1, \ldots, s_n \text{ dim and are distinct} \\ \Psi \vdash r \text{ dim and } r \neq s \\ \Psi; \Gamma \vdash A : \mathtt{Type} \\ \Psi - s_i; \Gamma\langle\varepsilon/s_i\rangle \vdash t_i^\varepsilon : A\langle\varepsilon/s_i\rangle \\ \Psi - s; \Gamma\langle\varepsilon/r\rangle \vdash b : A\langle r/s\rangle \\ \forall(\varepsilon, i, j | i \neq j) : t_i^\varepsilon\langle\varepsilon/s_j\rangle \equiv t_j^{\varepsilon'}\langle\varepsilon/s_i\rangle \\ \forall(i, \varepsilon) : t_i^\varepsilon\langle r\langle\varepsilon/s_i\rangle/s\rangle \equiv b\langle\varepsilon/s_i\rangle\end{array}}{\Psi; \Gamma \vdash \mathtt{fill}_{A,\underline{t}}^{s_1,\ldots,s_n;\, s:r\to\bullet}(b) : A}$$



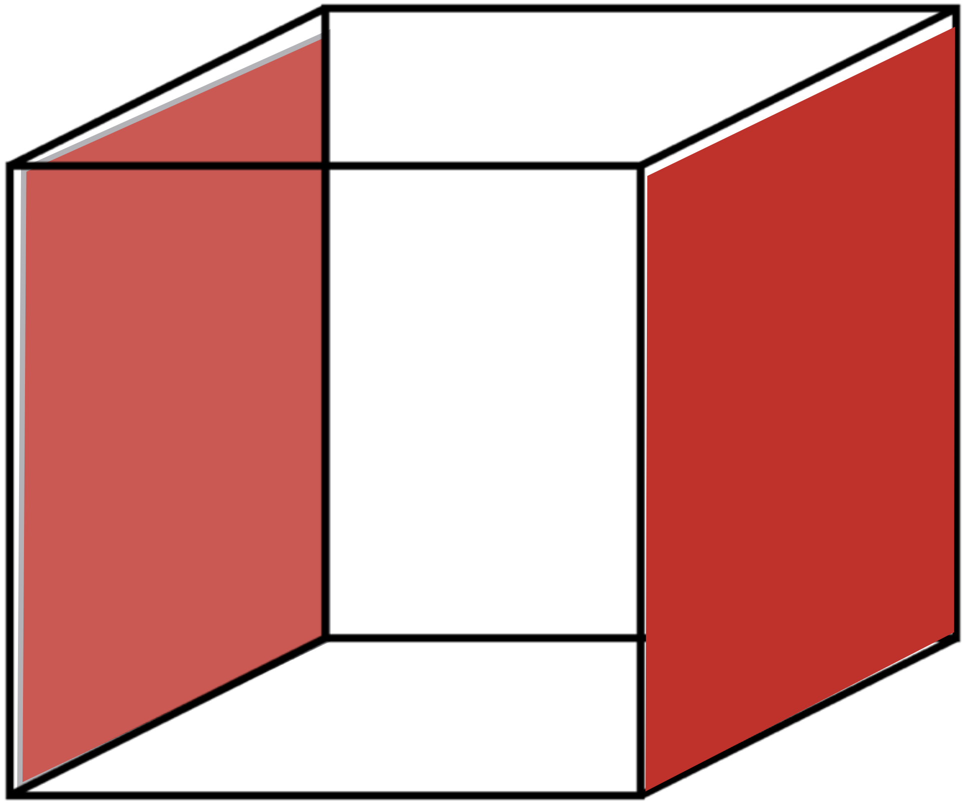


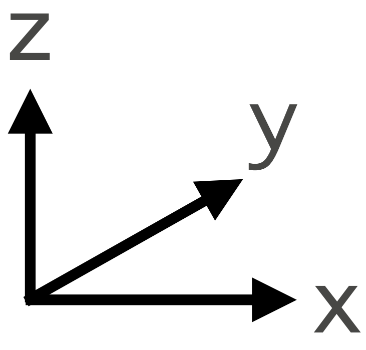
tube dimensions: x,z

filling dimension: y

transverse side: 1

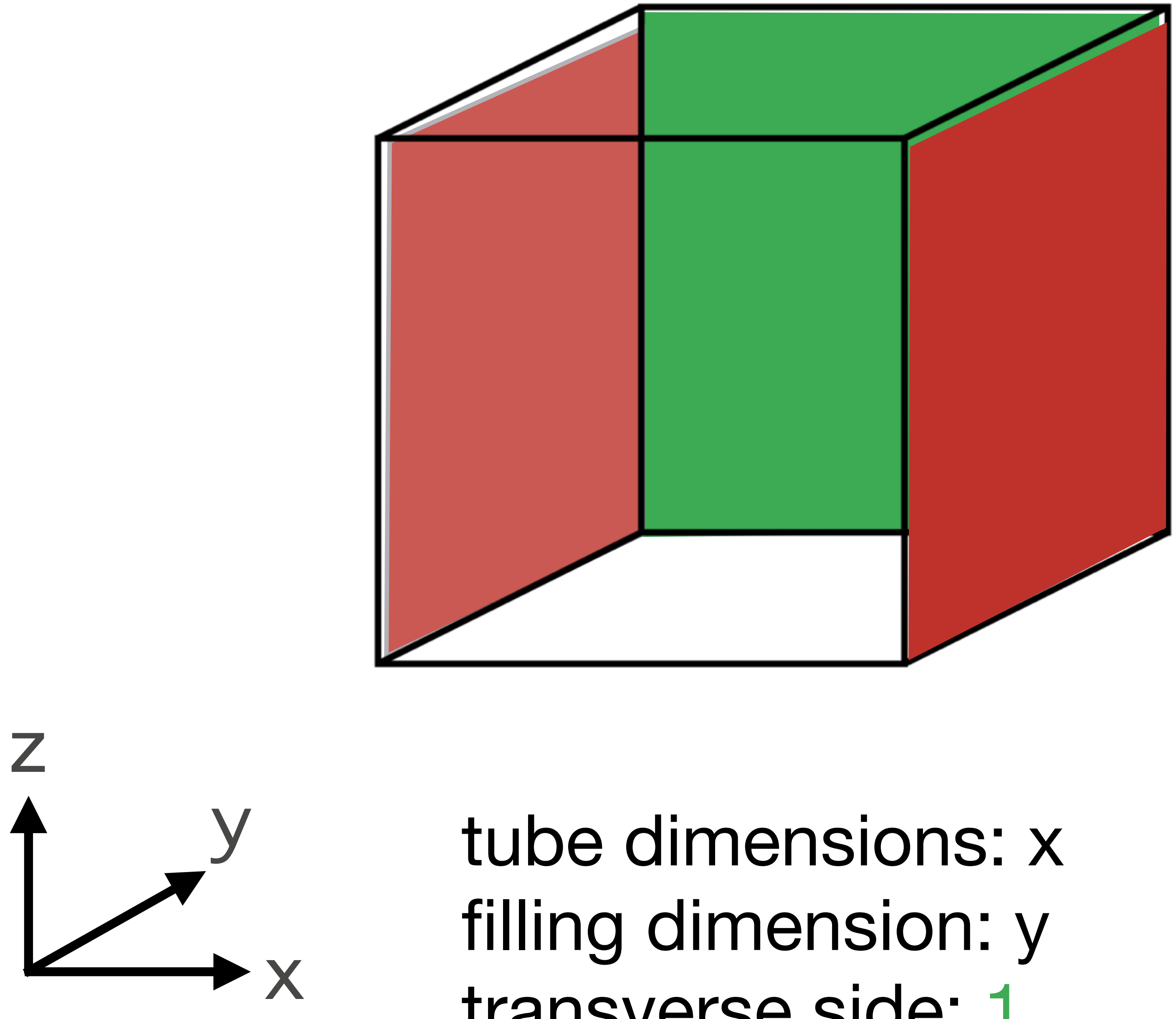
Generalized open boxes





