Eilenberg-MacLane Spaces in Homotopy Type Theory

Dan Licata

Wesleyan University

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Inria Paris Rocquencourt

Constructive Type Theory

Three senses of constructivity:

Constructive Type Theory

Three senses of constructivity:

- Non-affirmation of certain classical principles provides **axiomatic freedom**

Synthetic geometry

Euclid's postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. Given a line and a point not on it, there is exactly one line through the point that does not intersect the line

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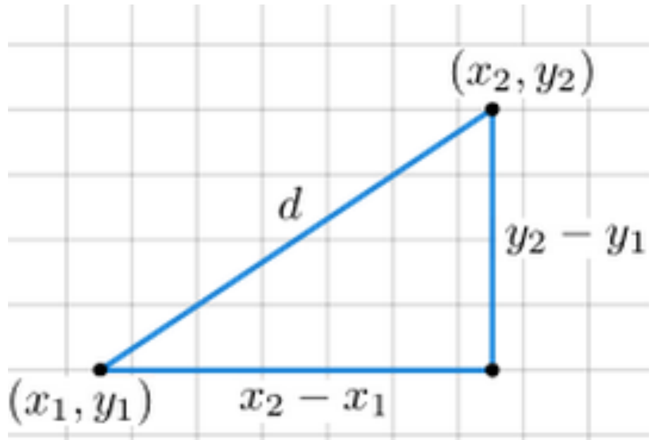
Cartesian



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models

←

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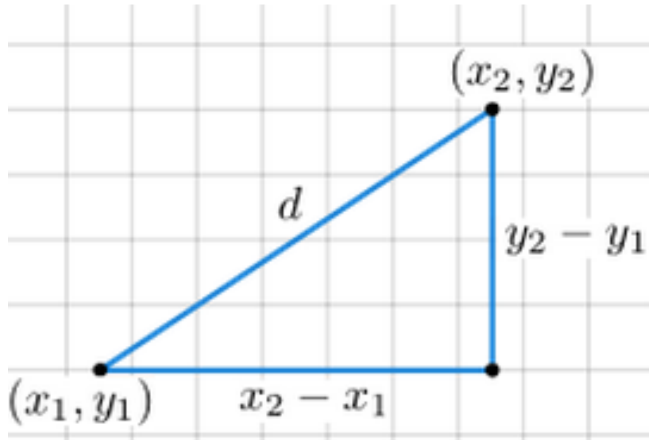

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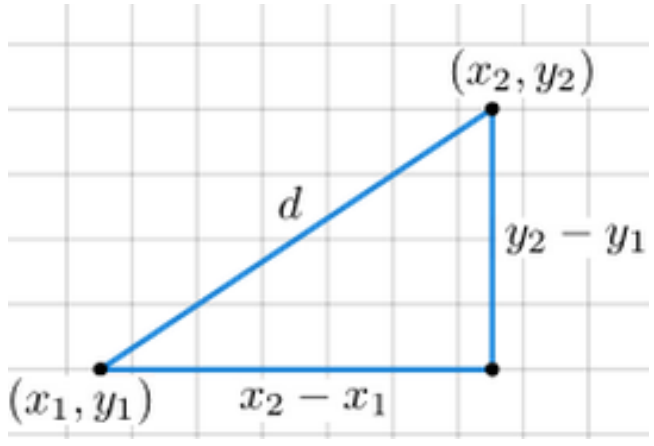

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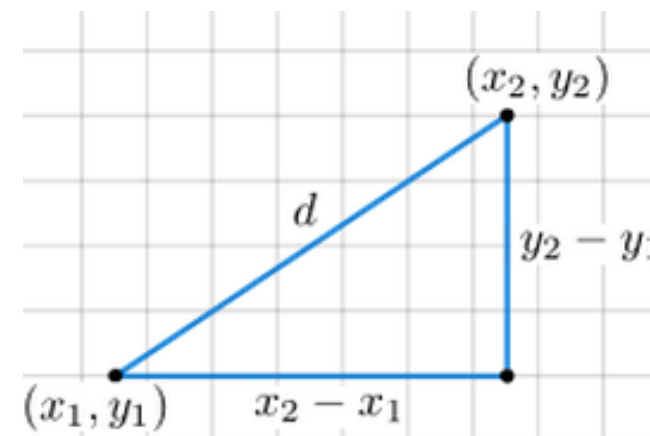
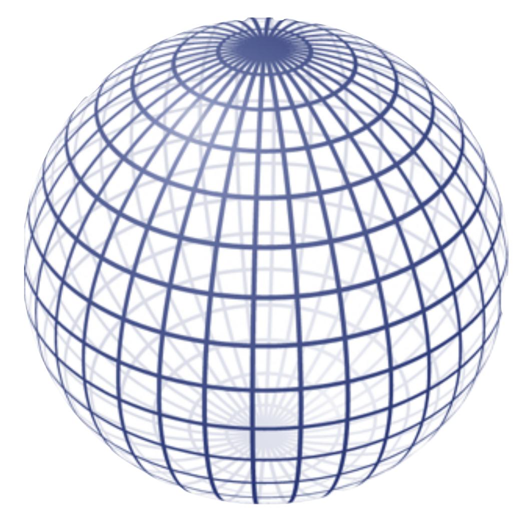


Spherical



Synthetic geometry

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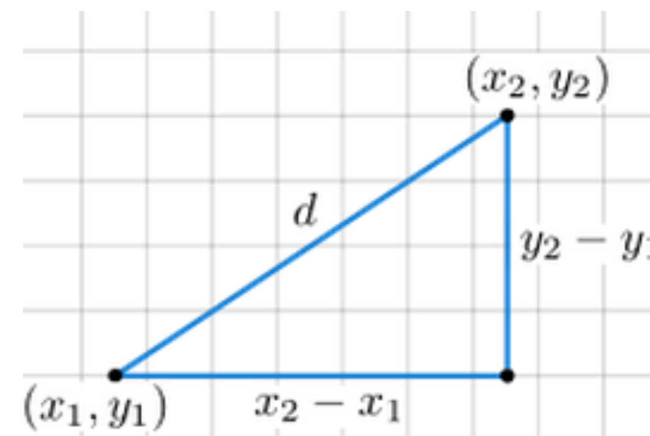
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. Two distinct lines meet at two antipodal points.

models

Cartesian



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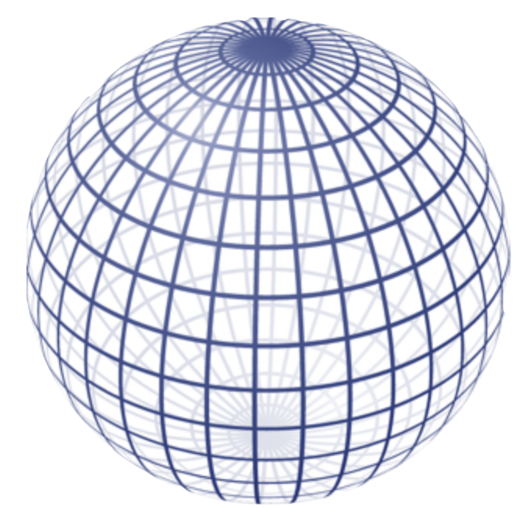

Synthetic mathematics

Type theory

1. $\tau ::= b \mid \tau_1 \to \tau_2$
2. $e ::= x \mid e_1\, e_2 \mid \lambda x.e$
3. $(\lambda x.e)e_2 = e[e_2/x]$

Synthetic mathematics

Set-theoretic functions

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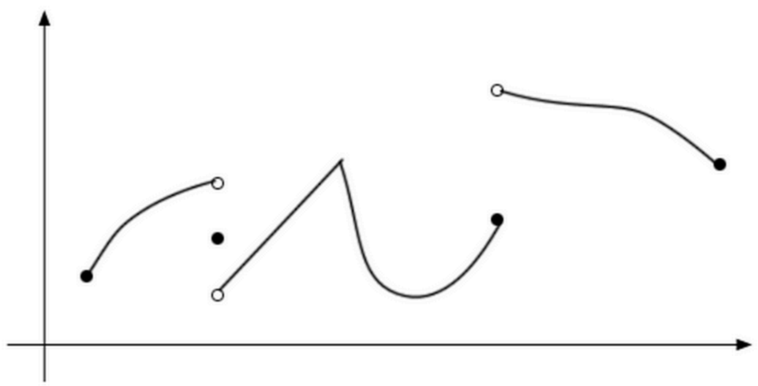

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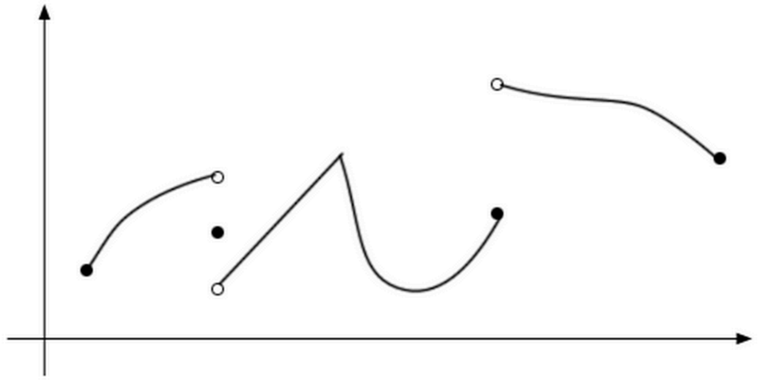
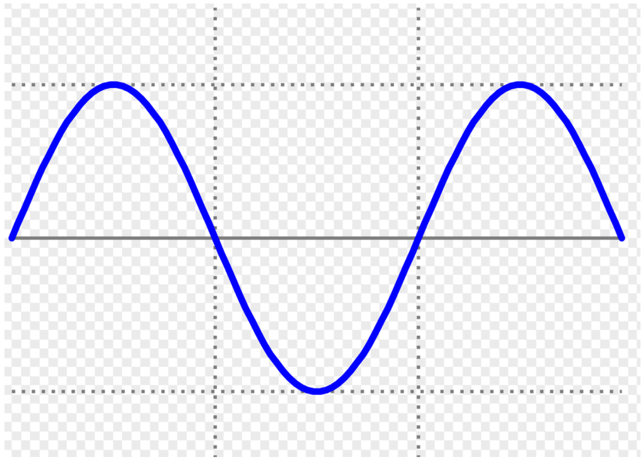
Set-theoretic functions



Domain-theoretic functions



Synthetic mathematics

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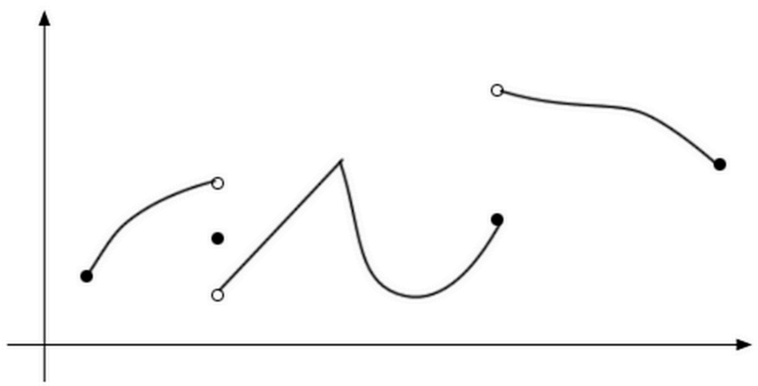

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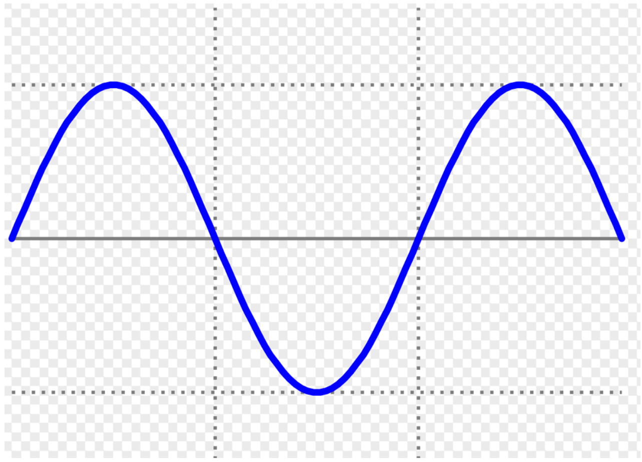
4. $Y(f) = f(Y(f))$

Domain-theoretic functions



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Three senses of constructivity:

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Three senses of constructivity:

- Non-affirmation of certain classical principles provides **axiomatic freedom**
- **Computational interpretation** supports software verification and proof automation

Computational Interpretation

There is an algorithm that,
given a closed term `e : bool`,
computes either
an equality `e = true`, or
an equality `e = false`.

Constructive Type Theory

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Constructive Type Theory

Three senses of constructivity:

- Non-affirmation of certain classical principles provides **axiomatic freedom**
- **Computational interpretation** supports software verification and proof automation
- Allows **proof-relevant mathematics**

Proof relevance

```
x : A
```

Proof relevance

$x : A$

$x =_A y$ equality type

Proof relevance

$x : A$

$p : x =_A y$ equality type

Proof relevance

`x : A`

`p : x =`$_A$` y` equality type

Any structure or property `C` can be transported along an equality

Proof relevance

$x : A$

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Any structure or property C can be transported along an equality

$$\mathrm{transport}_C(p) : C(x) \to C(y)$$

Proof relevance

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Any structure or property C can be transported along an equality

Leibniz's indiscernability of identicals

$transport_C(p) : C(x) \to C(y)$

Proof relevance

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Any structure or property C can be transported along an equality ← **Leibniz's indiscernability of identicals**

$\mathrm{transport}_C(p) : C(x) \to C(y)$

by a function: can it do real work?

