Synthetic Mathematics in Modal Dependent Type Theories

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Carnegie Mellon University

Tutorial 1

Joint work with/work by

- Lect 1,3,5: L., Mitchell Riley, Michael Shulman
- Lect 2: W., Urs Schreiber, Egbert Rijke,
  Shulman, Bas Spitters
- Lect 4: Shulman
- Lect 6: W., Schreiber, Jacob Gross, L., Max S. New, Ian Orton, Jennifer Paykin, Shulman
- Bas's talk on Thursday: Ranald Clouston, Bassel Mannaa, Rasmus Ejlers Møgelberg, Andrew M. Pitts, Spitters; L., Pitts, Ian Orton, and Spitters

Motivation

Synthetic homotopy theory

Synthetic homotopy theory

- Use types in HoTT to talk about ∞-groupoids: e.g. define $\mathbf{S}^1$ as higher inductive type with

  base : $\mathbf{S}^1$
  loop : Path $\mathbf{S}^1$ base base

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  base : $\mathbf{S}^1$
  loop : Path $\mathbf{S}^1$ base base

- "Calculate" homotopy groups: e.g. prove

$$\text{Path } \mathbf{S}^1 \text{ base base} \simeq \mathbb{Z}$$

  using $\mathbf{S}^1$-induction, univalence

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  any continuous map $\mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point

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- … but no "points" in HIT $\mathbf{D} \cong 1$
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- Internally "compile":
  formalize syntax of TT as QIIT,
  simplicial or cubical model, initiality,
  topological spaces, Quillen equivalence,
  quote HoTT proofs as encoded syntax…

Cohesive HoTT [Schreiber,Shulman]

synthetic homotopy theory
as in homotopy type theory

types are ∞-groupoids

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+
synthetic topology
as in *axiomatic cohesion*

types are ∞-groupoids

also have topological structure on every level

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relate HIT circle to $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ internally

Axiomatic cohesion [Lawvere]

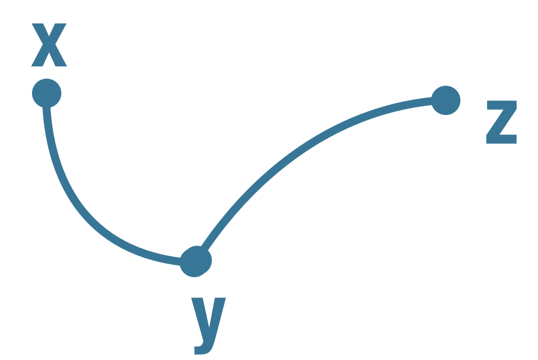
$$\begin{array}{c} \text{Spaces} \\ \Big\downarrow{\scriptstyle \Gamma} \\ \text{Sets} \end{array}$$

Axiomatic cohesion [Lawvere]

Spaces

$\downarrow \Gamma$

Sets



{x,y,z}

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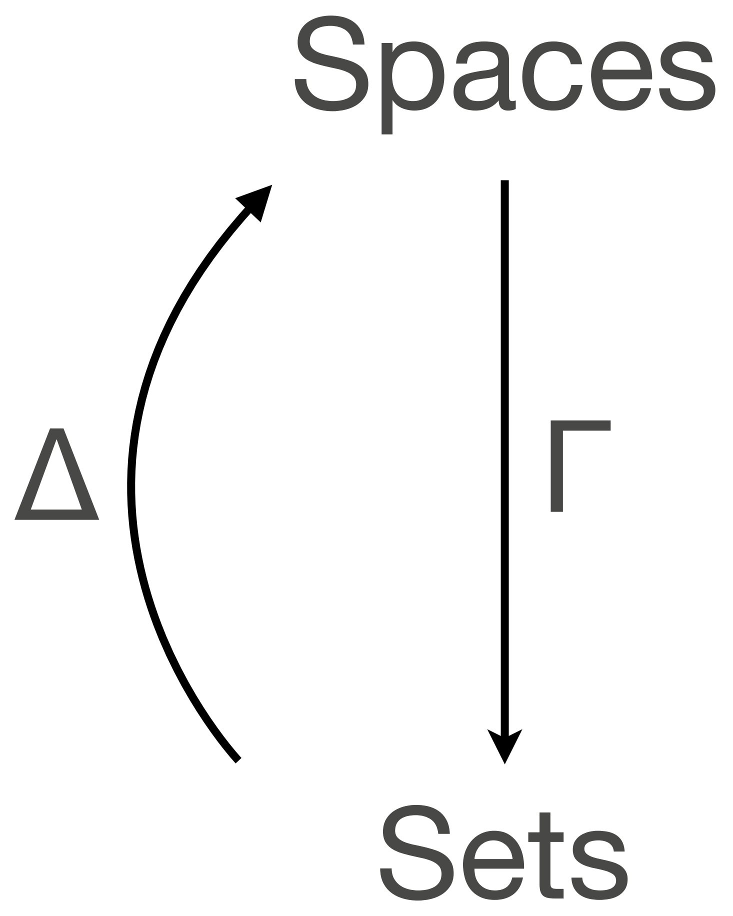