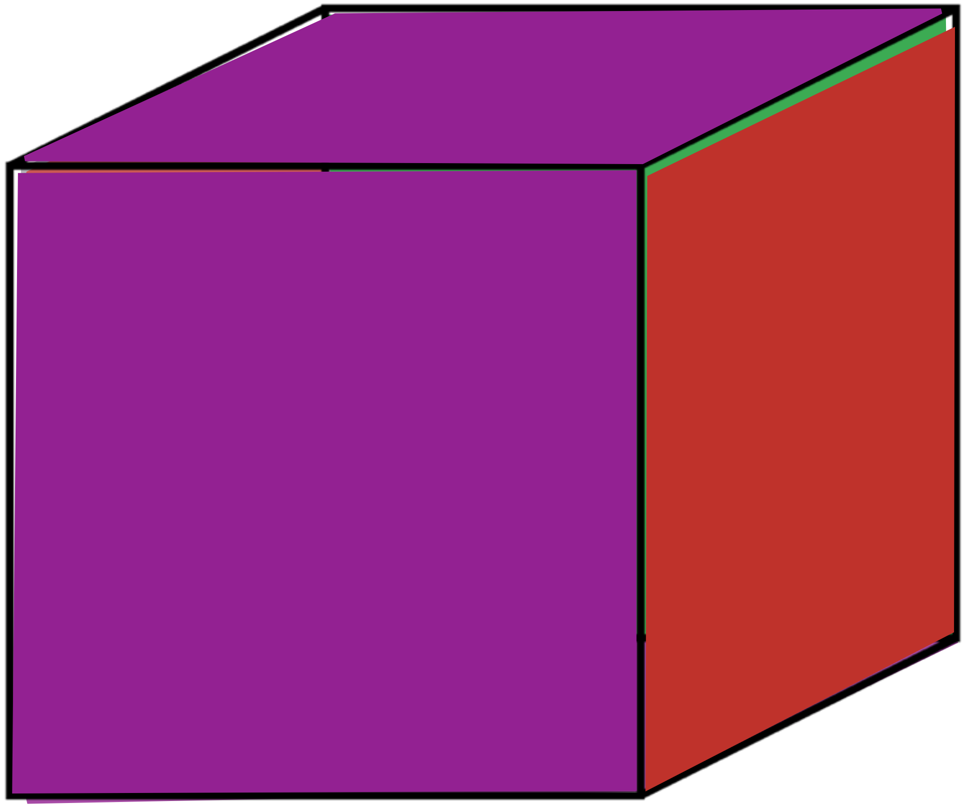
tube dimensions: x
filling dimension: y
transverse side: 1

Generalized open boxes



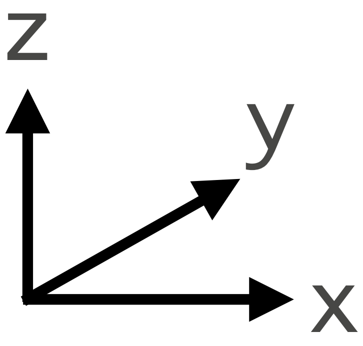
tube dimensions: x
filling dimension: y
transverse side: 1

Generalized open boxes



$$\mathtt{fill}_{A;t}{}^{x;y:1\to\bullet}(b)$$

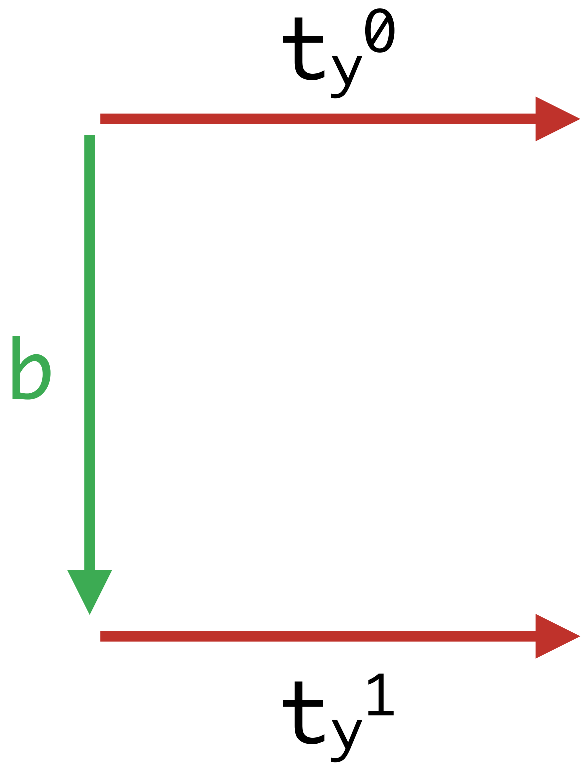
$$:^{x,y,z}\ A$$



tube dimensions: x
filling dimension: y
transverse side: 1

Kan composition

Filling “comes with” missing faces/diagonals:

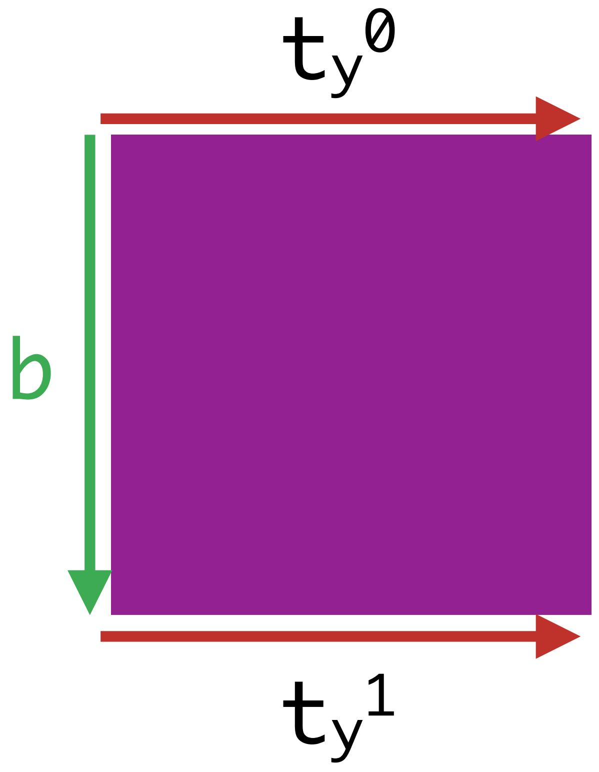


tube dimension: y
filling dimension: x
transverse side: 0

Kan composition

Filling “comes with” missing faces/diagonals:

$\mathtt{fill}_{A;ty1,ty0}^{y;x:y\to\bullet}(b)$

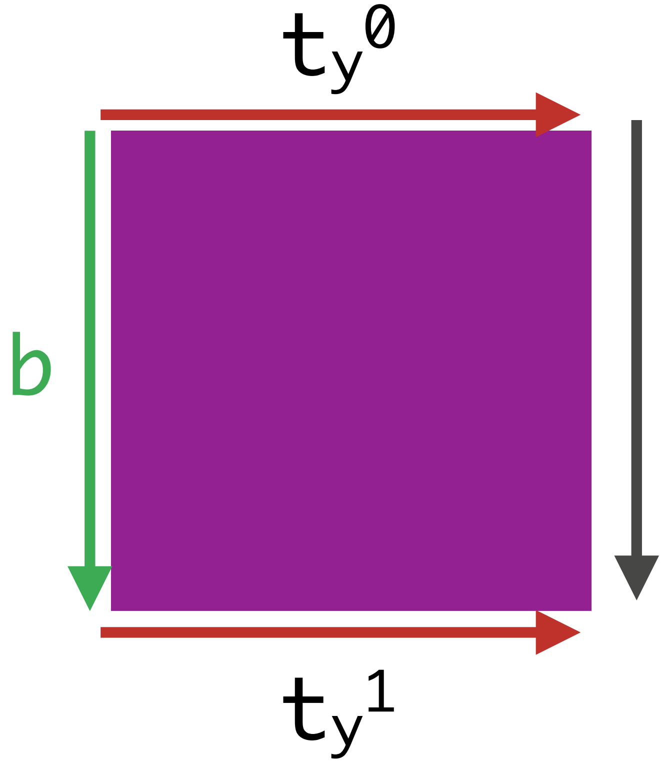


tube dimension: y
filling dimension: x
transverse side: 0

Kan composition

Filling “comes with” missing faces/diagonals:

$\mathtt{fill}_{A;\mathtt{ty1},\mathtt{ty0}}^{y;x:y\to\bullet}(\mathtt{b})$



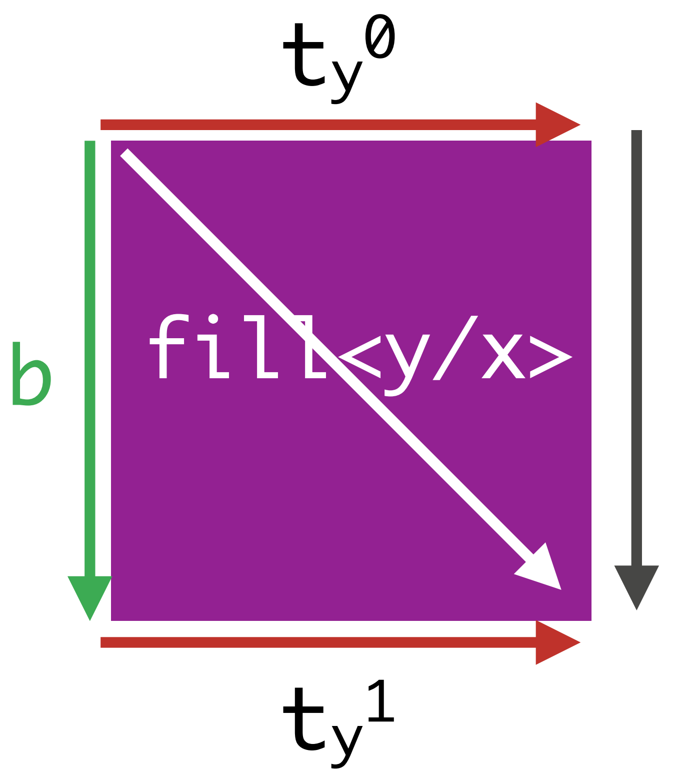
$\mathtt{fill}_{A;\mathtt{ty1},\mathtt{ty0}}^{y;x:y\to\bullet}(\mathtt{b})\ \texttt{<1/x>}$

tube dimension: y
filling dimension: x
transverse side: 0

Kan composition

Filling “comes with” missing faces/diagonals:

$\mathtt{fill}_{A;ty1,ty0}^{y;x:y\to\bullet}(b)$



t_y^0

b

$\mathtt{fill}\langle y/x\rangle$

t_y^1

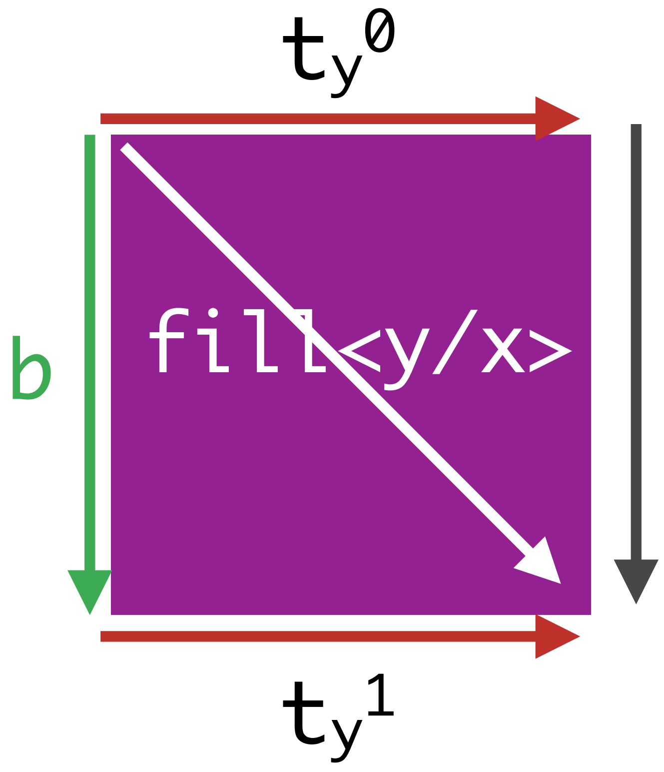
$\mathtt{fill}_{A;ty1,ty0}^{y;x:y\to\bullet}(b)\ \langle 1/x\rangle$

tube dimension: y
filling dimension: x
transverse side: 0

Kan composition

Filling “comes with” missing faces/diagonals:

$$\mathtt{fill}_{A;ty1,ty0}^{y;x:y\to\bullet}(b)$$



$$\overbrace{\mathtt{fill}_{A;ty1,ty0}^{y;x:y\to\bullet}(b)\ \langle 1/x\rangle}^{\mathtt{coe}_{A;ty1,ty0}^{y;x:y\to 1}(b)}$$