Proof relevance

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Proof relevance

$x : A$

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$p_1 =_{x=y} p_2$

Proof relevance

$x : A$

$p : x =_A y$ equality type

$q : p_1 =_{x=y} p_2$

Proof relevance

$x : A$

$p : x =_A y$ equality type

$q : p_1 =_{x=y} p_2$

$q_1 =_{p1=p2} q_2$

Proof relevance

$x : A$

$p : x =_A y$ equality type

$q : p_1 =_{x=y} p_2$

$r : q_1 =_{p1=p2} q_2$

Proof relevance

$x : A$

$p : x =_A y$ equality type

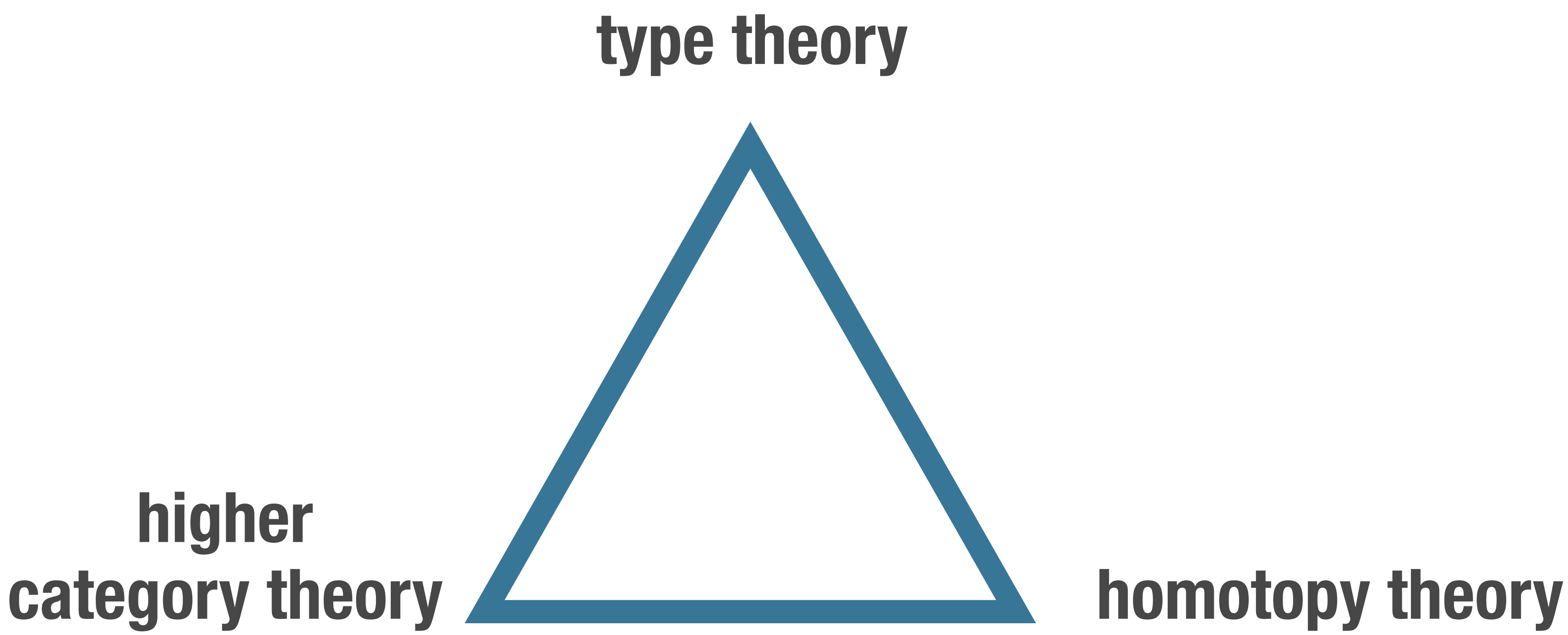
$q : p_1 =_{x=y} p_2$

$r : q_1 =_{p1=p2} q_2$

$\vdots$

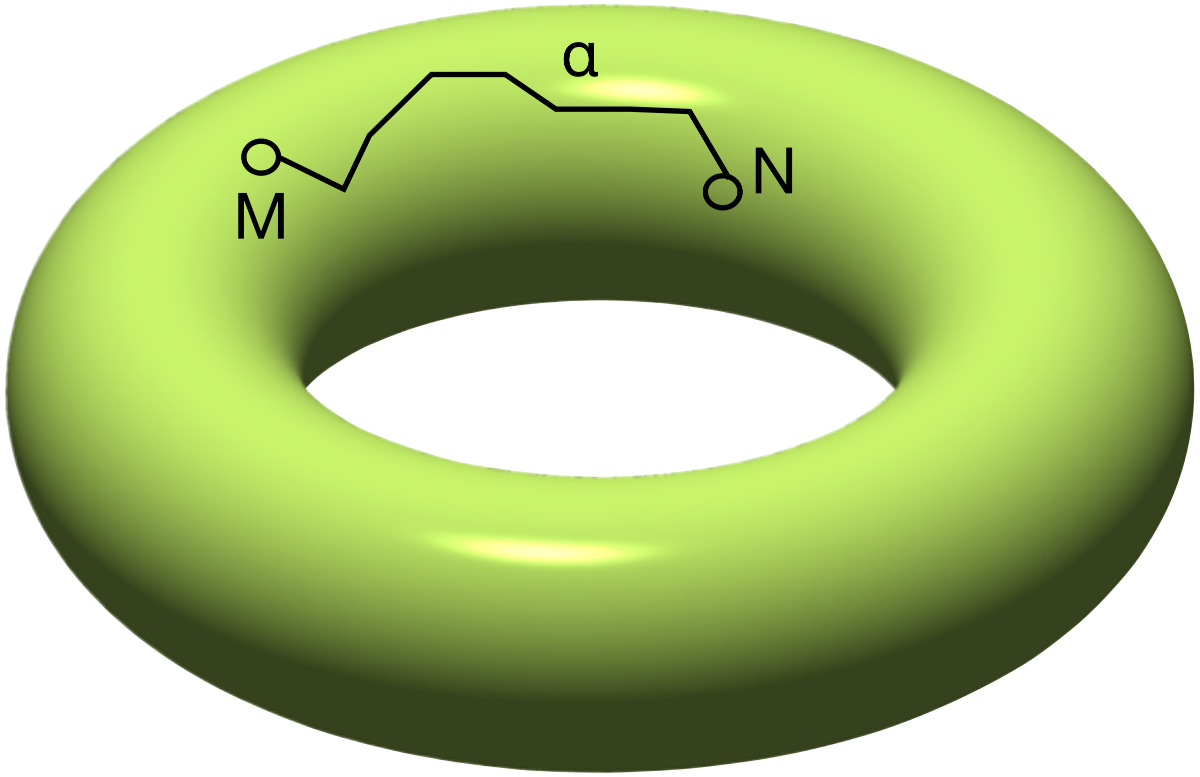
higher equalities radically expand the kind of math that can be done synthetically…

Homotopy Type Theory



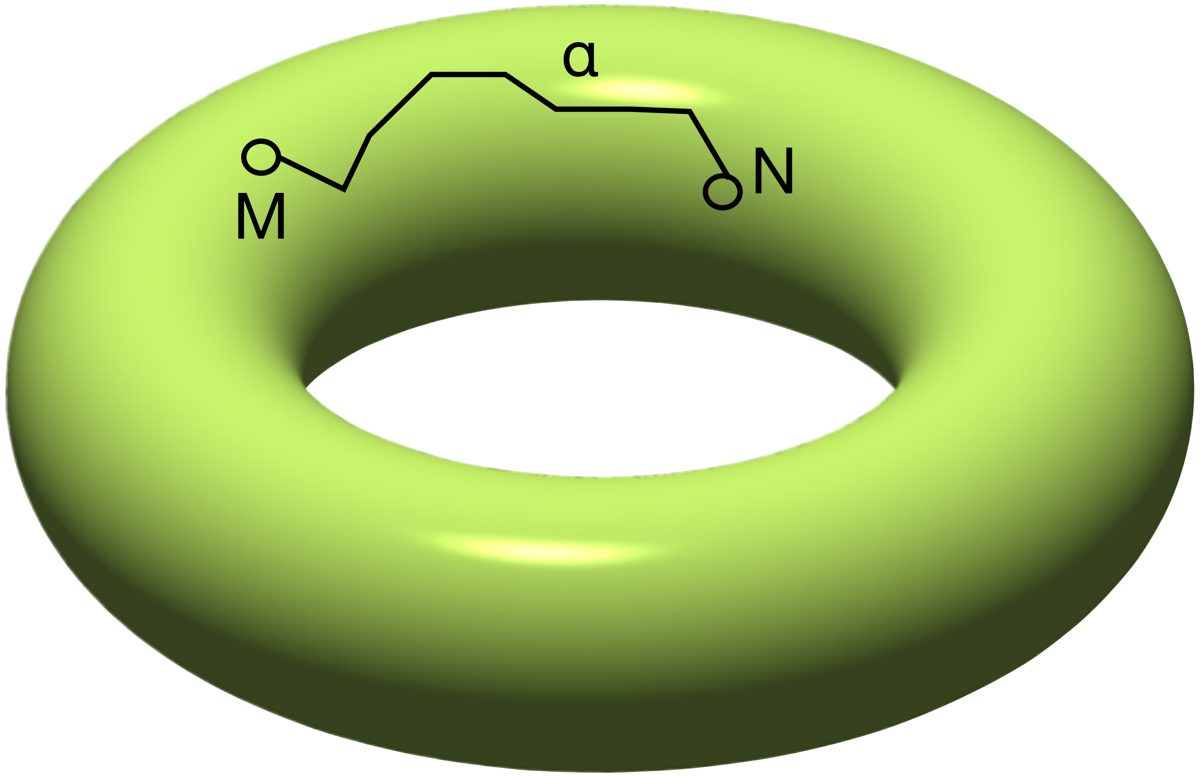
[Awodey,Warren,Voevodsky,,Streicher,Hofmann
Lumsdaine,Gambino,Garner,van den Berg]

Types as spaces



Types as spaces

type A is a space



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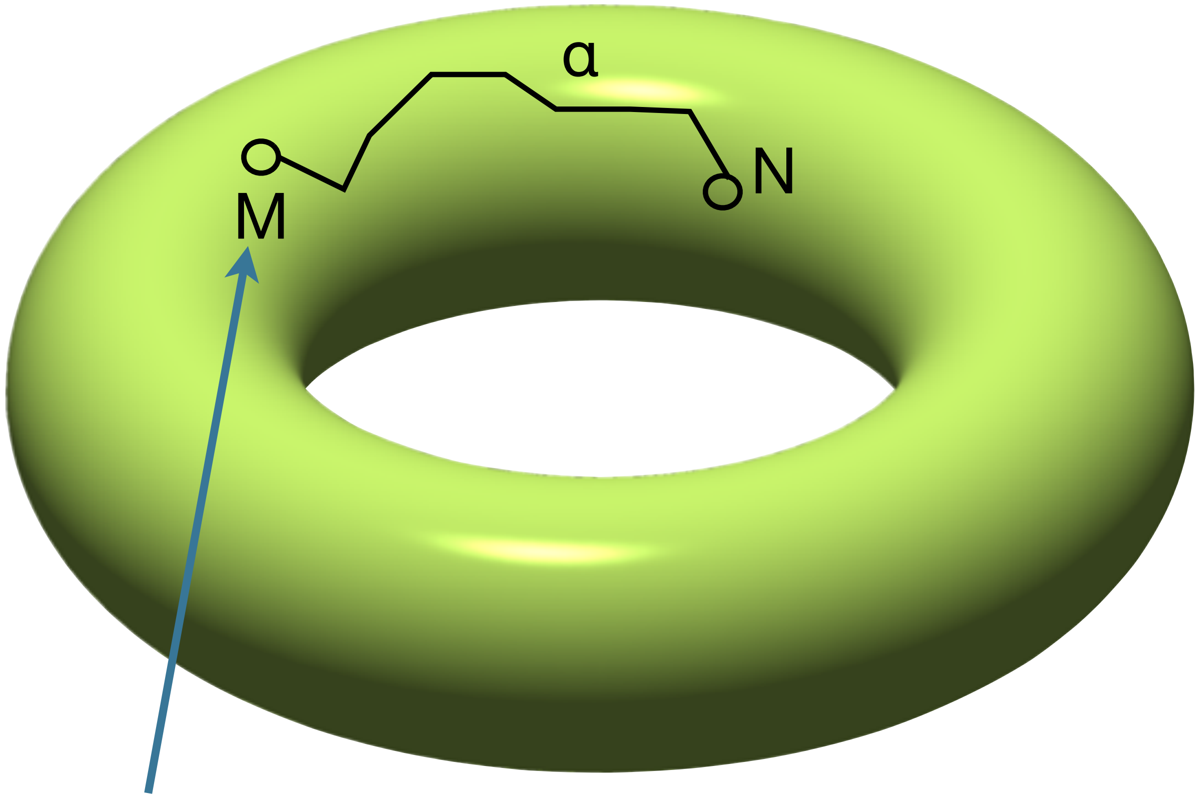
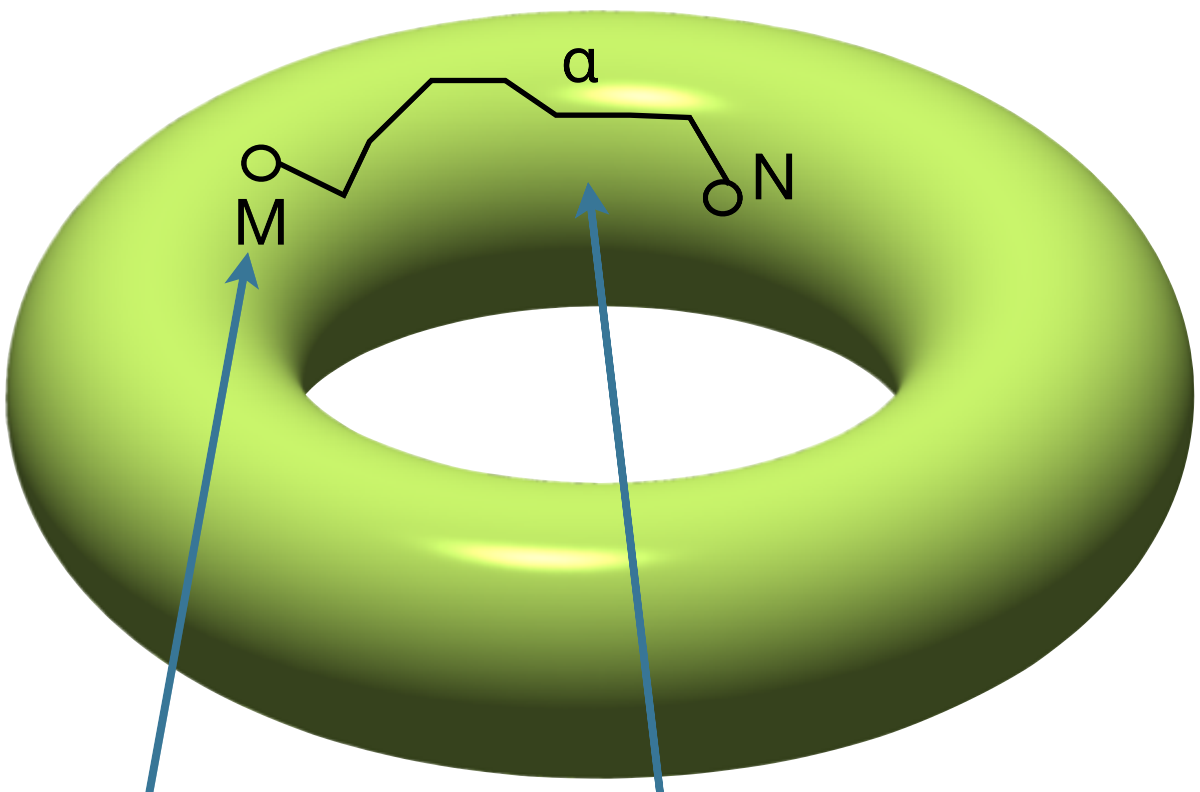


programs
M : A
are points

Types as spaces

type A **is a space**



programs
$M : A$
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proofs of equality
$\alpha \ : \ M =_A N$
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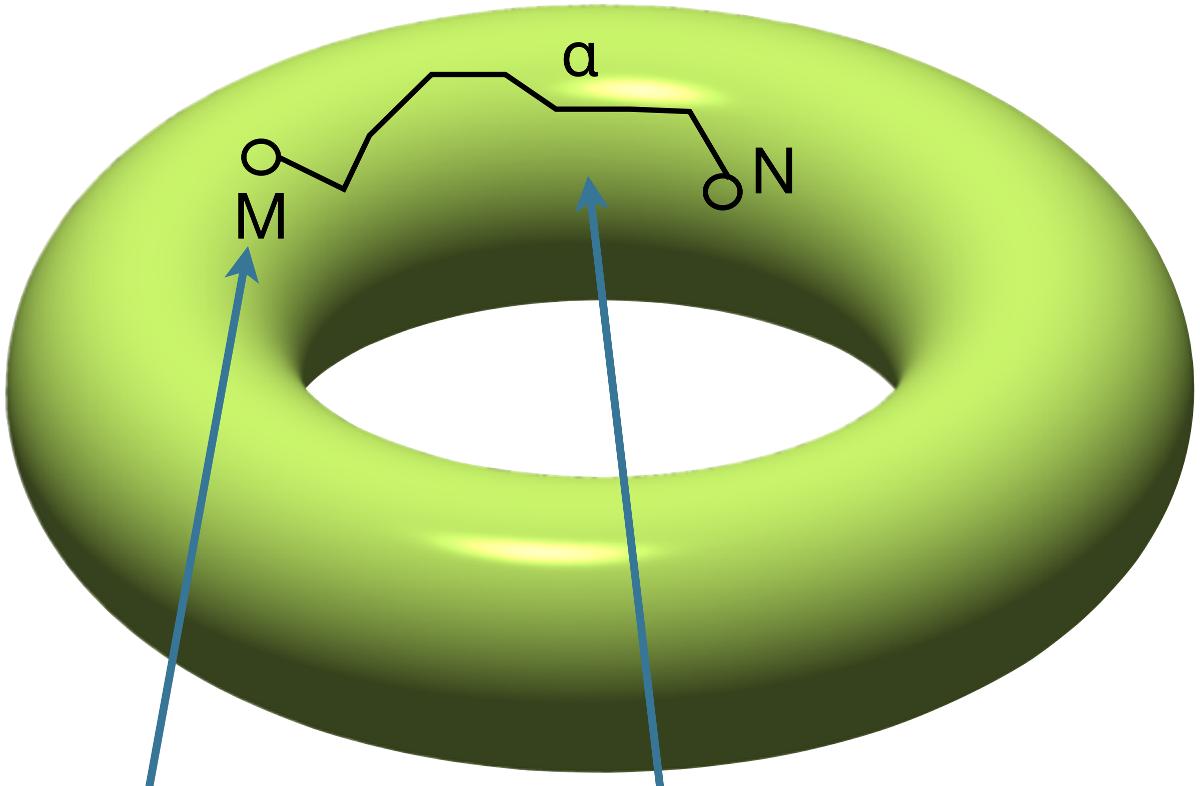