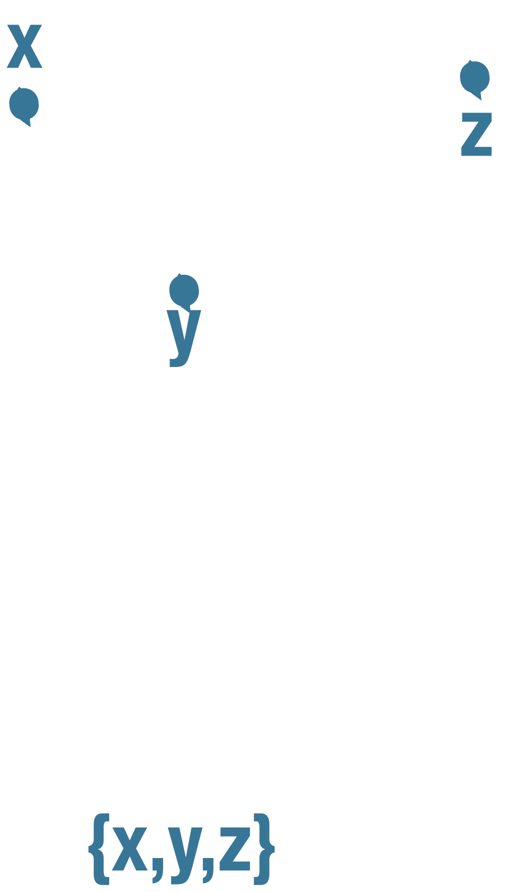

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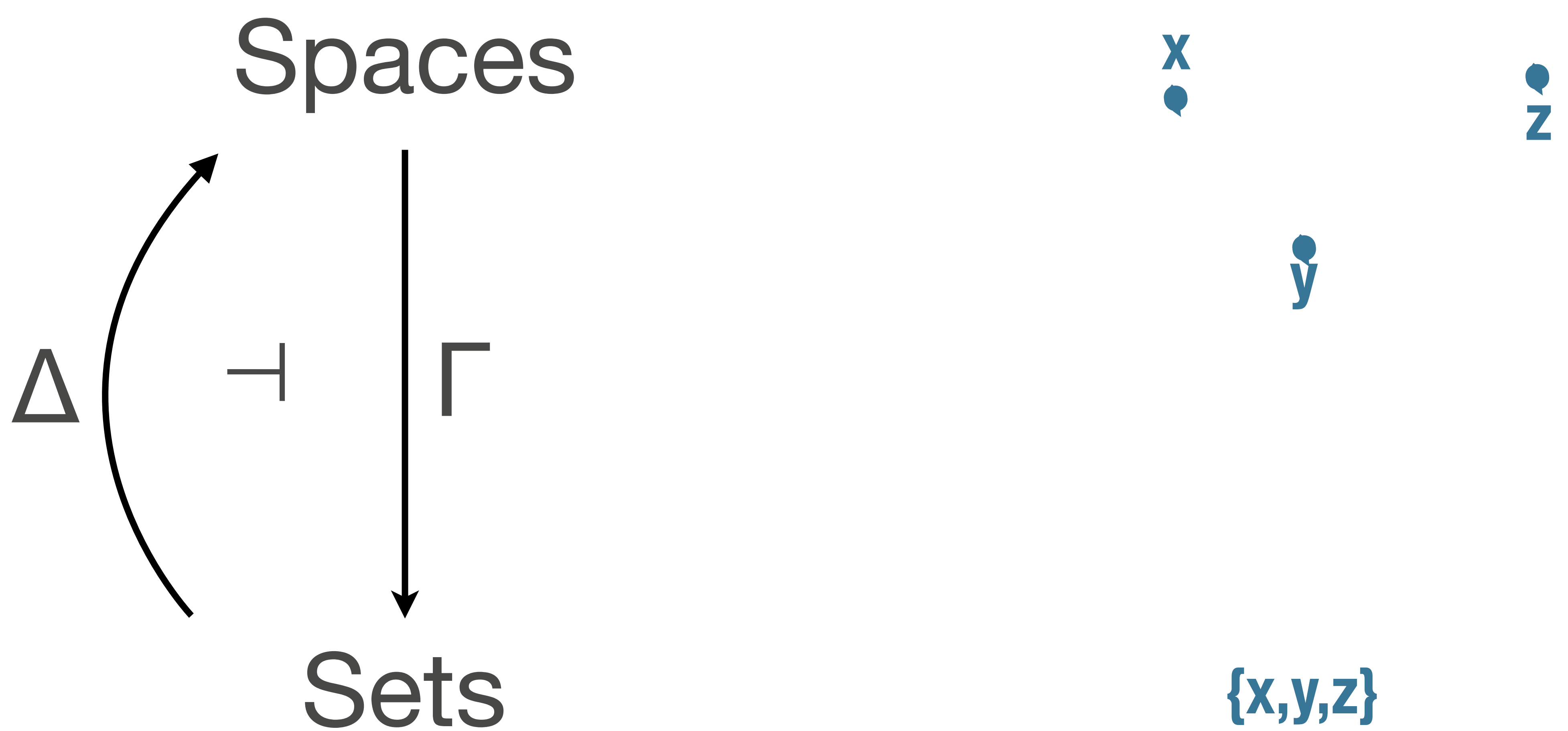

$$\frac{\Delta X \to_{\text{Spaces}} S}{X \to_{\text{Sets}} \Gamma S}$$

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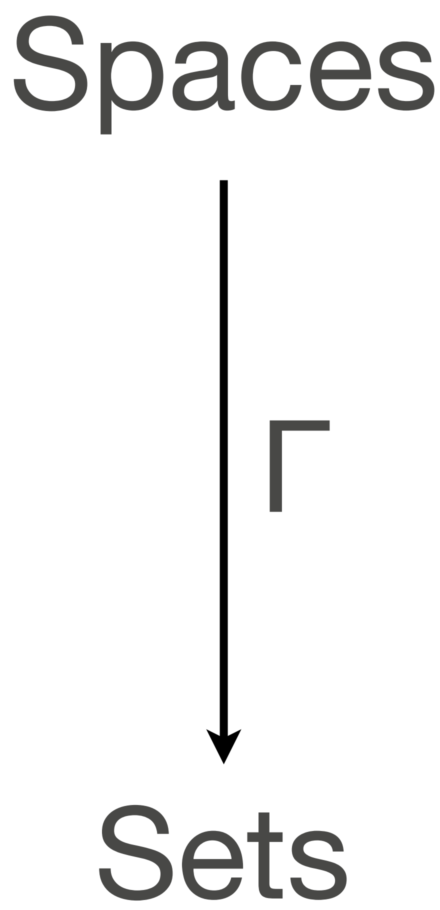