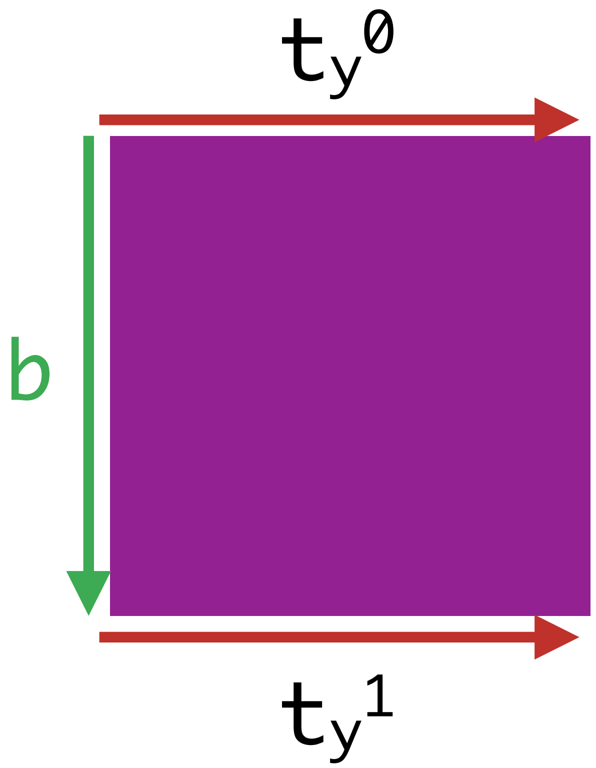
tube dimension: y
filling dimension: x
transverse side: 0

Boundary equations

$$(\mathtt{fill}^{s_1,\ldots,s_n;\; s:r\to\bullet}_{A,\underline{t}}(b))\langle\varepsilon/s_i\rangle \equiv t_i^\varepsilon$$

$$(\mathtt{fill}^{s_1,\ldots,s_n;\; s:r\to\bullet}_{A,\underline{t}}(b))\langle r/s\rangle \equiv b$$

$$(\mathtt{fill}^{s_1,\ldots,s_n;\; s:r\to\bullet}_{A,\underline{t}}(b))\langle r'/s\rangle \equiv \mathtt{coe}^{s_1,\ldots,s_n;\; s:r\to r'}_{A,\underline{t}}(b)$$