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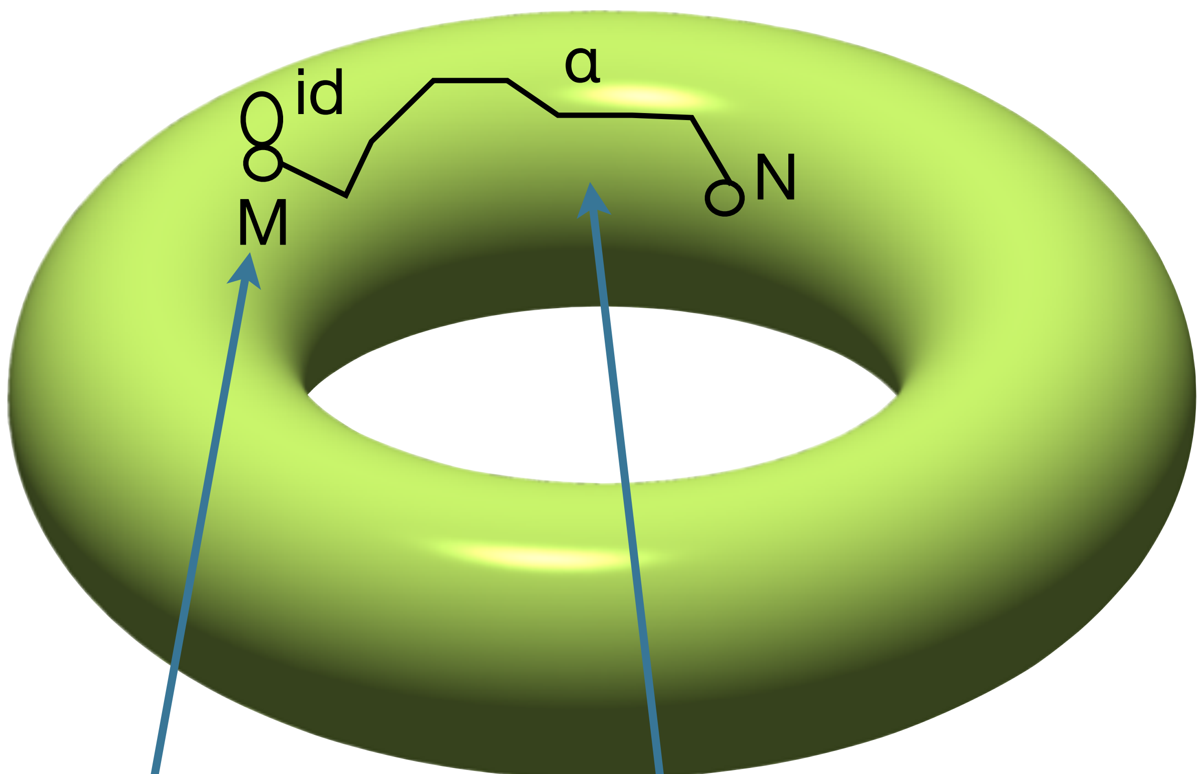

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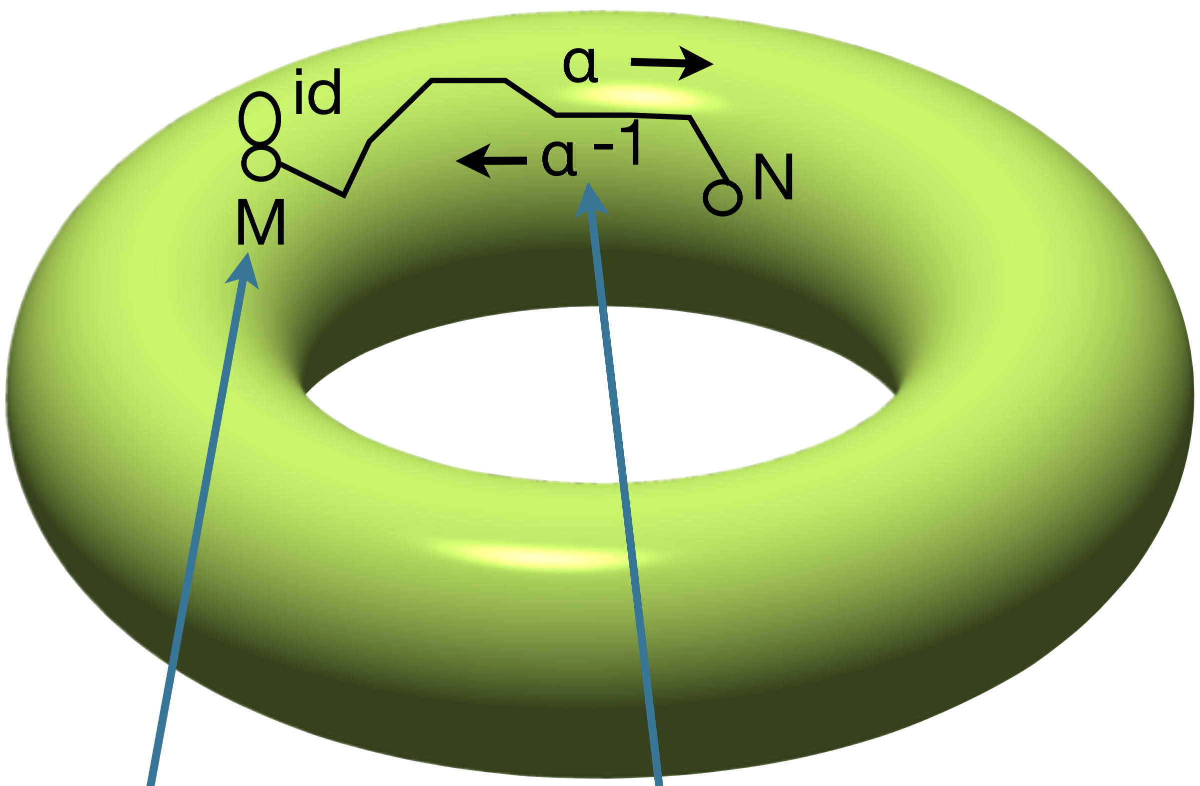
path operations

```
id      : M = M (refl)
```

Types as spaces

type A is a space



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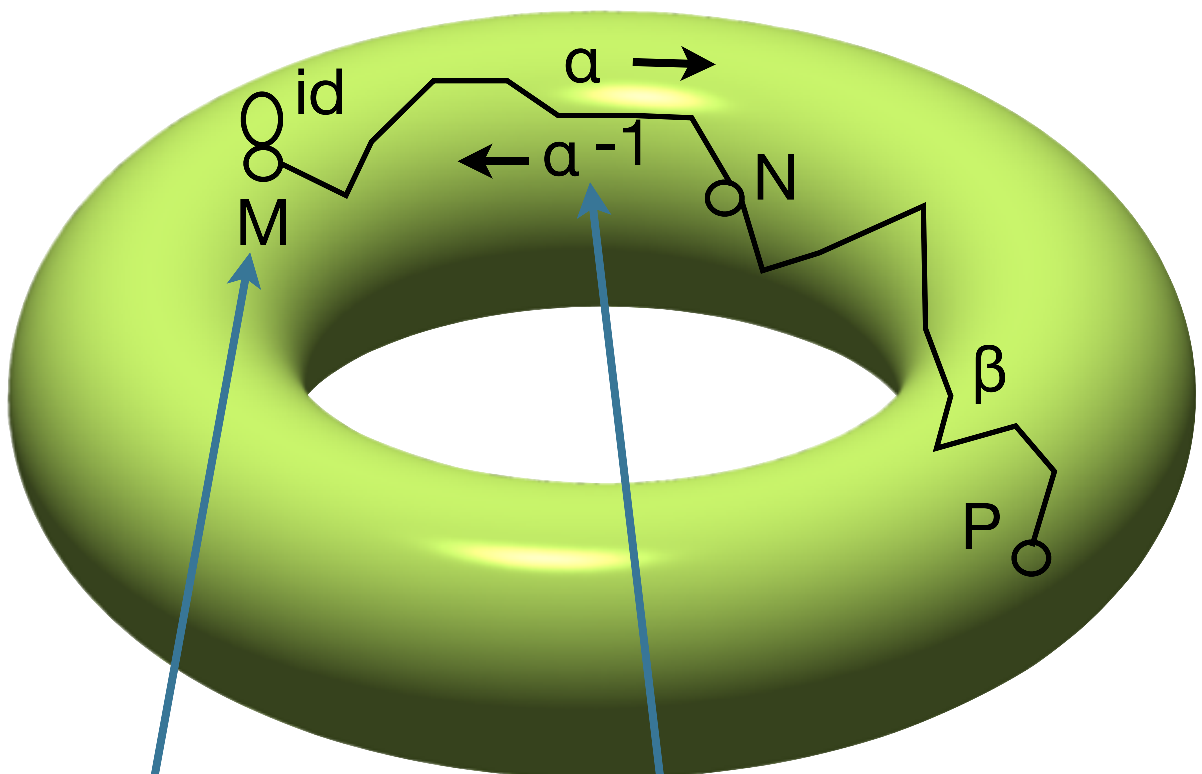

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path operations

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id     : M = M (refl)
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β o α  : M = P (trans)
```

Homotopy

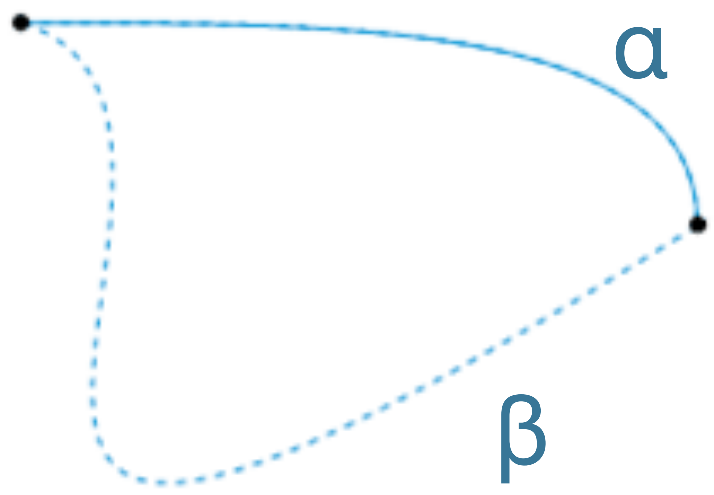
Deformation of one path into another

α

β

Homotopy

Deformation of one path into another



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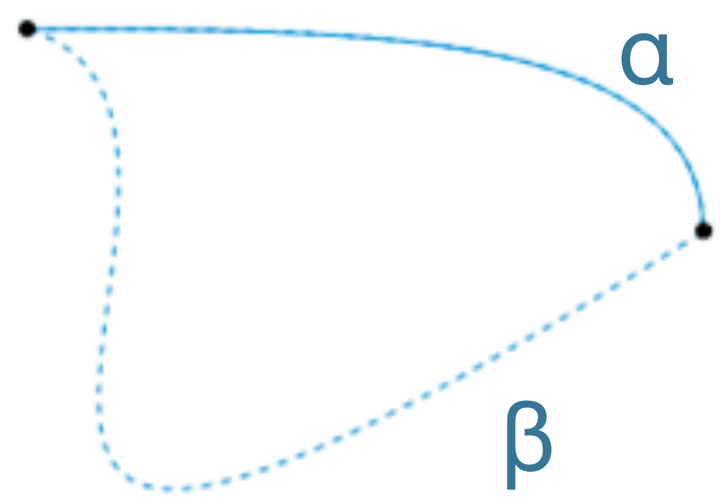
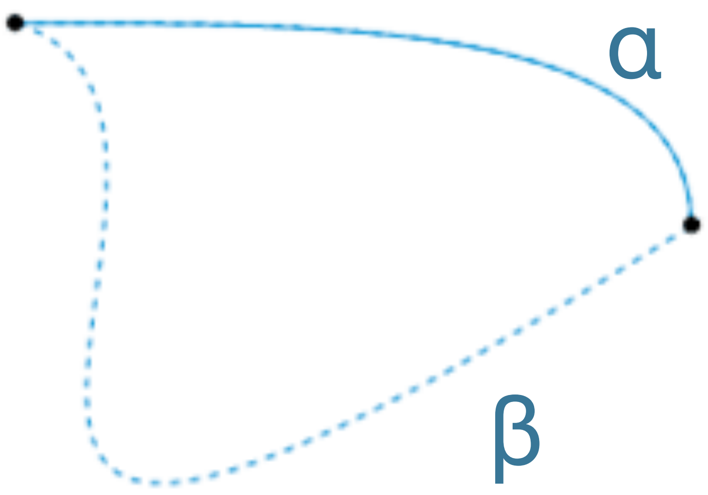


= 2-dimensional *path between paths*

Homotopy

Deformation of one path into another



$\delta : \alpha =_{x=y} \beta$

= 2-dimensional *path between paths*

Types as spaces

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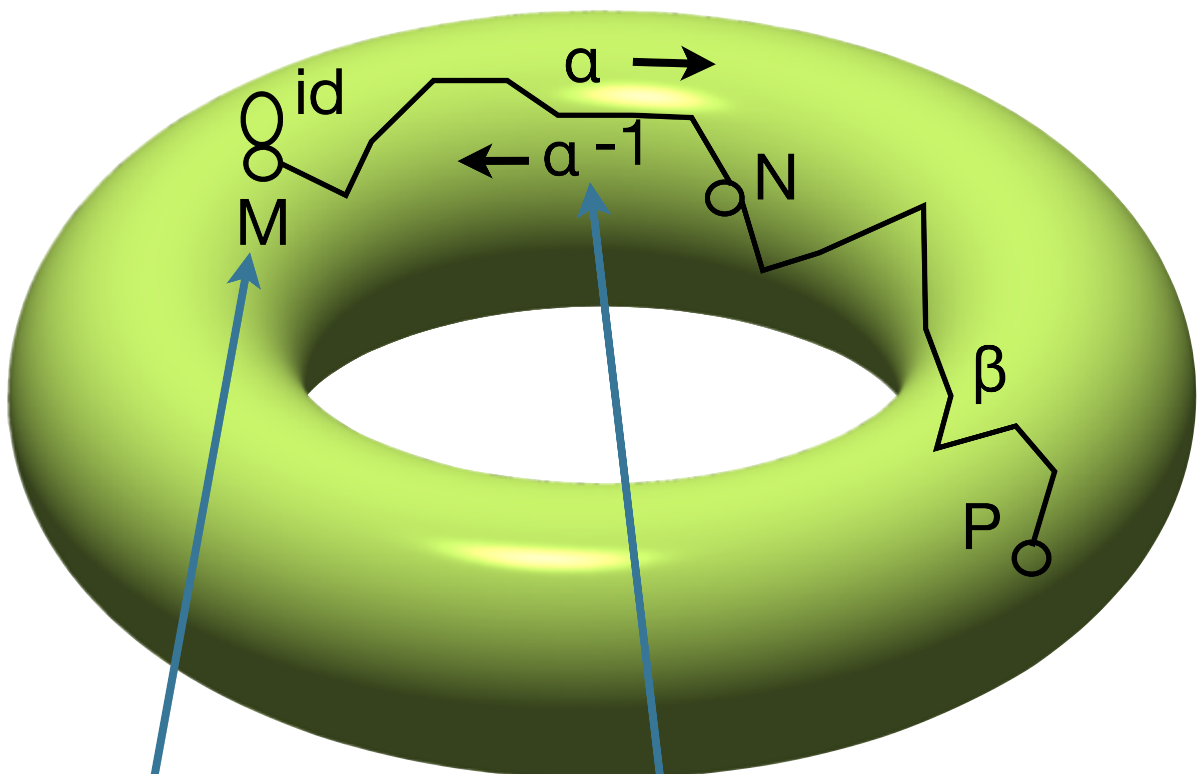

programs

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proofs of equality

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path operations

id : M = M (refl)

α^{-1} : N = M (sym)

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homotopies

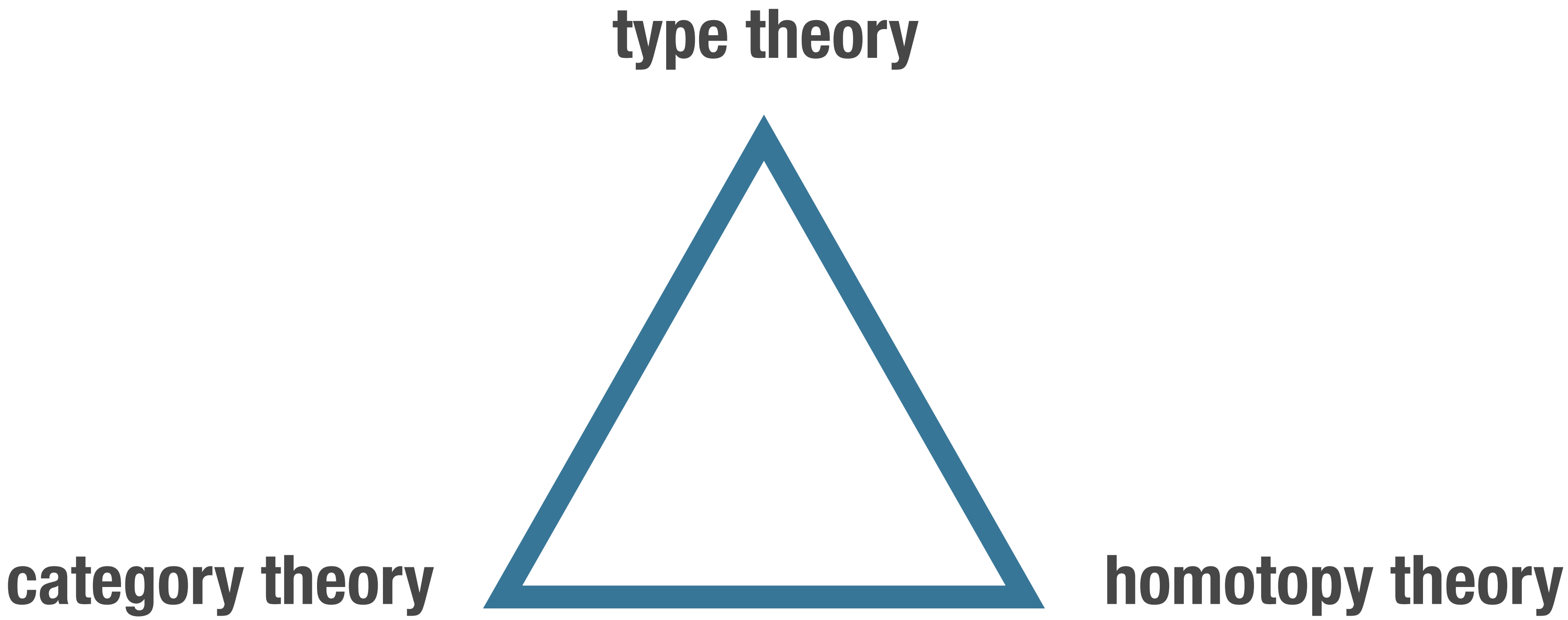
ul : id o α $=_{M=N}$ α

il : α^{-1} o α $=_{M=M}$ id

asc : γ o (β o α)

$=_{M=P}$ (γ o β) o α

Homotopy Type Theory



Types as ∞-groupoids

type A **is an ∞-groupoid**

- infinite-dimensional algebraic structure, with morphisms, morphisms between morphisms, ...
- each level has a groupoid structure, and they interact

morphisms