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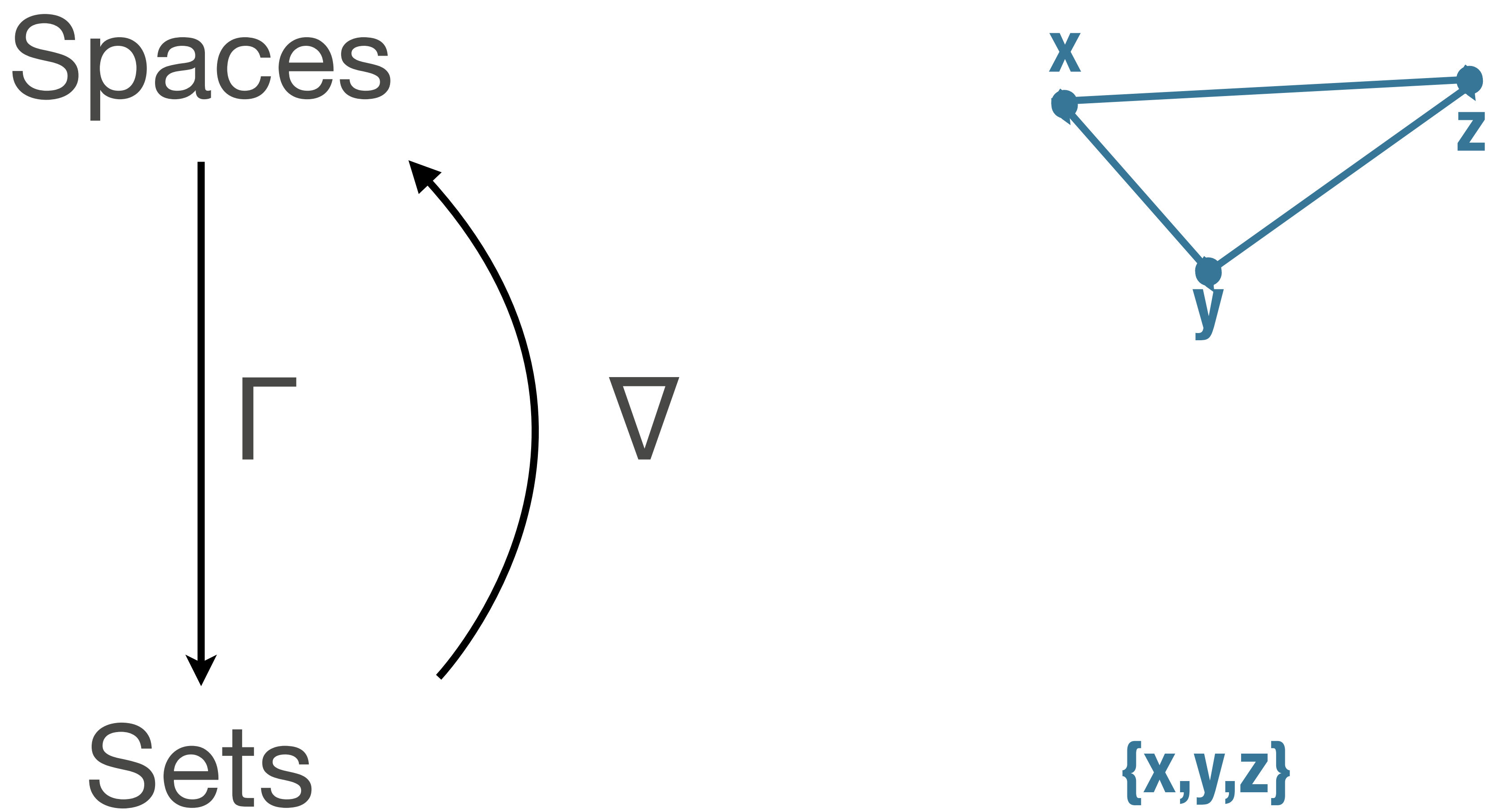
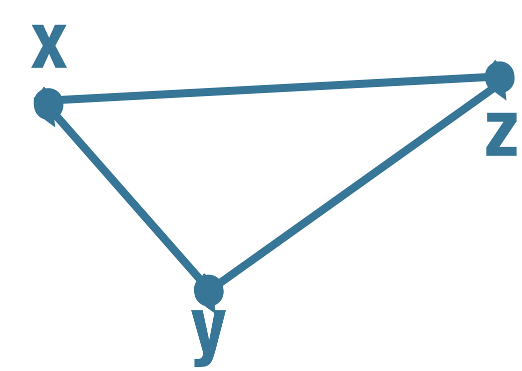


Axiomatic cohesion [Lawvere]

Spaces

$$\Gamma \dashv \nabla$$

Sets

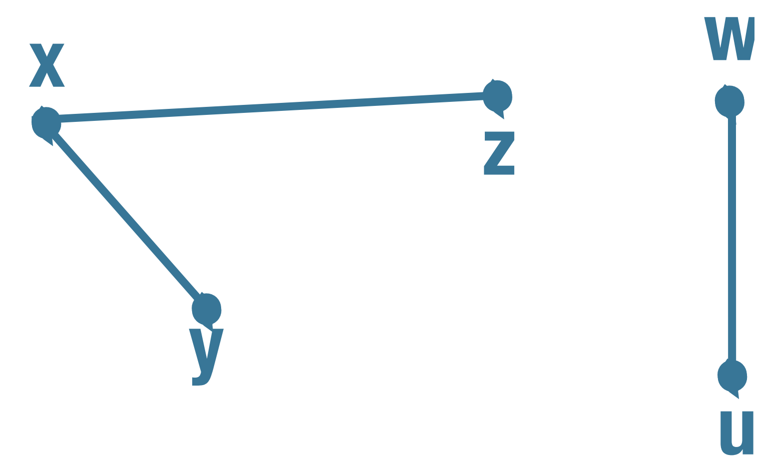


{x,y,z}

$$\frac{S \to_{\text{Spaces}} \nabla Y}{\Gamma S \to_{\text{Sets}} Y}$$

Axiomatic cohesion [Lawvere]

$$\text{Spaces} \; \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} \!\!\!\!\; \text{Sets}, \qquad \Delta \dashv \Gamma \dashv \nabla, \quad \Gamma : \text{Spaces} \to \text{Sets}, \quad \Delta, \nabla : \text{Sets} \to \text{Spaces}$$