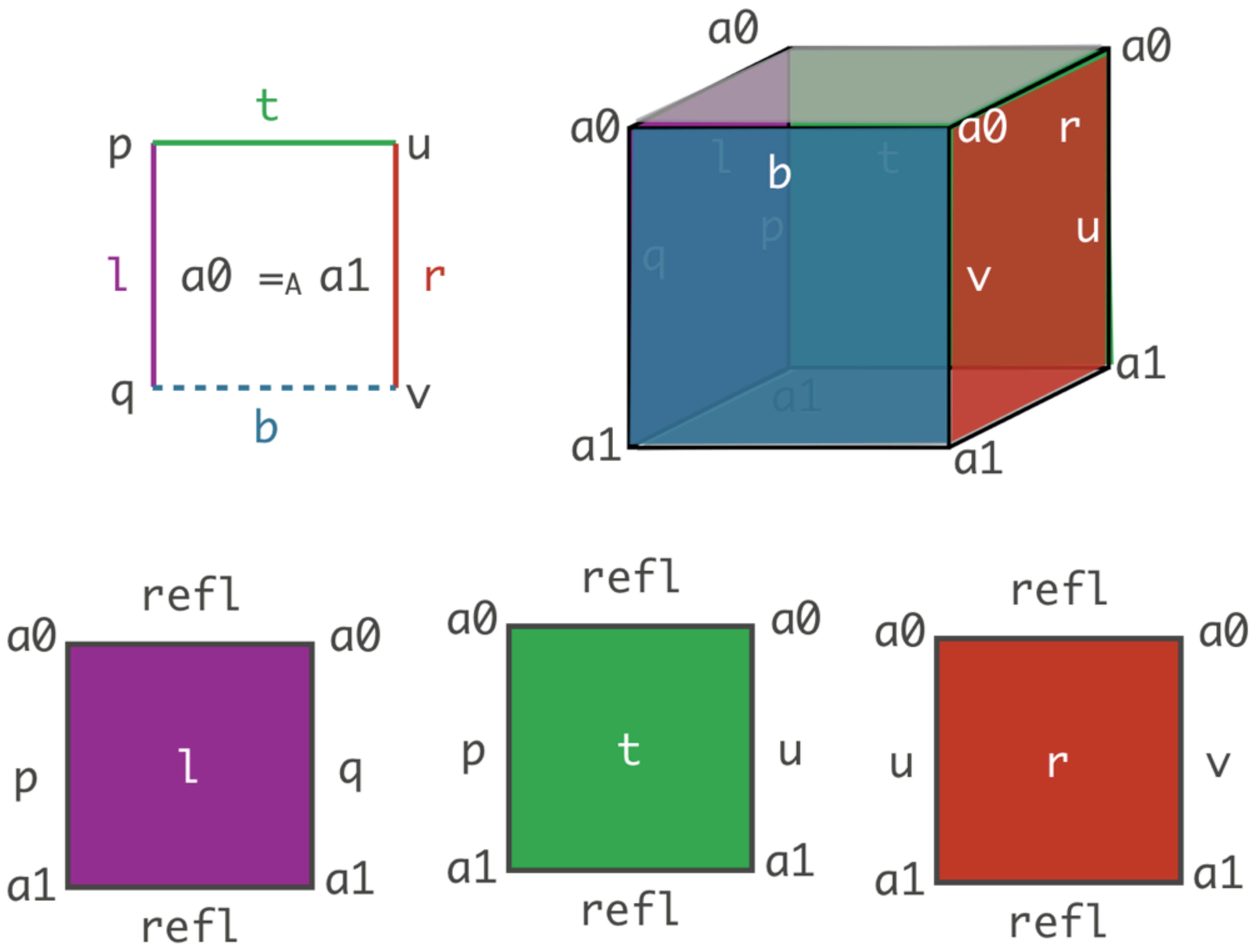


Uniformity

$$(\mathtt{fill}^{s_1,\ldots,s_n;\, s:r\to\bullet}_{A,\underline{t}}(b))\langle r_0/s_0\rangle \equiv \mathtt{fill}^{s_1,\ldots,s_n;\, s:r\to\bullet}_{A\langle r_0/s_0\rangle,\underline{t}\langle r_0/s_0\rangle}(b\langle r_0/s_0\rangle)$$

(something missing for substitutions involving tube/filling dimensions)

Composition in Id



$$\mathtt{coe}^{s_1,\ldots,s_n;\, s:r_s\to r_t}_{\mathtt{Id}^{[s_{n+1}]}_A(a_0,a_1),\underline{t}}(b) \equiv \langle s_{n+1}\rangle \mathtt{coe}^{s_1,\ldots,s_n,s_{n+1};\, s:r_s\to r_t}_{A,(\underline{t_i^\varepsilon @ s_{n+1}},a_0,a_1)}(b@s_{n+1})$$

Composition in Π

$$\mathsf{coe}^{s_1,\ldots,s_n;\, s:r_s\to r_t}_{\Pi_{x:A}B,\underline{t}}(b) \equiv$$

$$\lambda a.\mathsf{coe}^{s_1,\ldots,s_n;\, s:r_s\to r_t}_{B[\mathtt{fill}^{s:r_t\to\bullet}_{A}(a)/x],\mathbf{app}(t_i^{\varepsilon},\mathtt{fill}^{s:r_t\langle\varepsilon/s_i\rangle\to\bullet}_{A\langle\varepsilon/s_i\rangle}(a))}(\mathbf{app}(b, \mathsf{coe}^{s:r_t\to r_s}_{A}(a)))$$

Homogeneous Id type

$$\mathtt{refl}_A(a) := \langle s \rangle a$$

$$\mathtt{transport}_B(p; u) := \mathtt{coe}^{s:0\to 1}_{\mathsf{app}(B, p@s)}(b)$$

singleton contractibility:

$$\lambda x, p. \langle s_1 \rangle (\mathtt{coe}^{s_1; s_2:0\to 1}_{A,(a,p@s_2)}(a), \langle s_2 \rangle \mathtt{fill}^{s_1; s_2:0\to\bullet}_{A,(a,p@s_2)}(a))$$

The hard part

$$\mathtt{fill}^{s_1,\ldots,s_n;\, s:r_s\to\bullet}_{\Pi\, x:A.\, B,\underline{t}}(b)$$

$$\mathtt{ua}_s(A, B, (u, \ldots))$$

$$\mathtt{fill}^{s_1,\ldots,s_n;\, s:r\to\bullet}_{\mathtt{ua}_s(A,B,u),\underline{t}}(b)$$

$$\mathtt{fill}^{s_1,\ldots,s_n;\, s:r\to\bullet}_{\mathtt{Type},\underline{t}}(b)$$

Thanks!