```
id        : M = M (refl)
α⁻¹       : N = M (sym)
β o α     : M = P (trans)
```

morphisms between morphisms

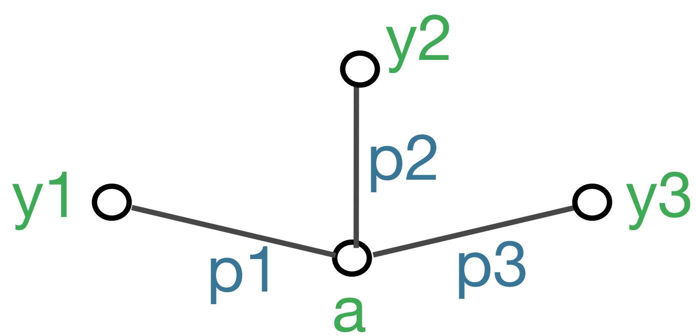
ul : id o α $=_{M=N}$ α

il : α^{-1} o α $=_{M=M}$ id

asc : γ o (β o α) $=_{M=P}$ (γ o β) o α

Path induction

Type of paths from a to somewhere



is inductively generated by

Path induction

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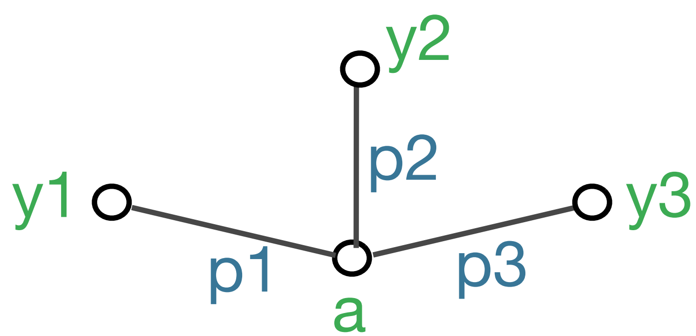


is inductively generated by

Fix a type A with element a:A.

For a family of types C(y:A, p:a=y),

to give an element of

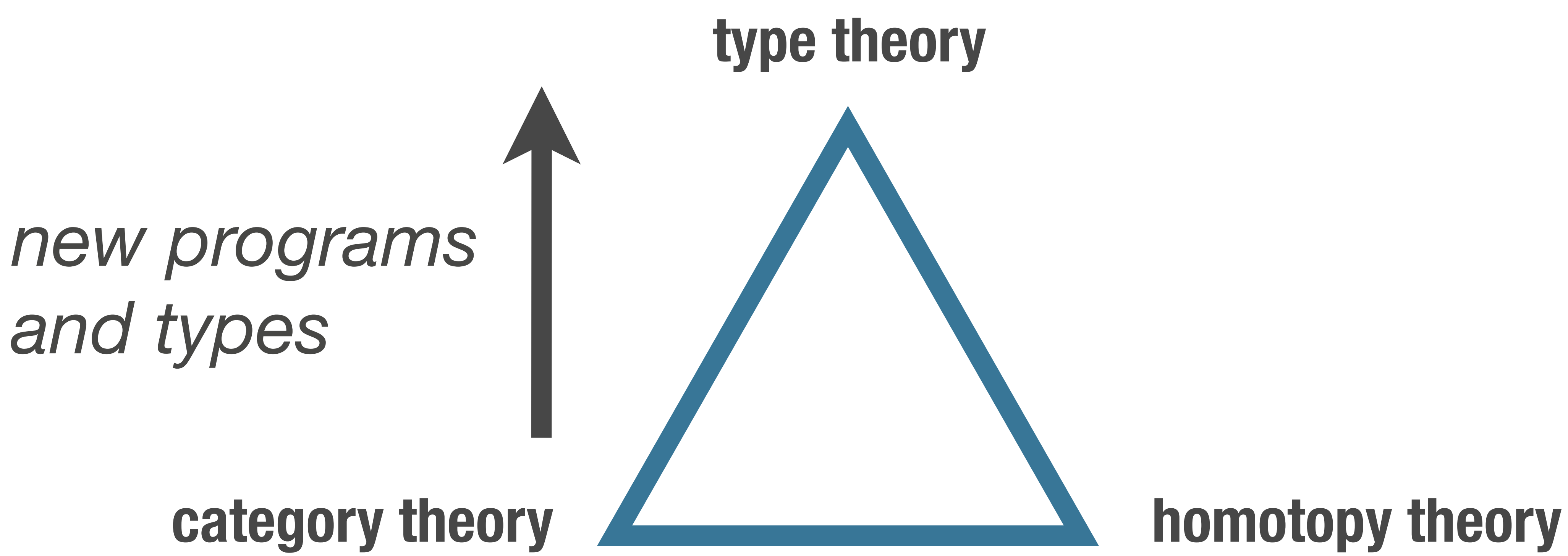
C(y,p) for all y and p:a=y,

suffices to give an element of

C(a,id)

Type theory is a synthetic theory of spaces/∞-groupoids

Homotopy Type Theory



Univalence [Voevodsky]

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- *Equivalence of types* is a generalization to spaces of bijection of sets

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- Univalence axiom:
  equality of types (`A =Type B`) is (equivalent to) equivalence of types (`Equiv A B`)

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- ∴ all structures/properties respect equivalence

Univalence [Voevodsky]

- *Equivalence of types* is a generalization to spaces of bijection of sets
- Univalence axiom:
  equality of types (`A =Type B`) is (equivalent to) equivalence of types (`Equiv A B`)
- ∴ all structures/properties respect equivalence
- Not by collapsing equivalence,
  but by exploiting proof-relevant equality:
  `transport` *does real work*

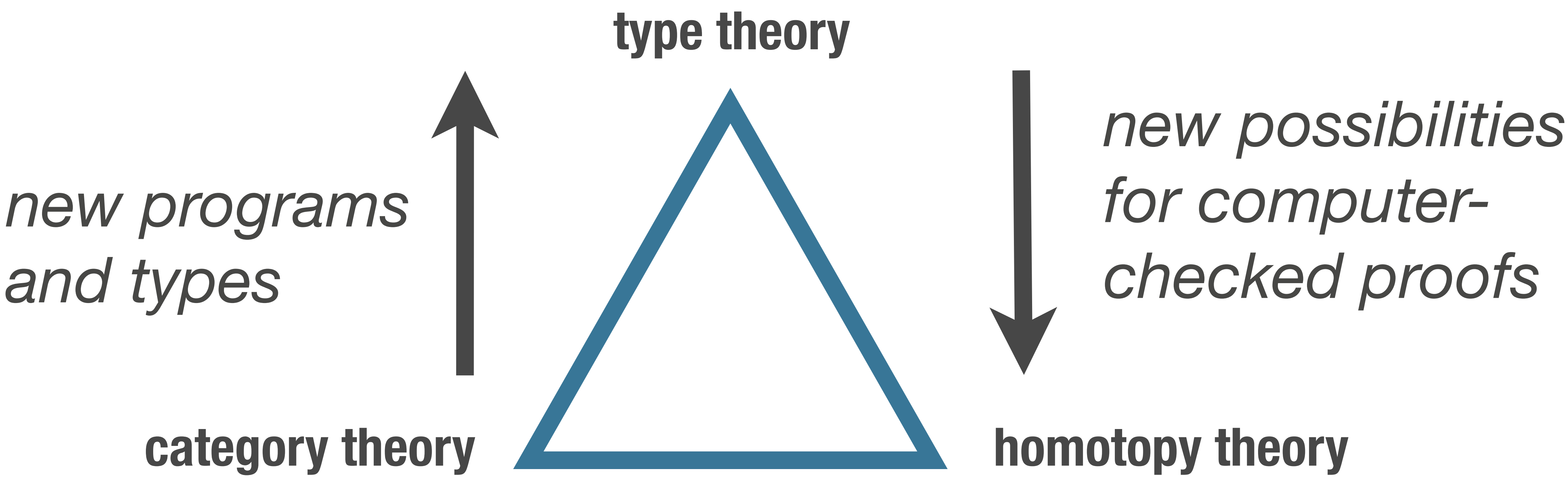
Higher inductive types

[Bauer,Lumsdaine,Shulman,Warren]

New way of forming types:

Inductive type specified by generators
not only for points (elements), but also for paths

Homotopy Type Theory



Basis for this formalization

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- Agda proof assistant [Norell, Abel, Danielsson]

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- 10,000 line HoTT library

Basis for this formalization

- Agda proof assistant [Norell, Abel, Danielsson]
- 10,000 line HoTT library
- essentially no automation

Outline

1. Eilenberg-MacLane spaces
2. K(G,1)
3. K(G,n)

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1. **Eilenberg-MacLane spaces**
2. K(G,1)
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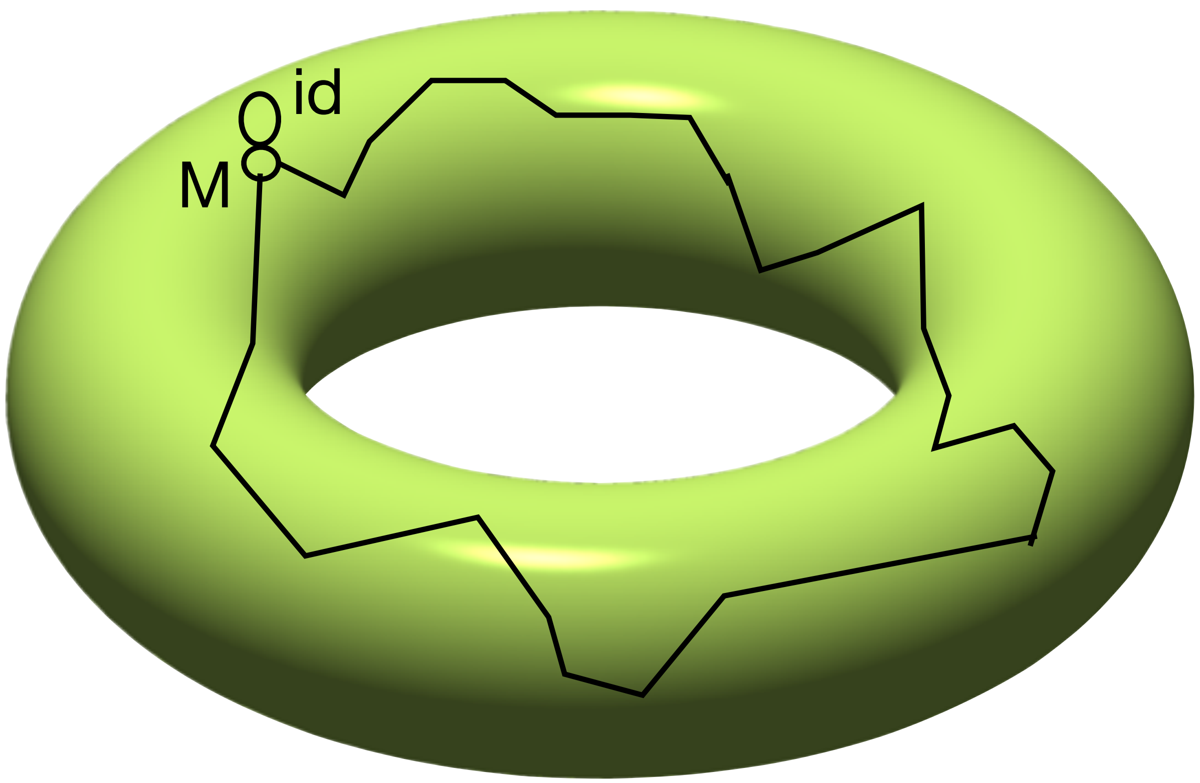
Types as spaces



loop operations

id : M = M (refl)

α^{-1} : M = M (sym)

β o α : M = M (trans)

homotopies

ul : id o α = α

il : α^{-1} o α = id

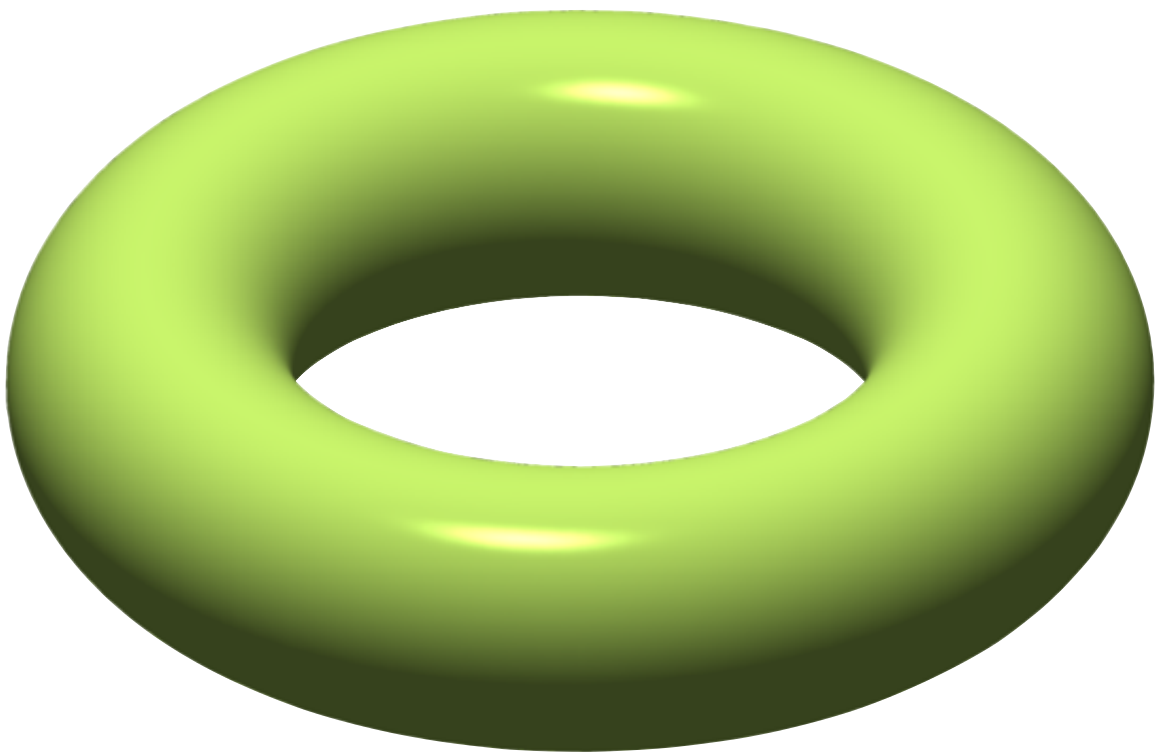
asc : γ o (β o α)
= (γ o β) o α

Homotopy Groups

Homotopy groups of a space X:

- $\pi_1(X)$ is fundamental group (group of loops)
- $\pi_2(X)$ is group of homotopies (2-dimensional loops)
- $\pi_3(X)$ is group of 3-dimensional loops
- ...

Telling spaces apart



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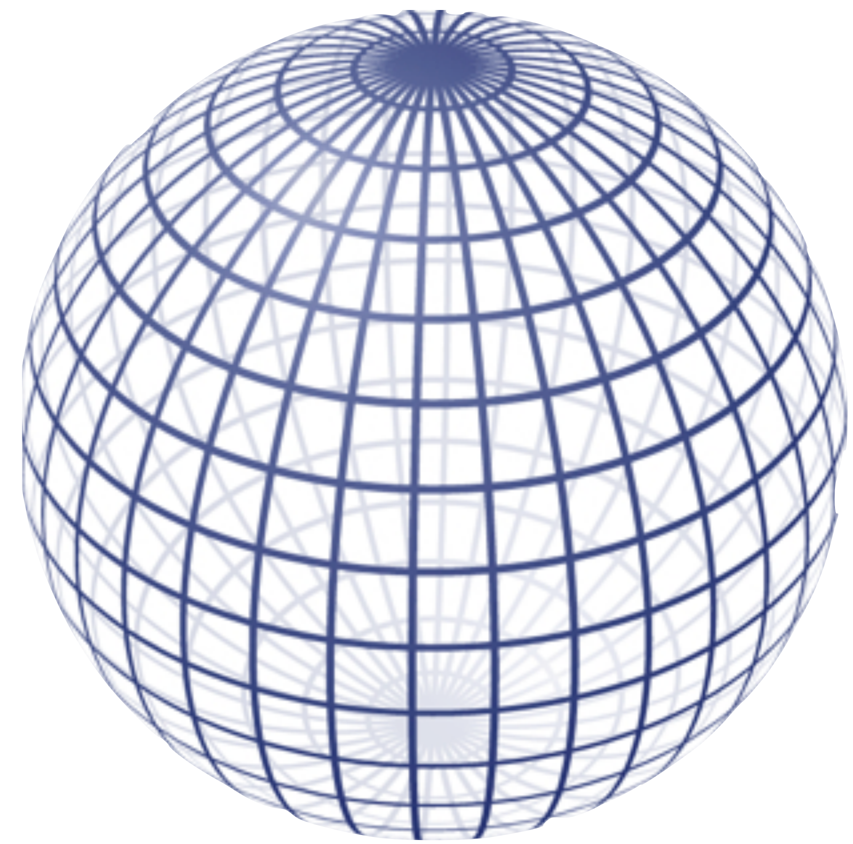

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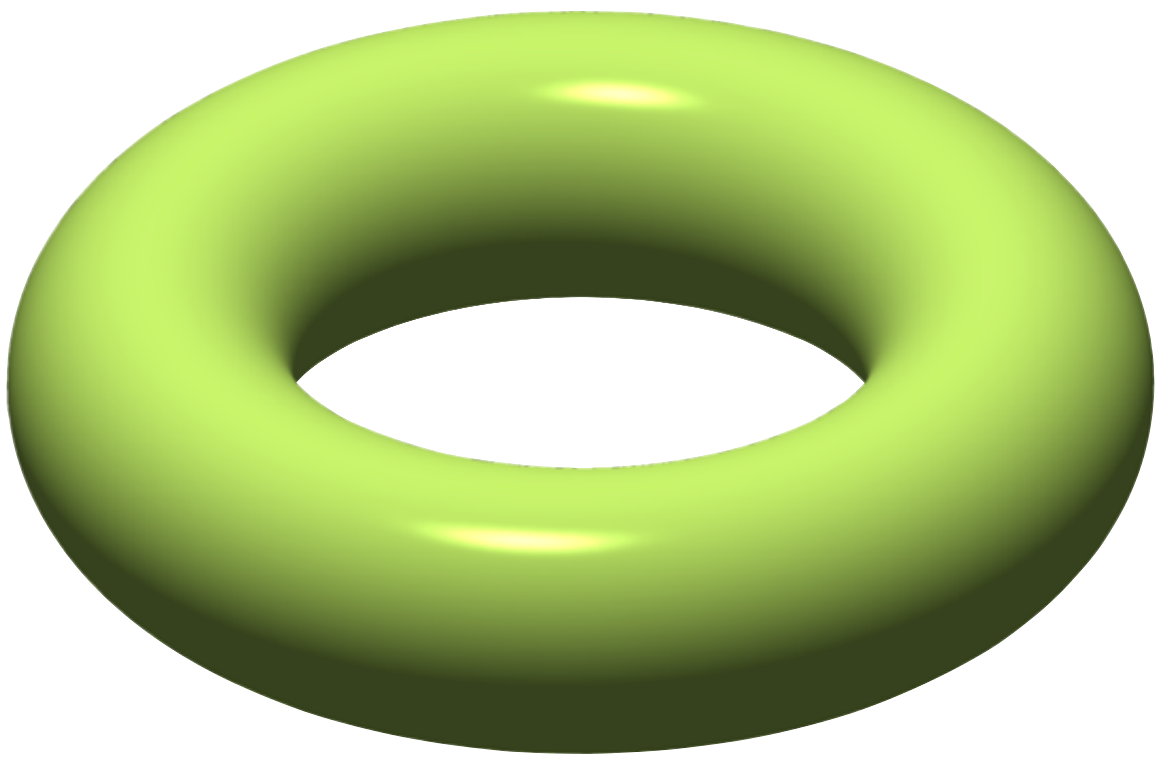
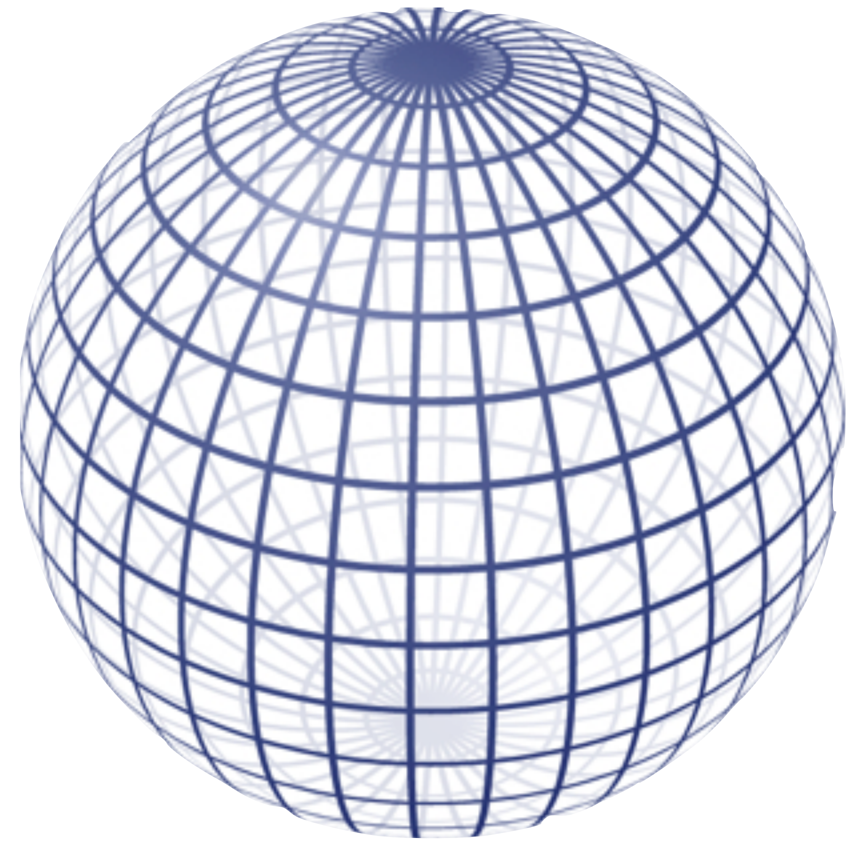


$\not\simeq$



fundamental group
is non-trivial ($\mathbb{Z} \times \mathbb{Z}$)

fundamental group
is trivial

Homotopy Groups

A

$x =_A x$ $\Omega(A,x)$

$id =_{x=x} id$ $\Omega^2(A,x)$

$id =_{\Omega 2(A,x)} id$ $\Omega^3(A,x)$

⋮

Homotopy Groups

A

$x =_A x$ $\qquad \Omega(A,x)$

$id =_{x=x} id$ $\qquad \Omega^2(A,x)$

$id =_{\Omega 2(A,x)} id$ $\qquad \Omega^3(A,x)$

$\vdots$

$$\pi_n(A,x) = ||\Omega^n(A,X)||_0$$

Homotopy Groups

A

$x =_A x$ $\Omega(A,x)$

$id =_{x=x} id$ $\Omega^2(A,x)$

$id =_{\Omega 2(A,x)} id$ $\Omega^3(A,x)$

⋮

$\pi_n(A,x) = ||\Omega^n(A,X)||_0$

0-truncation = set of connected components = all paths are equal

Eilenberg-MacLane Space

For a group G

K(G,1) is a space such that
$\pi_1(K(G,1)) = G$ and
$\pi_k(K(G,1)) = 1$ otherwise

K(G,n) is a space such that (G abelian)
$\pi_n(K(G,n)) = G$ and
$\pi_k(K(G,n)) = 1$ otherwise

Spaces with specified groups

Find X with $\pi_1(X) = G$ and $\pi_2(X) = H$

Define $X = K(G,1) \times K(H,2)$

$$\begin{aligned}\pi_1(X) &= \pi_1(K(G,1)) \times \pi_1(K(H,2)) \\ &= G \times 1 \\ &= G\end{aligned}$$

$$\begin{aligned}\pi_2(X) &= \pi_2(K(G,1)) \times \pi_2(K(H,2)) \\ &= 1 \times H \\ &= H\end{aligned}$$

Cohomology

Homotopy groups aren't the only invariant:
homology groups, cohomology groups

Define *ordinary cohomology with coefficients in G* by

$$H^n(A) = \|A \to K(G,n)\|_0$$

Cohomology

Homotopy groups aren't the only invariant:
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Define *ordinary cohomology with coefficients in G* by

$$H^n(A) = \|A \to K(G,n)\|_0$$

satisfies (constructive) Eilenberg-Steenrod axioms

Eilenberg-MacLane space

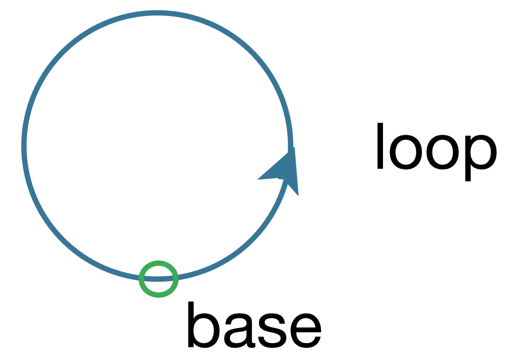
Can we build Eilenberg-MacLane spaces from higher inductive types?

Outline

1. Eilenberg-MacLane spaces
2. **K(G,1)**
3. K(G,n)
4. Proofs

Fundamental group of circle

How many different loops are there on the circle, up to homotopy?



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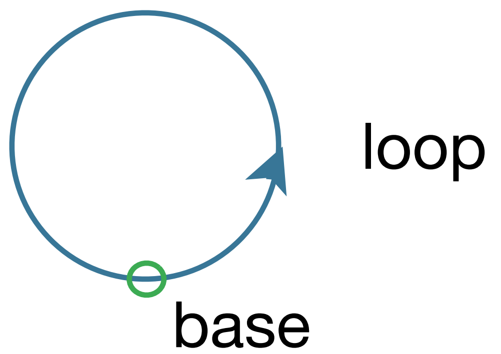
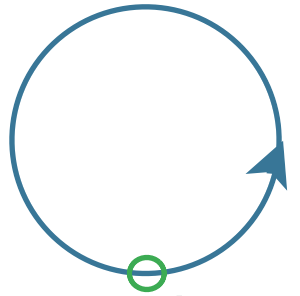