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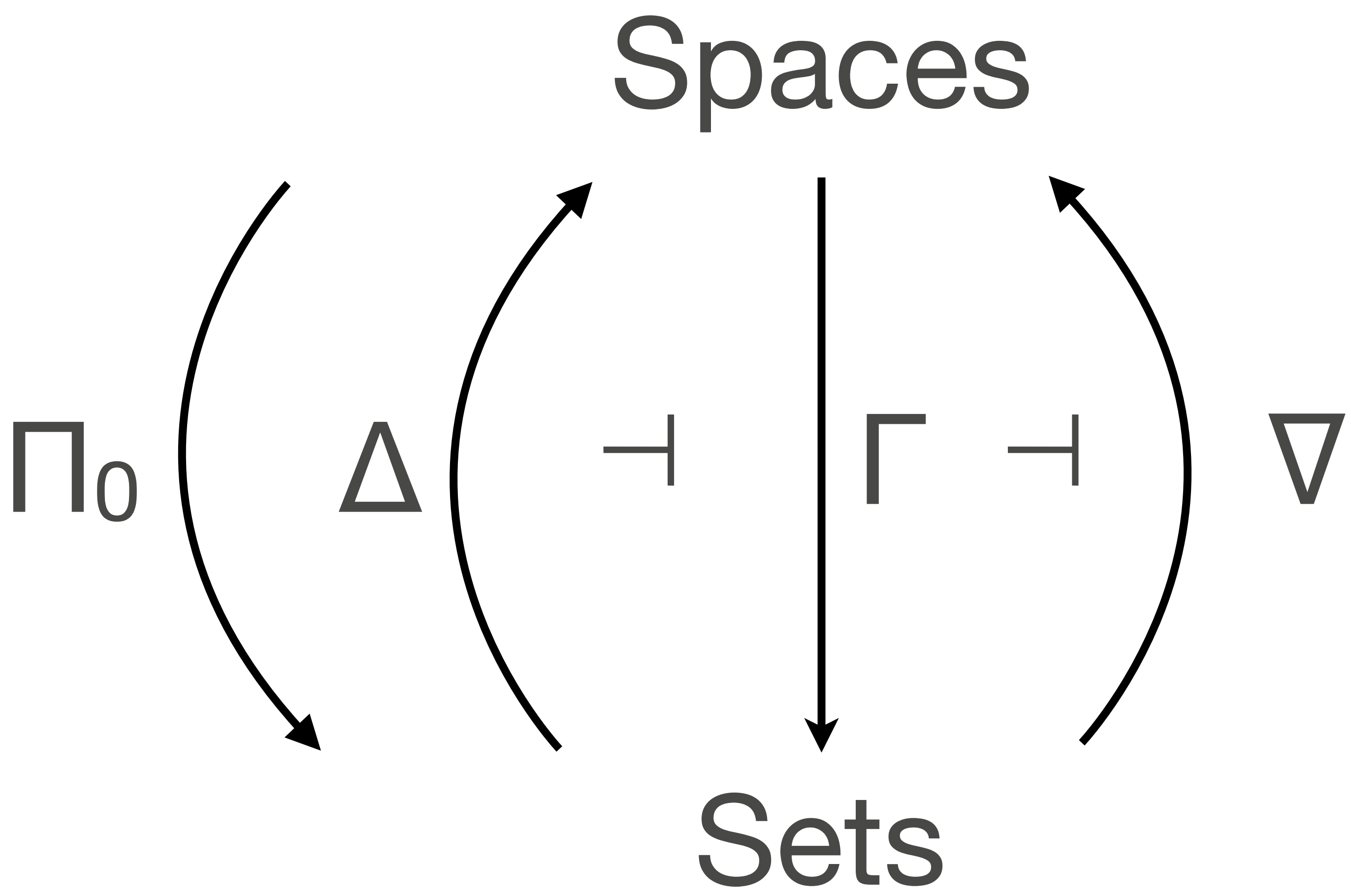

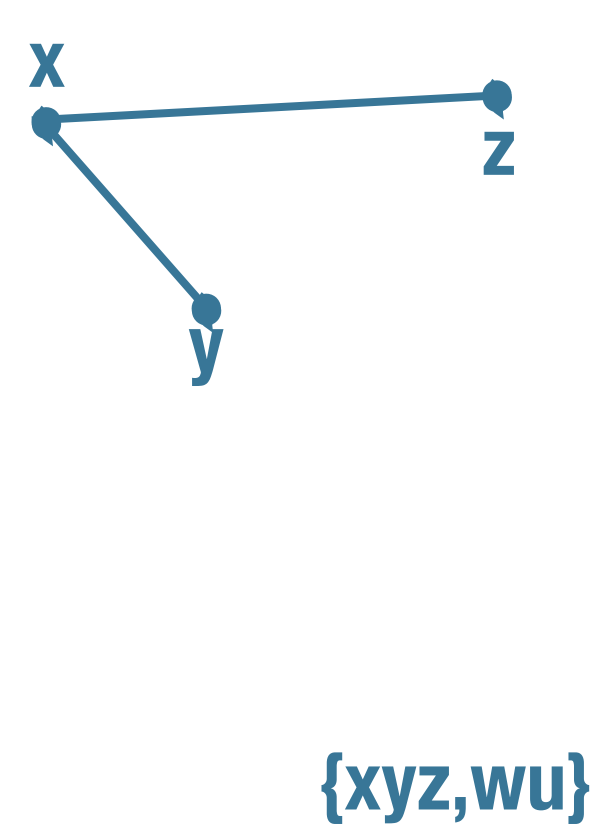

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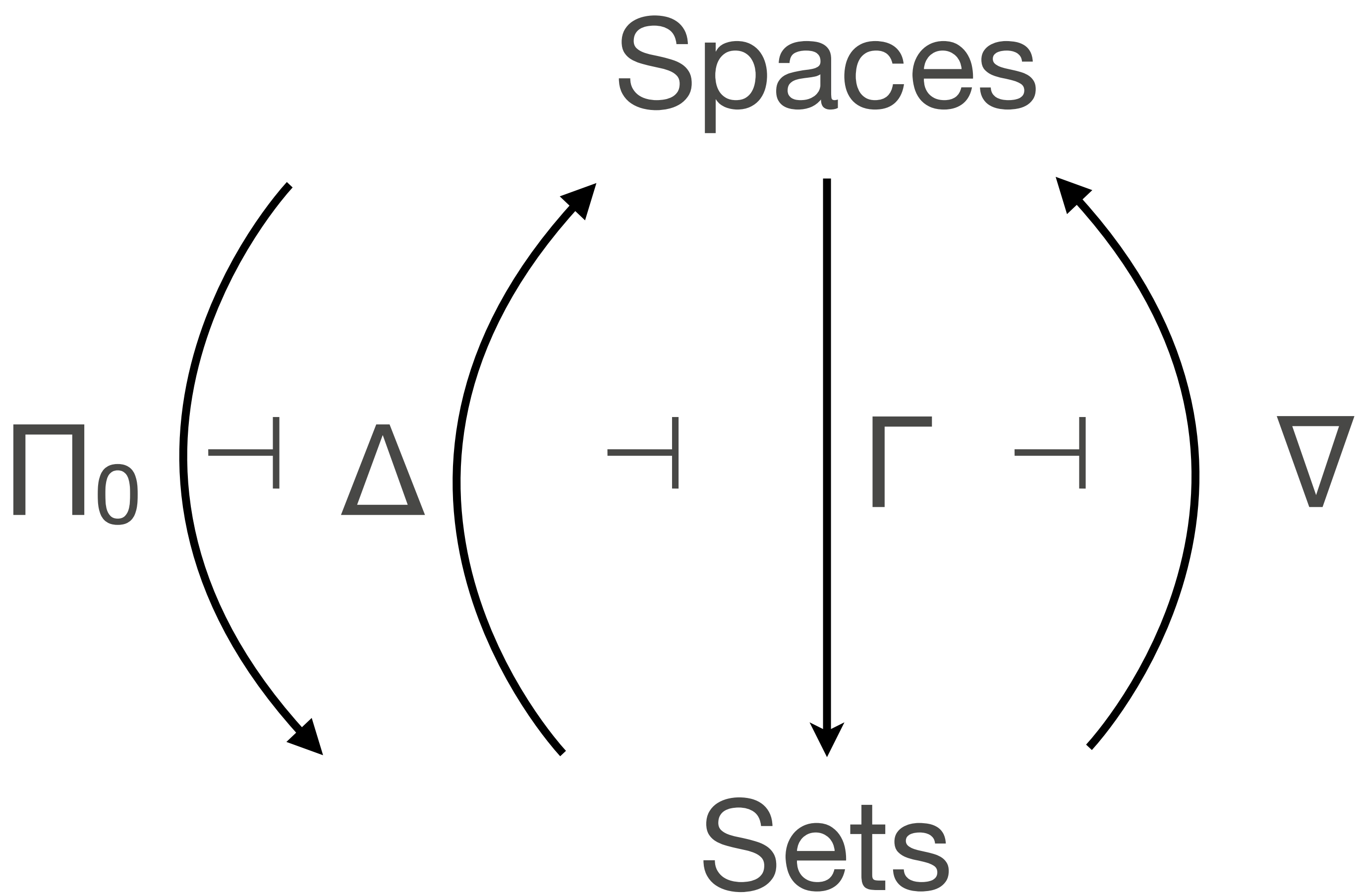
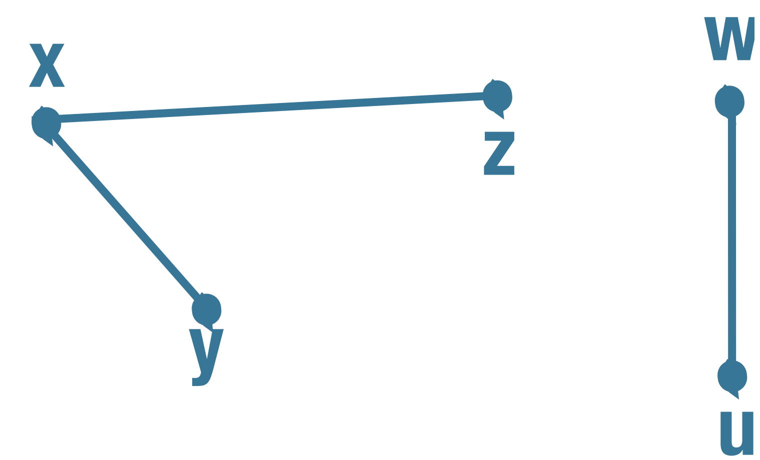


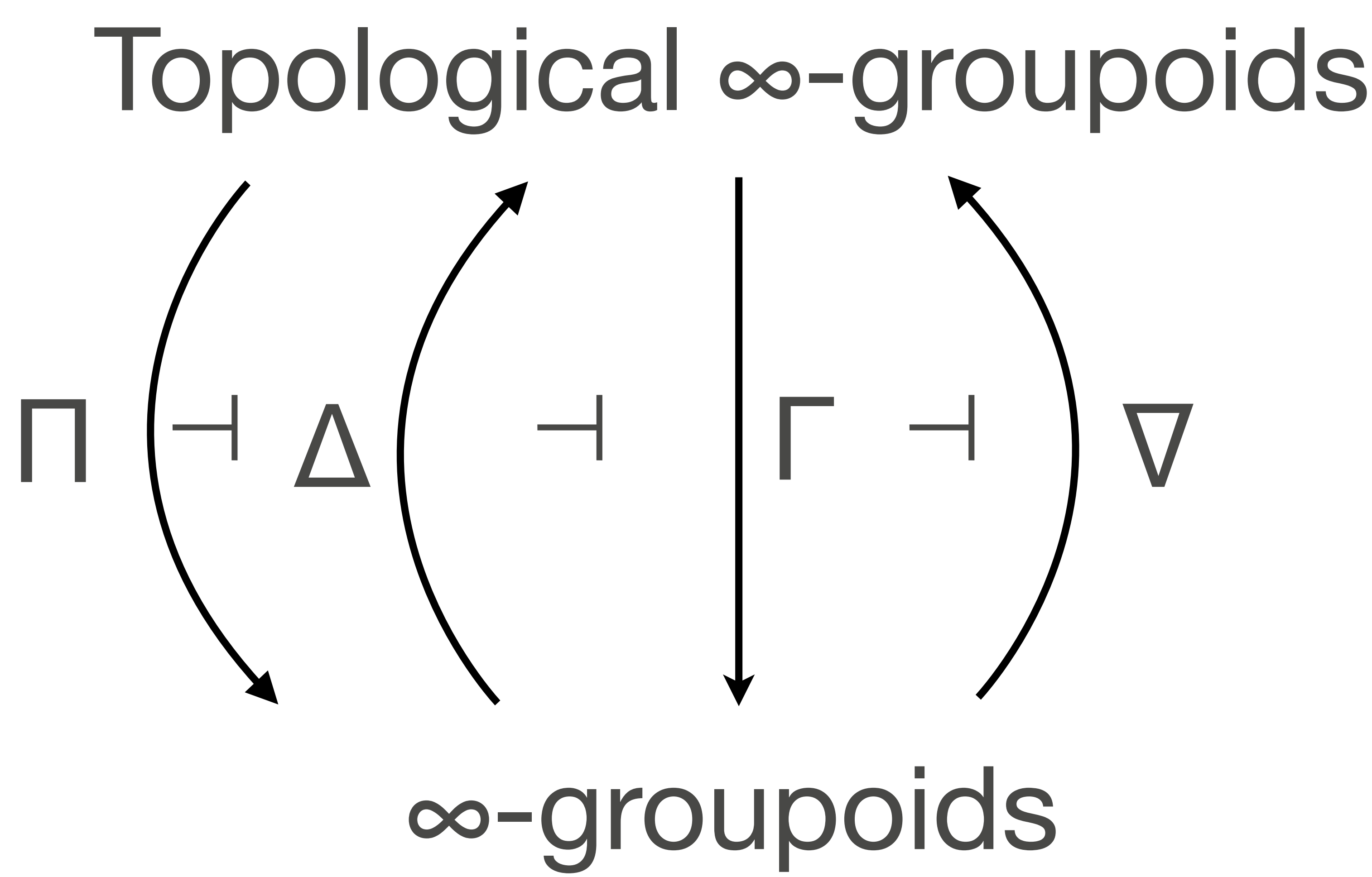


{xyz,wu}

$$\frac{S \to_{Spaces} \Delta\, Y}{\Pi_0\, S \to_{Sets} Y}$$

∞-categorical Cohesion

[Schreiber,Shulman]