```
id
```

Fundamental group of circle

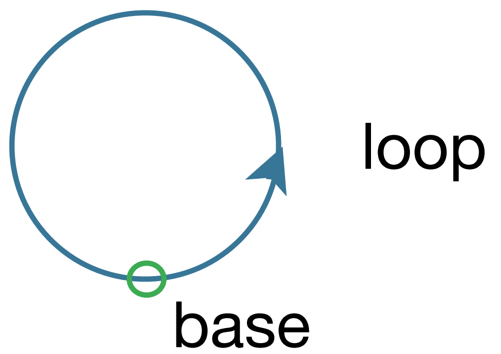
How many different loops are there on the circle, up to homotopy?




```
id
loop
```

Fundamental group of circle

How many different loops are there on the circle, up to homotopy?



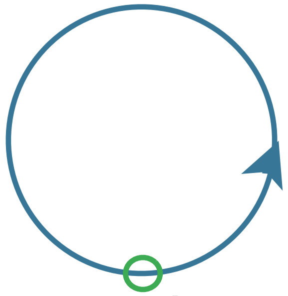
`id`

`loop`

`loop`$^{-1}$

Fundamental group of circle

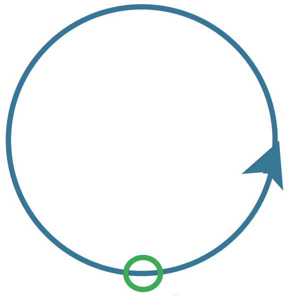
How many different loops are there on the circle, up to homotopy?




```
id
loop
loop⁻¹
loop o loop
```

Fundamental group of circle

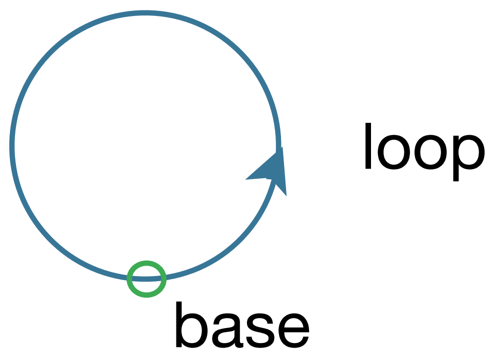
How many different loops are there on the circle, up to homotopy?




```
id
loop
loop⁻¹
loop o loop
loop⁻¹ o loop⁻¹
```

Fundamental group of circle

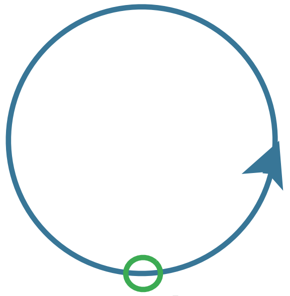
How many different loops are there on the circle, up to homotopy?




```
id
loop
loop⁻¹
loop o loop
loop⁻¹ o loop⁻¹
loop o loop⁻¹
```

Fundamental group of circle

How many different loops are there on the circle, up to homotopy?




```
id
loop
loop-1
loop o loop
loop-1 o loop-1
loop o loop-1  = id
```

$\texttt{id}$

$\texttt{loop}$

$\texttt{loop}^{-1}$

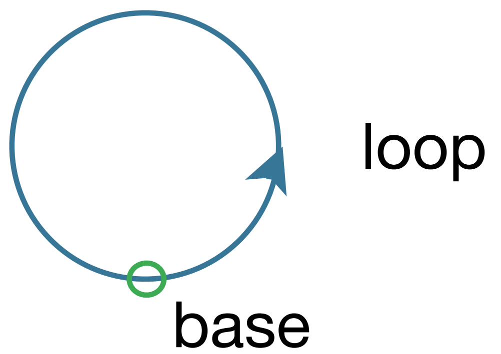
$\texttt{loop} \circ \texttt{loop}$

$\texttt{loop}^{-1} \circ \texttt{loop}^{-1}$

$\texttt{loop} \circ \texttt{loop}^{-1} = \texttt{id}$

Fundamental group of circle

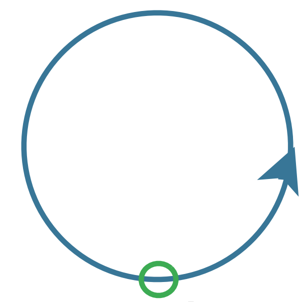
How many different loops are there on the circle, up to homotopy?




```
id                      0
loop
loop⁻¹
loop o loop
loop⁻¹ o loop⁻¹
loop o loop⁻¹   = id
```

Fundamental group of circle

How many different loops are there on the circle, up to homotopy?




```
id                      0
loop                    1
```
$\texttt{loop}^{-1}$

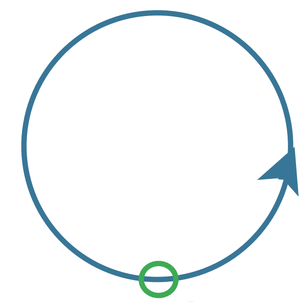
$\texttt{loop o loop}$

$\texttt{loop}^{-1}\ \texttt{o loop}^{-1}$

$\texttt{loop o loop}^{-1}$ = id

Fundamental group of circle

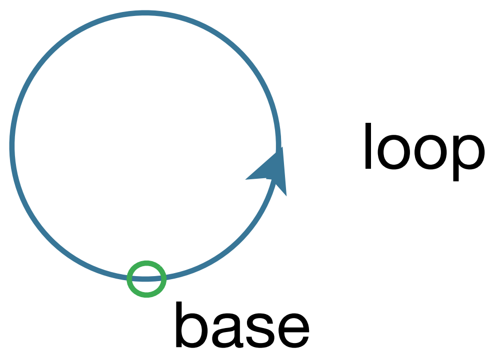
How many different loops are there on the circle, up to homotopy?



`id`	`0`
`loop`	`1`
$\texttt{loop}^{-1}$	`-1`
`loop o loop`	
$\texttt{loop}^{-1}\ \texttt{o}\ \texttt{loop}^{-1}$	
$\texttt{loop o loop}^{-1}\ = \texttt{id}$	

Fundamental group of circle

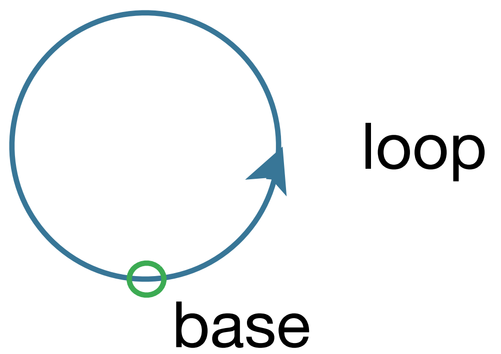
How many different loops are there on the circle, up to homotopy?



`id`	`0`
`loop`	`1`
$\texttt{loop}^{-1}$	`-1`
`loop o loop`	`2`
$\texttt{loop}^{-1}$ `o` $\texttt{loop}^{-1}$	
`loop o` $\texttt{loop}^{-1}$ `= id`	

Fundamental group of circle

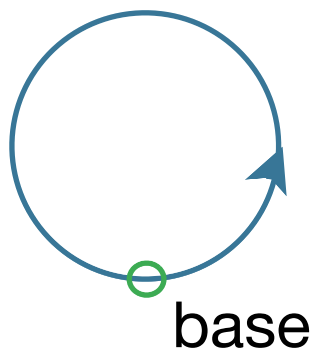
How many different loops are there on the circle, up to homotopy?



`id`	`0`
`loop`	`1`
`loop`$^{-1}$	`-1`
`loop o loop`	`2`
`loop`$^{-1}$ `o loop`$^{-1}$	`-2`
`loop o loop`$^{-1}$ `= id`	

Fundamental group of circle

How many different loops are there on the circle, up to homotopy?



`id`	`0`
`loop`	`1`
`loop`$^{-1}$	`-1`
`loop o loop`	`2`
`loop`$^{-1}$ `o loop`$^{-1}$	`-2`
`loop o loop`$^{-1}$ `= id`	`0`

Fundamental group of circle

The circle S^1 is a space such that

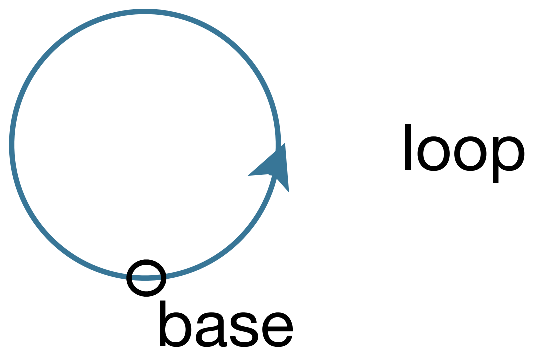
$\pi_1(S^1) = \mathbb{Z}$ and

$\pi_k(S^1) = 1$ otherwise

The circle is $K(\mathbb{Z},1)$

The Circle

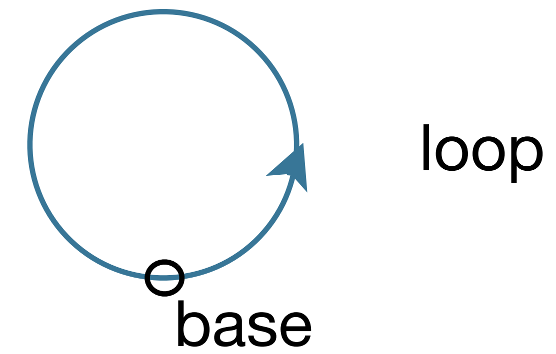
Circle S^1 is a **higher inductive type** generated by



The Circle

Circle S^1 is a **higher inductive type** generated by

```
base : S¹
loop : base = base
```

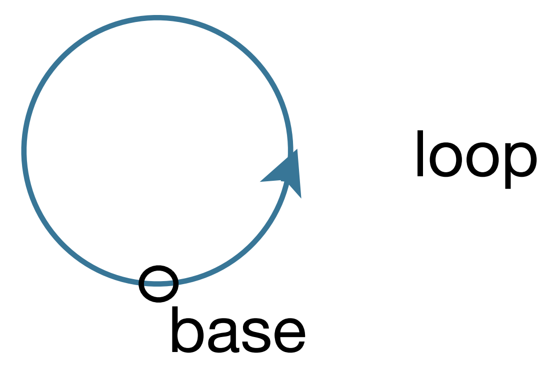


The Circle

Circle S^1 is a **higher inductive type** generated by

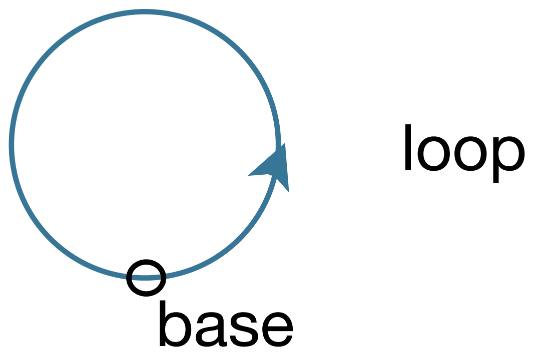
point `base : S¹`

`loop : base = base`



The Circle

Circle S^1 is a **higher inductive type** generated by

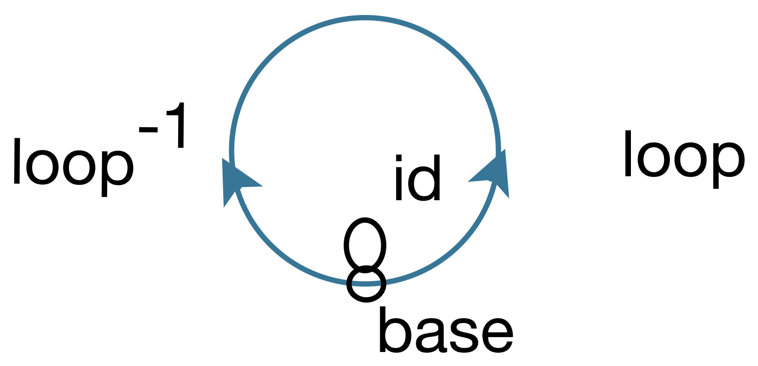


point `base : S¹`

path `loop : base = base`

The Circle

Circle S^1 is a **higher inductive type** generated by



point base : S^1

path loop : base = base

Free type: equipped with structure

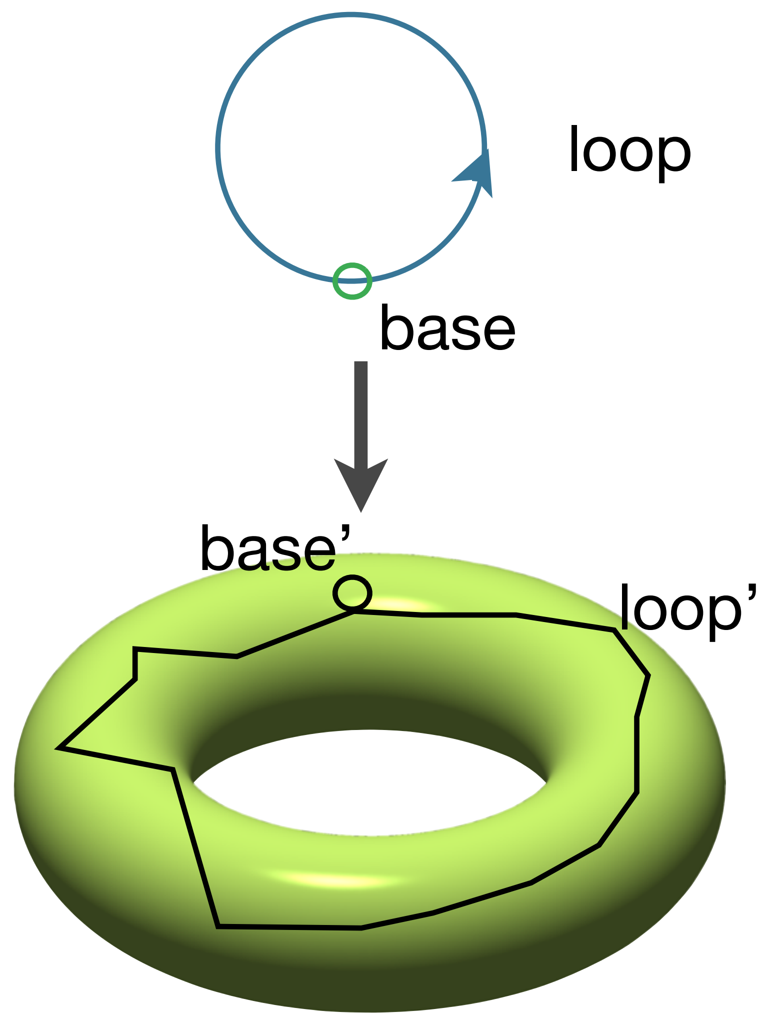
```
id                 inv : loop o loop⁻¹ = id
loop⁻¹             ...
loop o loop
```

The Circle

Circle recursion:

function $S^1 \to X$ determined by

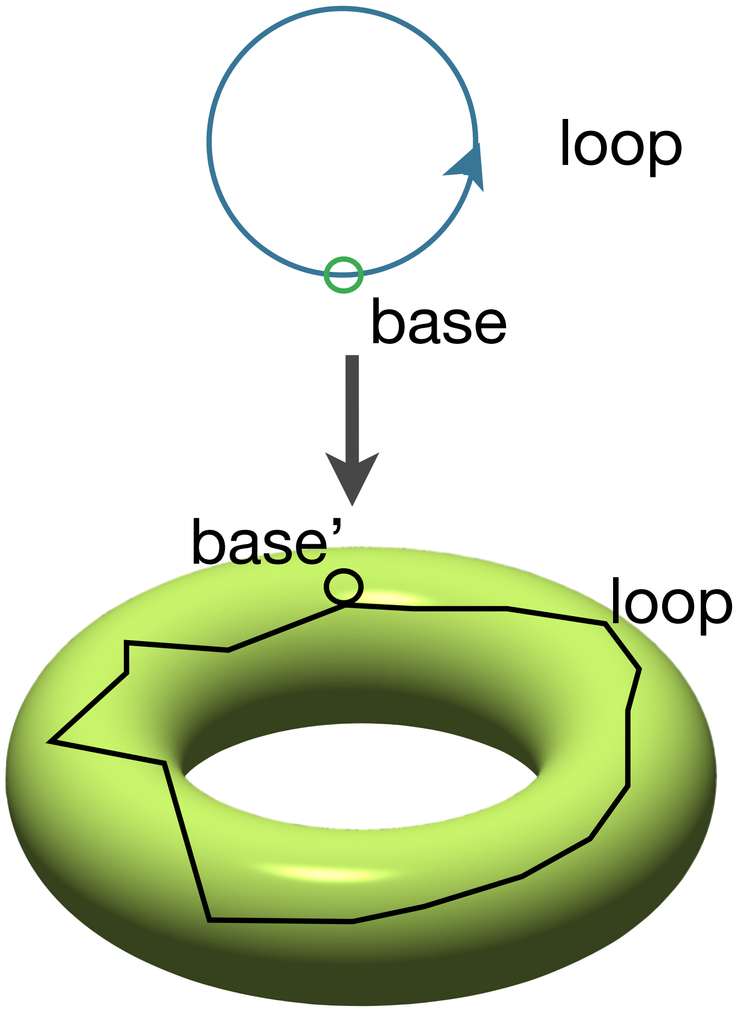
```
base' : X
loop' : base' = base'
```



The Circle

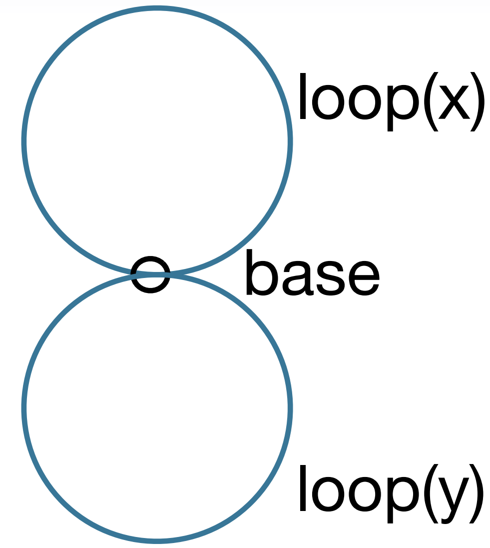
Circle recursion:
function $S^1 \rightarrow X$ determined by

```
base' : X
loop' : base' = base'
```



Circle induction: To prove a predicate P for all points on the circle, suffices to prove `P(base)`, continuously in the loop

K(G,1)



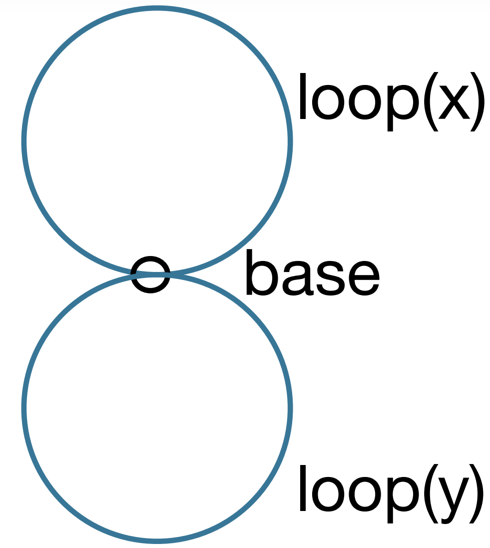
K(G,1) is a **higher inductive type** generated by

```
K(G,1) :   type

base : K(G,1)
loop : G → base=base
loop-ident : loop(1_G) = id
loop-comp : loop(x·_G y) = loop(x)·loop(y)
```

K(G,1)

K(G,1) is a **higher inductive type** generated by



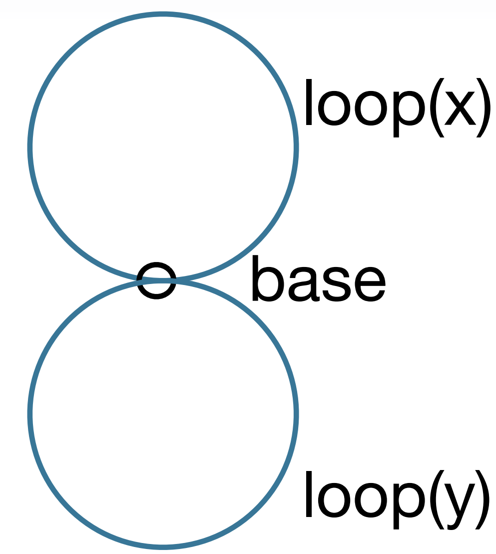

```
K(G,1) :   type

base : K(G,1)
loop : G → base=base
loop-ident : loop(1_G) = id
loop-comp : loop(x·_G y) = loop(x)·loop(y)
```

group homomorphism from G to Ω(K(G,1))

K(G,1)

K(G,1) is a **higher inductive type** generated by



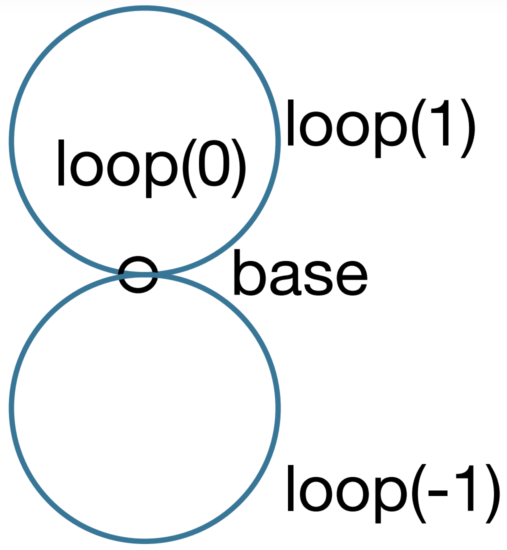

```
K(G,1) : 1-type

base : K(G,1)
loop : G → base=base
loop-ident : loop(1_G) = id
loop-comp : loop(x·_G y) = loop(x)·loop(y)
```

group homomorphism from G to Ω(K(G,1))

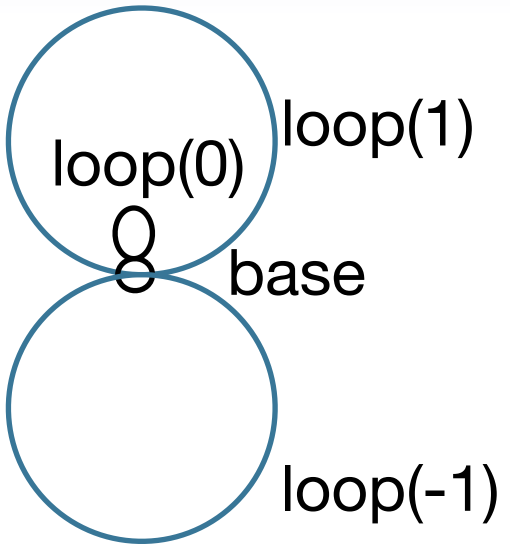
K($\mathbb{Z}$,1) revisited

K($\mathbb{Z}$,1) is *equivalent* to previous S^1



K($\mathbb{Z}$,1) revisited

K($\mathbb{Z}$,1) is *equivalent* to previous S^1



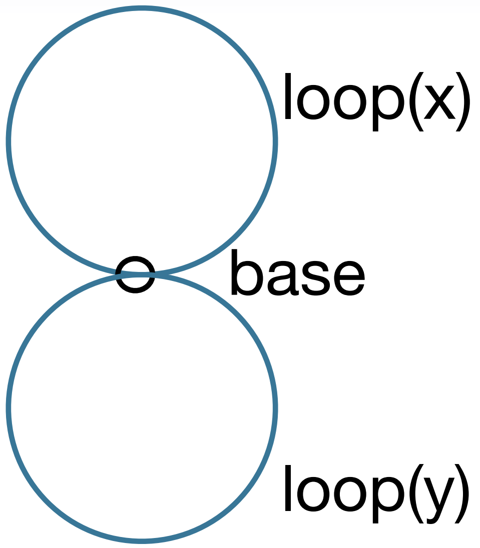

```
loop(0) = id

loop(1)