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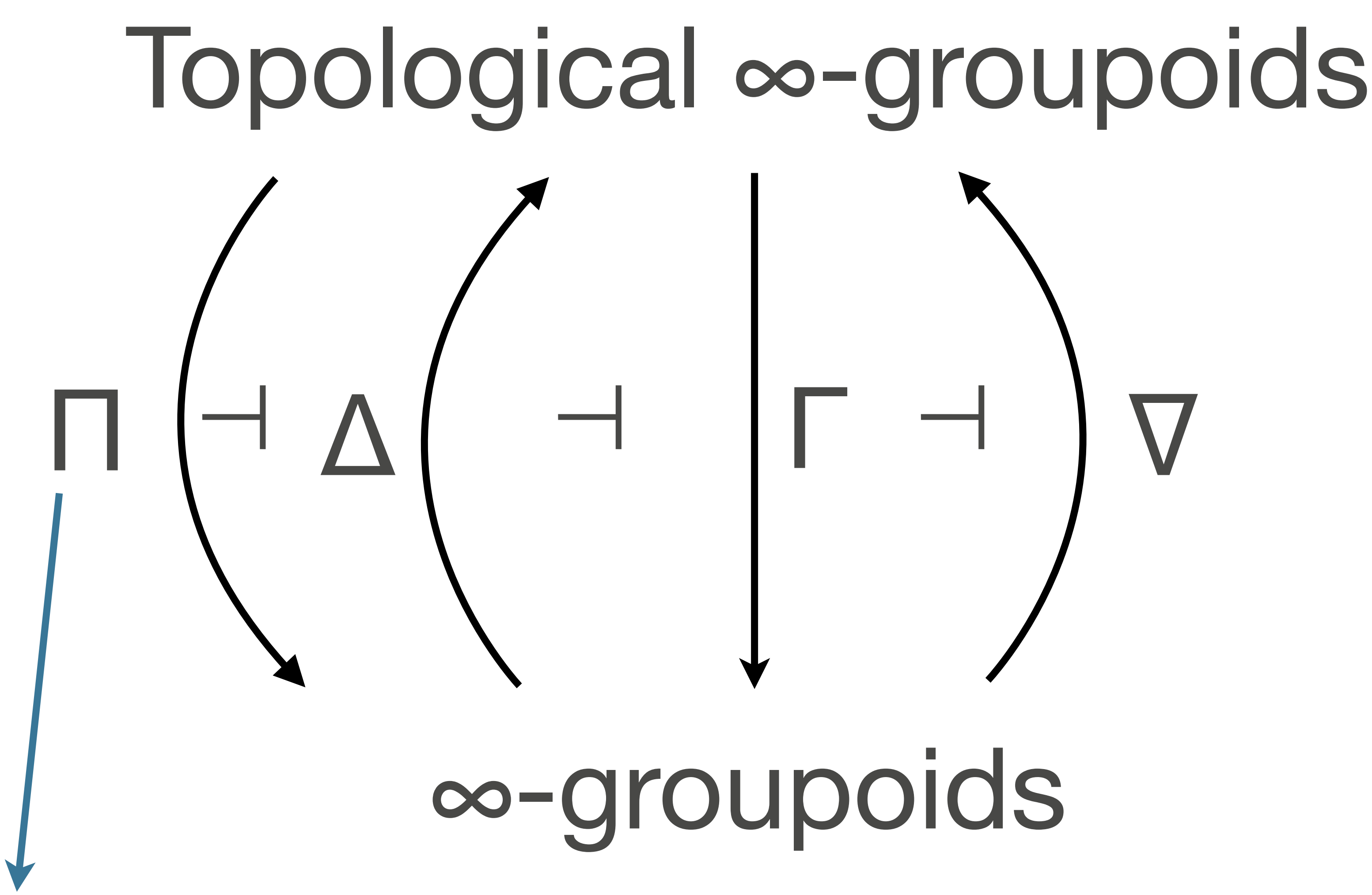

fundamental ∞-groupoid! e.g. ΔΠ(topological $\mathbb{S}^1$) = HIT S^1

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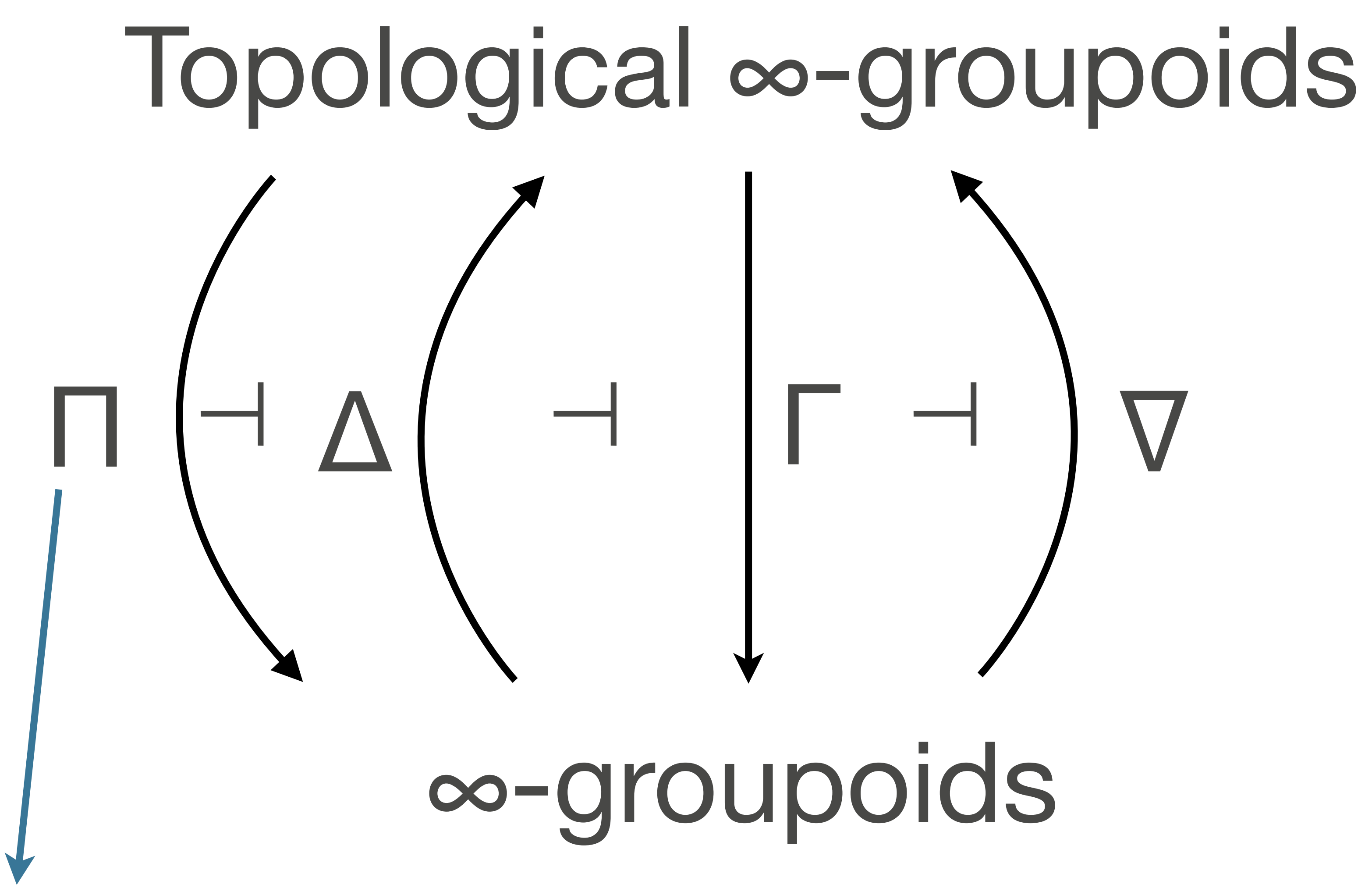