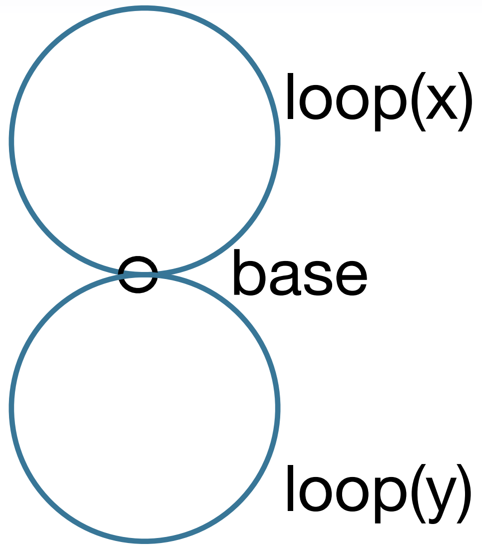
loop(-1) = !loop(1)

loop(2) = loop(1) · loop(1)
```

K(G,1) recursion



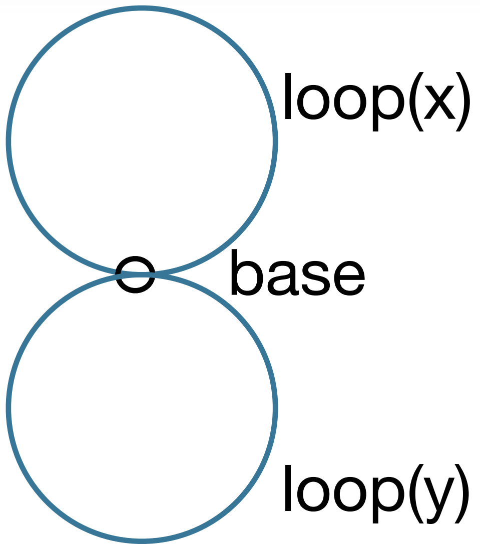
K(G,1) recursion



To define f : K(G,1) → C

- show C is a 1-type
- give f(base) : C
- f(loop) : group homomorphism from G to Ω(C,f(base))

$\pi_1(K(G,1)) = G$

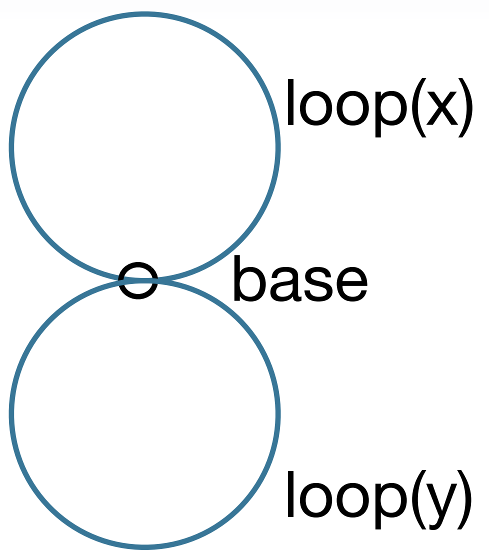


$\pi_1(K(G,1)) = G$

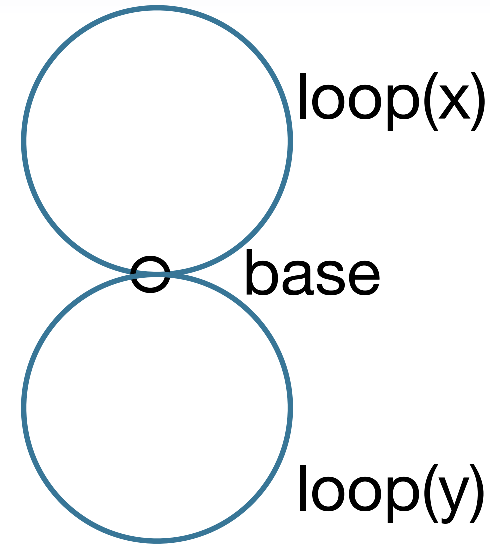
1. `Codes : K(G,1) → 1-Type`

   `Codes(base) = G`

   `Codes(loop(x)) =`
   “multiplication by x”



$\pi_1(K(G,1)) = G$



1. `Codes : K(G,1) → 1-Type`

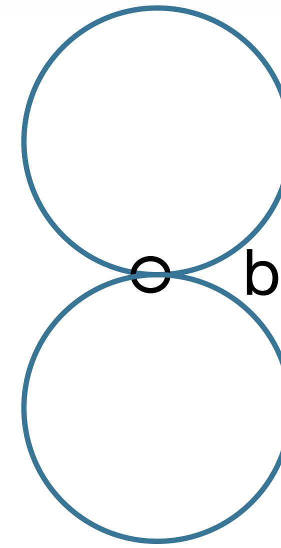
   `Codes(base) = G`

   `Codes(loop(x)) =`
   "multiplication by x"

2. `encode : Ω(K(G,1)) → G`

   `encode(p) =` $\texttt{transport}_{\texttt{Codes}}(\texttt{p}, 1_{\texttt{G}})$

$\pi_1(K(G,1)) = G$



1. `Codes : K(G,1) → 1-Type`

   `Codes(base) = G`

   `Codes(loop(x)) =`
   "multiplication by x"

2. `encode : Ω(K(G,1)) → G`

   `encode(p) =` $\texttt{transport}_{\texttt{Codes}}(\texttt{p}, \texttt{1}_{\texttt{G}})$

3. Calculate that `encode` and `loop` are mutually inverse using encode-decode method

```
comp-equiv : ∀ g -> Equiv El El
comp-equiv a = (improve (hequiv (\ x -> comp x a)
                                (\ x -> comp x (inv a))
                                (λ x → (unitr x ∘ ap (λ y → comp x y) (invr a)) ∘ assoc x a (inv a))
                                (λ x → (unitr x ∘ ap (λ y → comp x y) (invl a)) ∘ assoc x (inv a) a)))

decode' : El → Path{KG1} KG1.base KG1.base
decode' = KG1.loop

module Codes where

  f : ∀ g → (El , El-level) ≃ (El , El-level)
  f =  λ g → coe (Path-NTypes (tl 0)) (ua (comp-equiv g))

  abstract
      pri : f ident ≃ id
      pri = coe (! (Path2-NTypes (tl 0) _ _))
                (type≃-ext (ua (comp-equiv ident)) id
                           (λ x → unitr x ∘ ap≃ (type≃β (comp-equiv ident)) {x})
                 ∘ Path-NTypesβ (tl 0) (ua (comp-equiv ident)))

      prc : ∀ g1 g2 -> f (comp g1 g2) ≃ f g2 ∘ f g1
      prc g1 g2 = coe (! (Path2-NTypes (tl 0) _ _))
                      (! (ap-∘ fst (f g2) (f g1)) ∘
                       ! (ap (λ x → x ∘ ap fst (f g1)) (Path-NTypesβ (tl 0) (ua (comp-equiv g2)))) ∘
                       ! (ap (λ x → ua (comp-equiv g2) ∘ x) (Path-NTypesβ (tl 0) (ua (comp-equiv g1)))) ∘
                       type≃-ext (ua (comp-equiv (comp g1 g2))) (ua (comp-equiv g2) ∘ ua (comp-equiv g1))
                                 (λ g → ! (ap≃ (transport-∘ (λ x → x) (ua (comp-equiv g2)) (ua (comp-equiv g1)))) ∘
                                        (! (ap≃ (type≃β (comp-equiv g2))) ∘ ! (ap (λ x → fst (comp-equiv g2) x) (ap≃ (type≃β (comp-equiv g1)))) ∘
                                        ! (assoc g g1 g2)) ∘
                                        ap≃ (type≃β (comp-equiv (comp g1 g2))))
                       ∘ Path-NTypesβ (tl 0) (ua (comp-equiv (comp g1 g2))))

Codes : KG1 → NTypes (tl 0)
Codes = KG1-rec (NTypes-level (tl 0))
                (El , El-level)
                (record { f = Codes.f;
                          pres-ident = Codes.pri ;
                          pres-comp = Codes.prc })


abstract
  transport-Codes-loop : ∀ g g' -> (transport (fst o Codes) (KG1.loop g) g') ≃ comp g' g
  transport-Codes-loop g g' = transport (fst o Codes) (KG1.loop g) g' ≃⟨ ap≃ (transport-ap-assoc' fst Codes (KG1.loop g)) ⟩
                              transport fst (ap Codes (KG1.loop g)) g' ≃⟨ ap (λ x → transport fst x g') (KG1.KG1-rec/βloop{_}{NTypes-level (tl 0)}
                                                                         (record {f = Codes.f; pres-ident = Codes.pri; pres-comp = Codes.prc })) ⟩
                              transport fst (coe (Path-NTypes (tl 0)) (ua (comp-equiv g))) g' ≃⟨ ap≃ (transport-ap-assoc fst (coe (Path-NTypes (tl 0))
                                                                                                   (ua (comp-equiv g)))) ⟩
                              coe (fst≃ (coe (Path-NTypes (tl 0)) (ua (comp-equiv g)))) g' ≃⟨ ap (λ x → coe x g') (Path-NTypesβ (tl 0) (ua (comp-equiv g))) ⟩
                              coe (ua (comp-equiv g)) g' ≃⟨ ap≃ (type≃β (comp-equiv g)) ⟩
                              comp g' g ∎

encode : {x : KG1} -> Path KG1.base x -> fst (Codes x)
encode α = transport (fst o Codes) α ident

abstract
  encode-decode' : ∀ x -> encode (decode' x) ≃ x
  encode-decode' x = encode (decode' x) ≃⟨ id ⟩
                     encode (KG1.loop x) ≃⟨ id ⟩
                     transport (fst o Codes) (KG1.loop x) ident ≃⟨ transport-Codes-loop x ident ⟩
                     comp ident x ≃⟨ unitl x ⟩
                     x ∎

decode : {x : _} -> fst (Codes x) -> Path KG1.base x
decode {x} = KG1-elim (λ x' → (fst (Codes x') → Path KG1.base x') , Πlevel (λ _ → path-preserves-level KG1.level))
                      decode'
                      loop'
                      (λ _ → HSet-UIP (Πlevel (λ _ → use-level KG1.level _ _)) _ _ _ _)
                      (λ _ _ → HSet-UIP (Πlevel (λ _ → use-level KG1.level _ _)) _ _ _ _)
                      x where
  abstract
    loop' : ∀ g -> transport (\x -> fst (Codes x) -> Path KG1.base x) (KG1.loop g) decode' ≃ decode'
    loop' = (λ g → transport-→-from-square (fst o Codes) (Path KG1.base) (KG1.loop g) decode' decode'
                        (λ≃ (\ g' ->
                          (transport (Path KG1.base) (KG1.loop g) (decode' g') ≃⟨ id ⟩
                           transport (Path KG1.base) (KG1.loop g) (KG1.loop g') ≃⟨ transport-Path-right (KG1.loop g) (KG1.loop g') ⟩
                           (KG1.loop g) ∘ (KG1.loop g') ≃⟨ ! (KG1.loop-comp g' g) ⟩
                           KG1.loop (comp g' g) ≃⟨ ap KG1.loop (! (transport-Codes-loop g g')) ⟩
                           KG1.loop (transport (fst o Codes) (KG1.loop g) g') ≃⟨ id ⟩
                           decode' (transport (fst o Codes) (KG1.loop g) g') ∎))))


decode-encode : ∀ {x} (α : Path KG1.base x) -> decode (encode α) ≃ α
decode-encode id = KG1.loop-ident

Ω1[KG1]-Equiv-G : Equiv (Path{KG1} KG1.base KG1.base) El
Ω1[KG1]-Equiv-G = improve (hequiv encode decode decode-encode encode-decode')
```

Eilenberg-MacLane space

So far: For a group G, can define a space K(G,1) such that

$\pi_1(K(G,1)) = G$ and

$\pi_k(K(G,1)) = 1$ otherwise

Outline

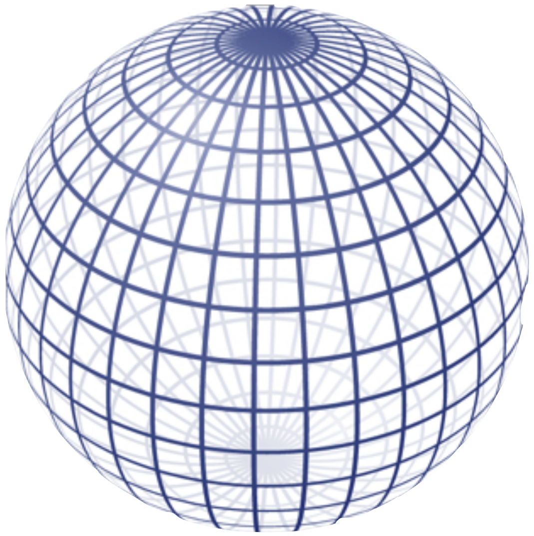
1. Eilenberg-MacLane spaces

2. $K(G,1)$

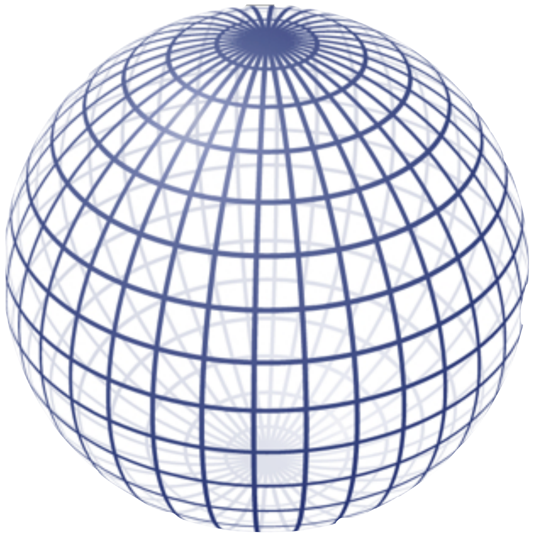
3. **$K(G,n)$**

Sphere S^2

Sphere S^2



Sphere S^2



- π_1 is trivial: inside of any loop can be filled
- π_2 is $\mathbb{Z}$: 2-paths on sphere = paths on circle
- $\pi_{k>2}$ is …

Homotopy Groups of S^2

k[th] homotopy group

n-dimensional sphere

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^3	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^4	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{12}$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{24}\times\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{120}\times\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^5$
S^5	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{30}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{72}\times\mathbf{Z}_2$
S^6	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{24}\times\mathbf{Z}_2$	$\mathbf{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{120}$	$\mathbf{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{120}$

[image from wikipedia]

Homotopy Groups of S^2

k[th] homotopy group

n-dimensional sphere

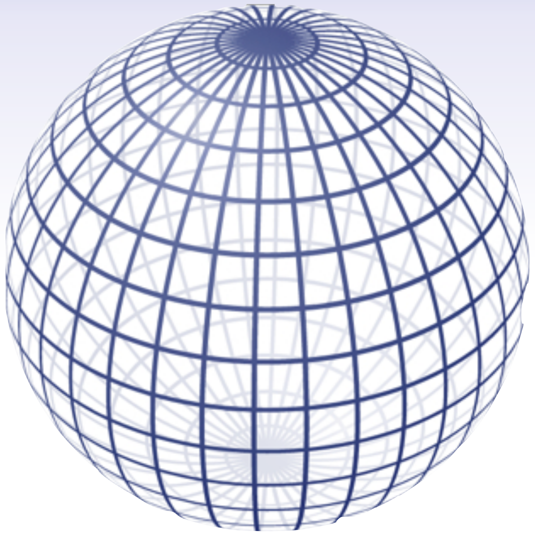
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

[image from wikipedia]

K($\mathbb{Z}$,2)

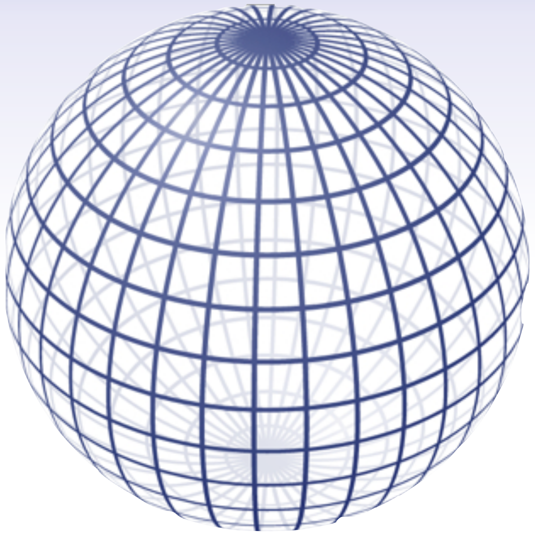
K($\mathbb{Z}$,2)

$K(\mathbb{Z},2)$



Define $K(\mathbb{Z},2) = ||S^2||_2$

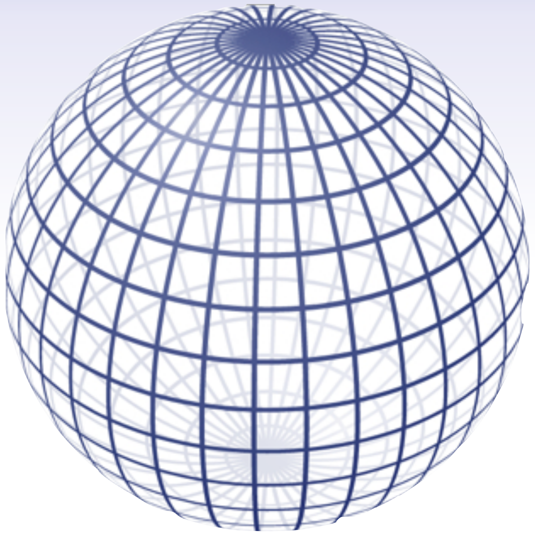
K($\mathbb{Z}$,2)



Define K($\mathbb{Z}$,2) = $\|S^2\|_2$

2-truncation = kill all paths at level higher than 2

$K(\mathbb{Z},2)$



Define $K(\mathbb{Z},2) = ||S^2||_2$

2-truncation = kill all paths at level higher than 2

- π_1 is trivial: same as S^2
- π_2 is $\mathbb{Z}$: same as S^2
- $\pi_{k>2}$ is trivial

$\pi_n(S^n)$

k^th homotopy group

n-dimensional sphere

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^3	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^4	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{12}$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{24}\times\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{120}\times\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^5$
S^5	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{30}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{72}\times\mathbf{Z}_2$
S^6	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{24}\times\mathbf{Z}_2$	$\mathbf{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{120}$	$\mathbf{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{120}$

[image from wikipedia]

K($\mathbb{Z}$,n)

Define K($\mathbb{Z}$,n) = $||S^n||_n$

- $\pi_{k<n}$ is trivial
- π_n is $\mathbb{Z}$
- $\pi_{k>n}$ is trivial

$K(\mathbb{Z},n)$

Define $K(\mathbb{Z},n) = ||S^n||_n$

- $\pi_{k<n}$ is trivial
- π_n is $\mathbb{Z}$
- $\pi_{k>n}$ is trivial

[HoTT proofs: L.,Brunerie,Lumsdaine]

K($\mathbb{Z}$,n)

Define K($\mathbb{Z}$,n) = $\|S^n\|_n$

- $\pi_{k<n}$ is trivial
- π_n is $\mathbb{Z}$
- $\pi_{k>n}$ is trivial

Generalize to other groups G?

[HoTT proofs: L.,Brunerie,Lumsdaine]

Suspension

ΣA is a higher inductive type generated by

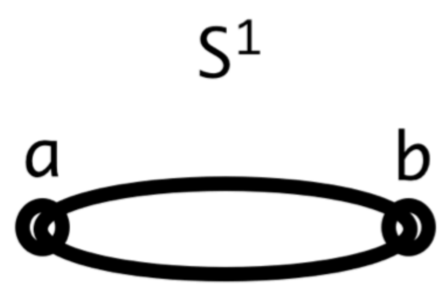
Suspension

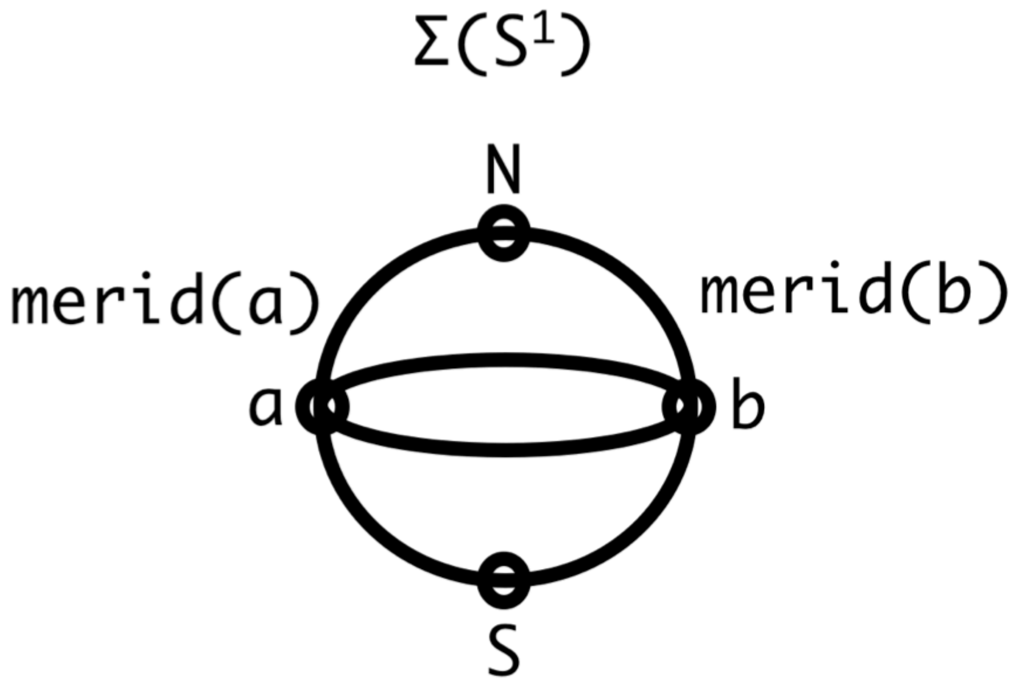
Σ A is a higher inductive type generated by