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Δ and ∇ full and faithful…

∞-categorical cohesion

Topological ∞-groupoids

$$\int \dashv \flat \dashv \sharp$$

Topological ∞-groupoids

$$\int = \Delta\Pi$$

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∞-categorical cohesion

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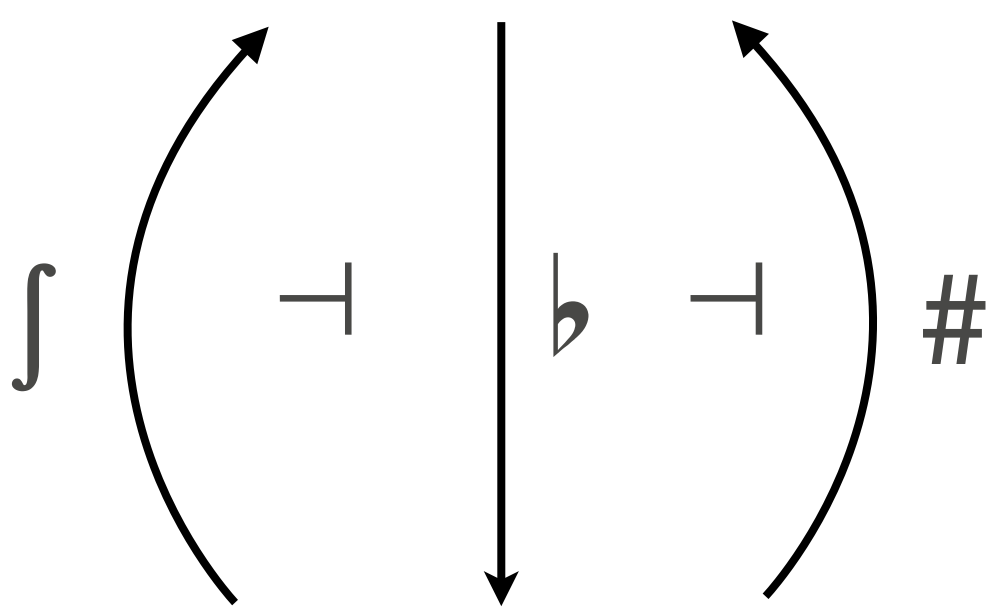

Topological ∞-groupoids

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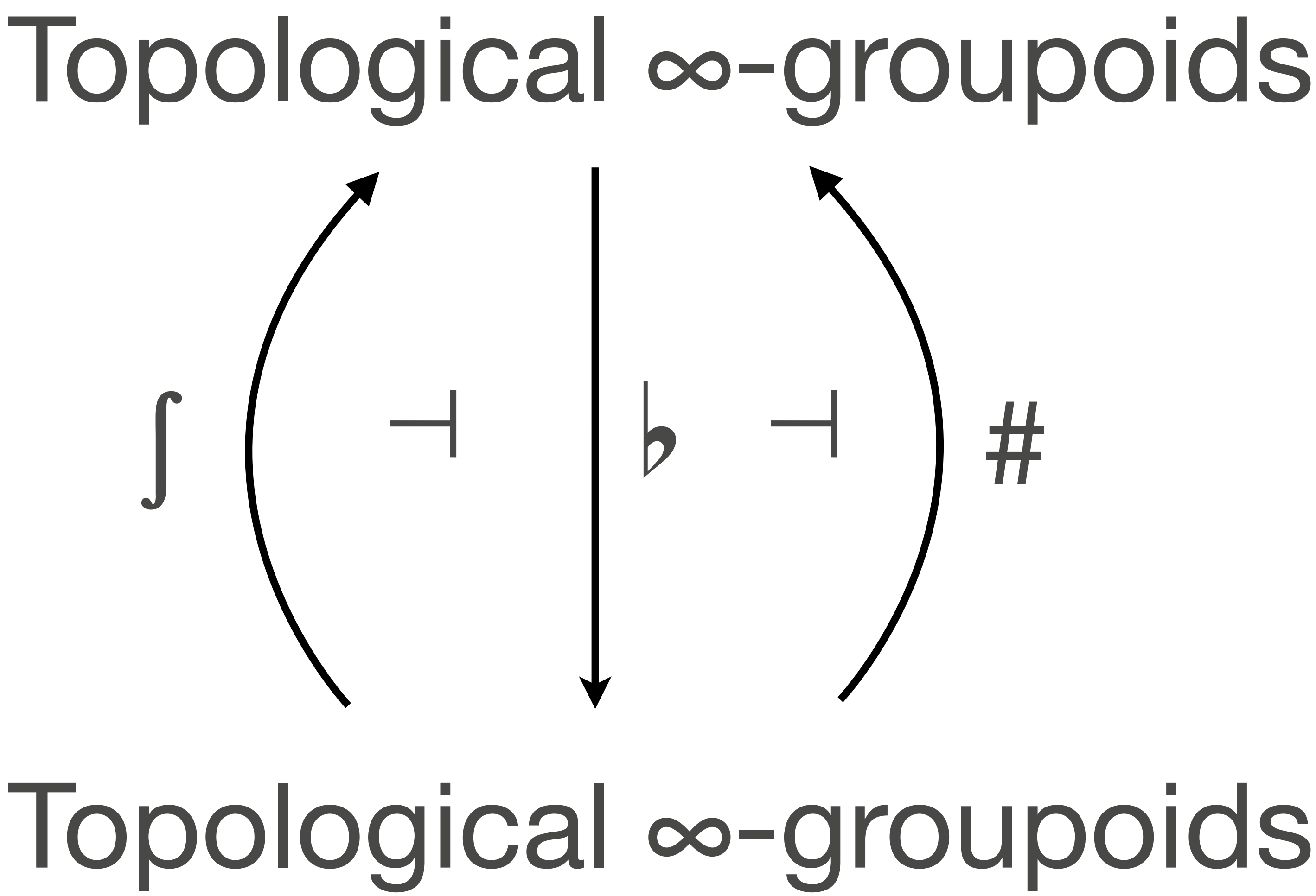
∞-categorical cohesion