```
N : Σ A
S : Σ A
merid : A → Σ A
```

Spheres

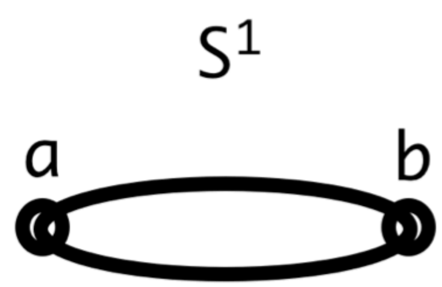
$S^2 = \Sigma\, S^1$

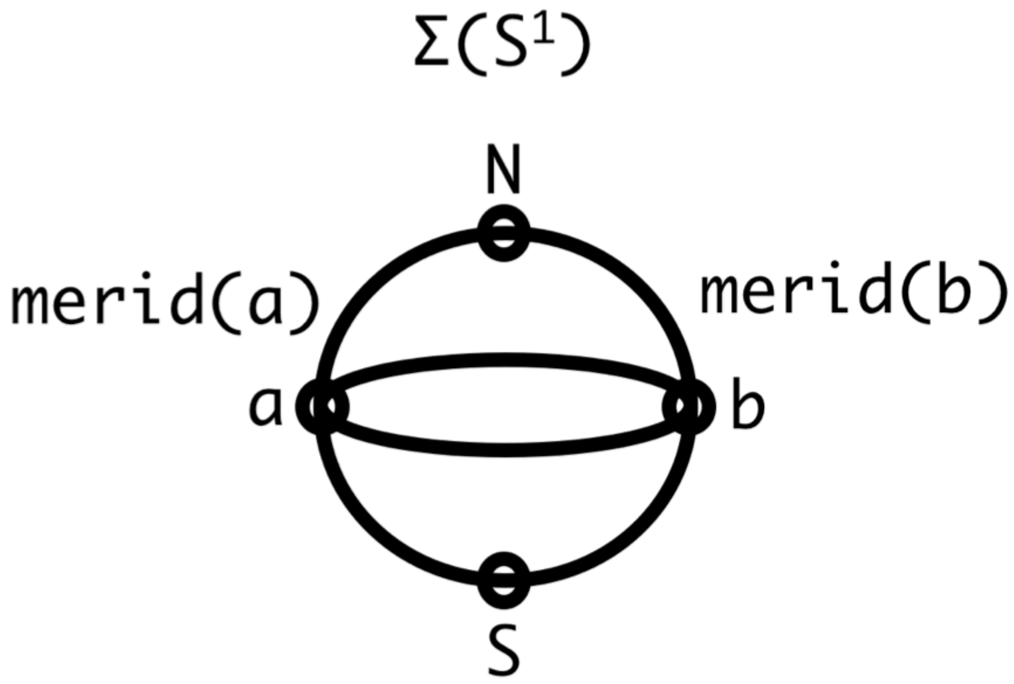




Spheres

$S^2 = \Sigma\ S^1$





$S^n = \Sigma^{n-1}\ S^1$

K(G,n)

Define $K(G,n) = ||\Sigma^{n-1} K(G,1)||_n$

Theorem:

- $\pi_{k<n}$ is trivial
- π_n is G
- $\pi_{k>n}$ is trivial

$\pi_n(K(G,n)) = G$

Define $K(G,n) = \|\Sigma^{n-1} K(G,1)\|_n$

$\pi_n(K(G,n)) = G$

Define $K(G,n) = ||\Sigma^{n-1} K(G,1)||_n$

$\pi_{n+1}(K(G,n+1)) = \pi_{n+1} ||\Sigma^n K(G,1)||_{n+1}$

$\pi_n(K(G,n)) = G$

Define $K(G,n) = \|\Sigma^{n-1} K(G,1)\|_n$

$$\pi_{n+1}(K(G,n+1)) = \pi_{n+1} \|\Sigma^n K(G,1)\|_{n+1}$$
$$= \pi_{n+1} \; \Sigma^n K(G,1)$$

$\pi_n(K(G,n)) = G$

Define $K(G,n) = ||\Sigma^{n-1} K(G,1)||_n$

$\pi_{n+1}(K(G,n+1)) = \pi_{n+1} ||\Sigma^n K(G,1)||_{n+1}$

$= \pi_{n+1} \ \Sigma^n K(G,1)$

$= \pi_n \ \Sigma^{n-1} K(G,1)$

$\pi_n(K(G,n)) = G$

Define $K(G,n) = ||\Sigma^{n-1} K(G,1)||_n$

$$\begin{aligned}\pi_{n+1}(K(G,n+1)) &= \pi_{n+1} ||\Sigma^n K(G,1)||_{n+1} \\ &= \pi_{n+1} \; \Sigma^n K(G,1) \\ &= \pi_n \; \Sigma^{n-1} K(G,1)\end{aligned}$$

Freudenthal suspension theorem; use HoTT proof by Lumsdaine

$\pi_n(K(G,n)) = G$

Define $K(G,n) = \|\Sigma^{n-1} K(G,1)\|_n$

$$\begin{aligned}
\pi_{n+1}(K(G,n+1)) &= \pi_{n+1} \|\Sigma^n K(G,1)\|_{n+1} \\
&= \pi_{n+1}\ \Sigma^n K(G,1) \\
&= \pi_n\ \Sigma^{n-1} K(G,1) \\
&= \pi_n\ \|\ \Sigma^{n-1} K(G,1)\ \|_n
\end{aligned}$$

Freudenthal suspension theorem; use HoTT proof by Lumsdaine

$\pi_n(K(G,n)) = G$

Define $K(G,n) = \|\Sigma^{n-1} K(G,1)\|_n$

$$\begin{aligned}
\pi_{n+1}(K(G,n+1)) &= \pi_{n+1} \|\Sigma^n K(G,1)\|_{n+1} \\
&= \pi_{n+1} \; \Sigma^n K(G,1) \\
&= \pi_n \; \Sigma^{n-1} K(G,1) \\
&= \pi_n \; \| \Sigma^{n-1} K(G,1) \|_n \\
&= \pi_n \; K(G,n)
\end{aligned}$$

Freudenthal suspension theorem; use HoTT proof by Lumsdaine

```
stable2 : π (k +1) (KG (n +1)) (base^ (n +1)) ≃ π k (KG n) (base^ n)
stable2 = π (k +1) (KG (n +1)) (base^ (n +1))              ≃⟨ (π<=Trunc (k +1) (n +1) (<=SCong lte) (FS.base'^ (n +1))) ⟩
          π (k +1) (Susp^ (S n -1pn) KG1) (FS.base'^ (n +1))  ≃⟨ ! (FS.Stable.stable k n (k<=n->k<=2n-2 k n indexing)) ⟩
          π k (Susp^ (n -1pn) KG1) (FS.base'^ n)              ≃⟨ ! (π<=Trunc k n lte (FS.base'^ n)) ⟩
          π k (KG n) (base^ n) ∎
```

Define $K(G,n) = \|\Sigma^{n-1} K(G,1)\|_n$

$$
\begin{aligned}
\pi_{n+1}(K(G,n+1)) &= \pi_{n+1} \|\Sigma^n K(G,1)\|_{n+1} \\
&= \pi_{n+1} \; \Sigma^n K(G,1) \\
&= \pi_n \; \Sigma^{n-1} K(G,1) \\
&= \pi_n \; \| \Sigma^{n-1} K(G,1) \|_n \\
&= \pi_n \; K(G,n)
\end{aligned}
$$

Freudenthal suspension theorem; use HoTT proof by Lumsdaine

```
stable2 : π (k +1) (KG (n +1)) (base^ (n +1)) ≃ π k (KG n) (base^ n)
stable2 = π (k +1) (KG (n +1)) (base^ (n +1))              ≃⟨ (π<=Trunc (k +1) (n +1) (<=SCong lte) (FS.base'^ (n +1))) ⟩
          π (k +1) (Susp^ (S n -1pn) KG1) (FS.base'^ (n +1))  ≃⟨ ! (FS.Stable.stable k n (k<=n->k<=2n-2 k n indexing)) ⟩
          π k (Susp^ (n -1pn) KG1) (FS.base'^ n)              ≃⟨ ! (π<=Trunc k n lte (FS.base'^ n)) ⟩
          π k (KG n) (base^ n) ∎
```

Define $K(G,n) = \|\Sigma^{n-1} K(G,1)\|_n$

$$\begin{aligned}
\pi_{n+1}(K(G,n+1)) &= \pi_{n+1} \|\Sigma^n K(G,1)\|_{n+1} \\
&= \pi_{n+1} \; \Sigma^n K(G,1) \\
&= \pi_n \; \Sigma^{n-1} K(G,1) \\
&= \pi_n \; \| \Sigma^{n-1} K(G,1) \|_n \\
&= \pi_n \; K(G,n)
\end{aligned}$$

merid : A → Ω (Σ A) is an equivalence?

Freudenthal suspension theorem; use HoTT proof by Lumsdaine

Freudenthal Suspension Thm

k^th homotopy group

n-dimensional sphere

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^3	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^4	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{12}$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{24}\times\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{120}\times\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^5$
S^5	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{30}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{72}\times\mathbf{Z}_2$
S^6	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{24}\times\mathbf{Z}_2$	$\mathbf{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{120}$	$\mathbf{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{120}$

[image from wikipedia]

Freudenthal

Theorem: If A is n-connected (trivial up to level n), then $||A||_{2n} = ||\Omega\ (\Sigma\ A)||_{2n}$

HoTT Proof [Lumsdaine]: generalization of encode-decode method

Freudenthal

Theorem: If A is n-connected (trivial up to level n), then $||A||_{2n} = ||\Omega\ (\Sigma\ A)||_{2n}$

HoTT Proof [Lumsdaine]: generalization of encode-decode method

Corollary:

$$\ldots = \pi_4(K(G,4)) = \pi_3(K(G,3)) = \pi_2(K(G,2))$$

$\pi_n(K(G,n)) = G$

By Freudenthal:

$\ldots = \pi_4(K(G,4)) = \pi_3(K(G,3) = \pi_2(K(G,2))$

$\pi_n(K(G,n)) = G$

By Freudenthal:

$\ldots = \pi_4(K(G,4)) = \pi_3(K(G,3) = \pi_2(K(G,2))$

By above:

$\pi_1(K(G,1)) = G$

$\pi_n(K(G,n)) = G$

By Freudenthal:

$\ldots = \pi_4(K(G,4)) = \pi_3(K(G,3) = \pi_2(K(G,2))$

By another encode-decode proof:

$\pi_2(K(G,2)) = \pi_1(K(G,1))$

By above:

$\pi_1(K(G,1)) = G$

Formalization

Formalization

* 10,000 line HoTT library

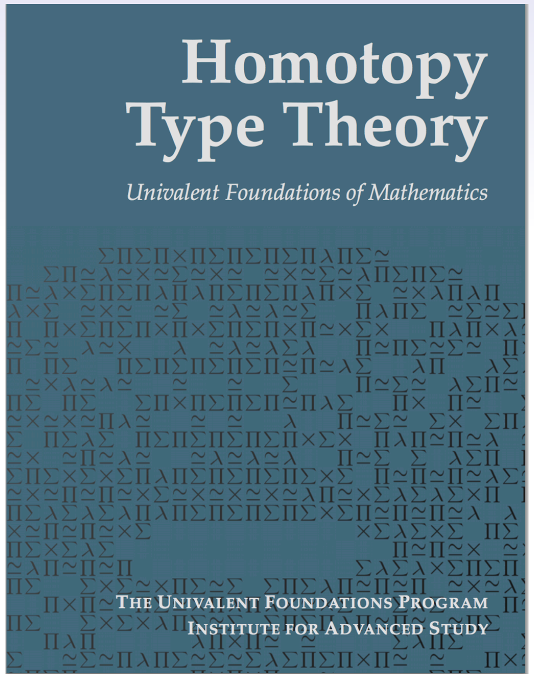
Formalization

- 10,000 line HoTT library
- + 250 lines for Freudenthal Suspension Theorem

Formalization

- 10,000 line HoTT library
- + 250 lines for Freudenthal Suspension Theorem
- + 750 lines for K(G,n)

Reading list



1. The HoTT Book

2. Homotopy theory in Agda:
   Fundamental group of the circle [LICS'13]
   $\pi_n(S^n) = \mathbb{Z}$ [CPP'13]
   K(G,n) [LICS'14]
   `github.com/dlicata335/`
   `github.com/hott/hott-agda`

3. Blog: `homotopytypetheory.org`