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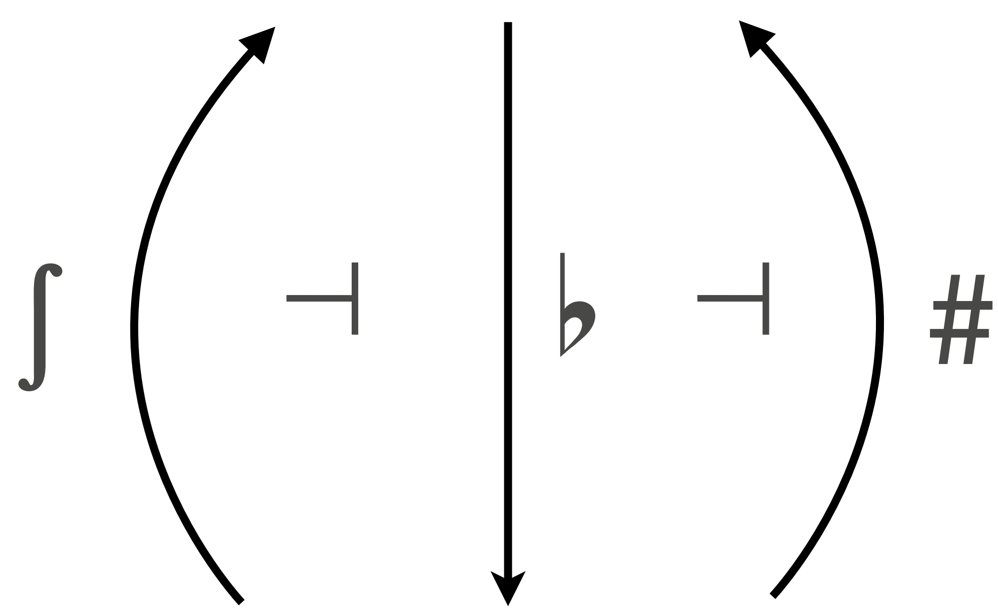

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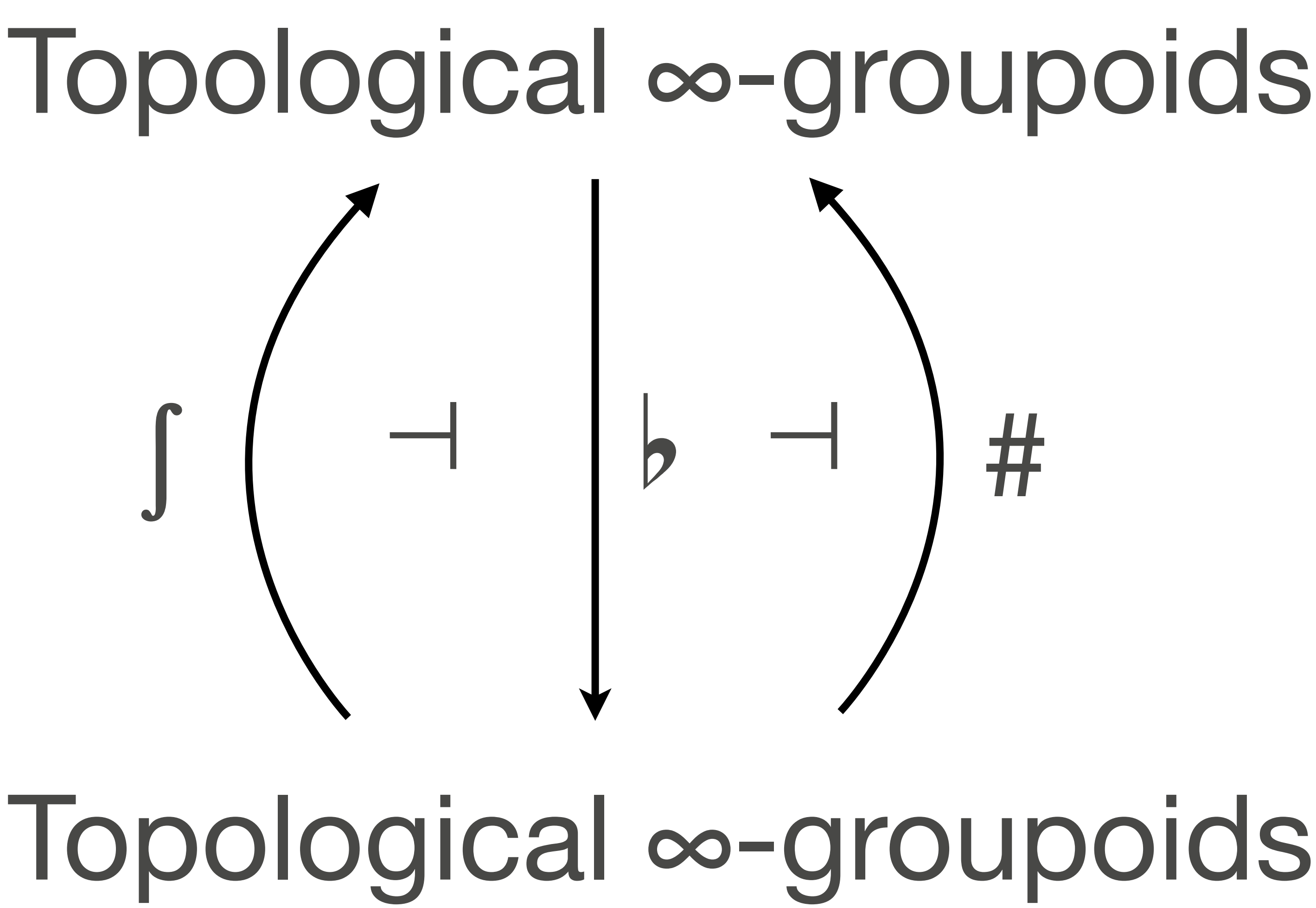

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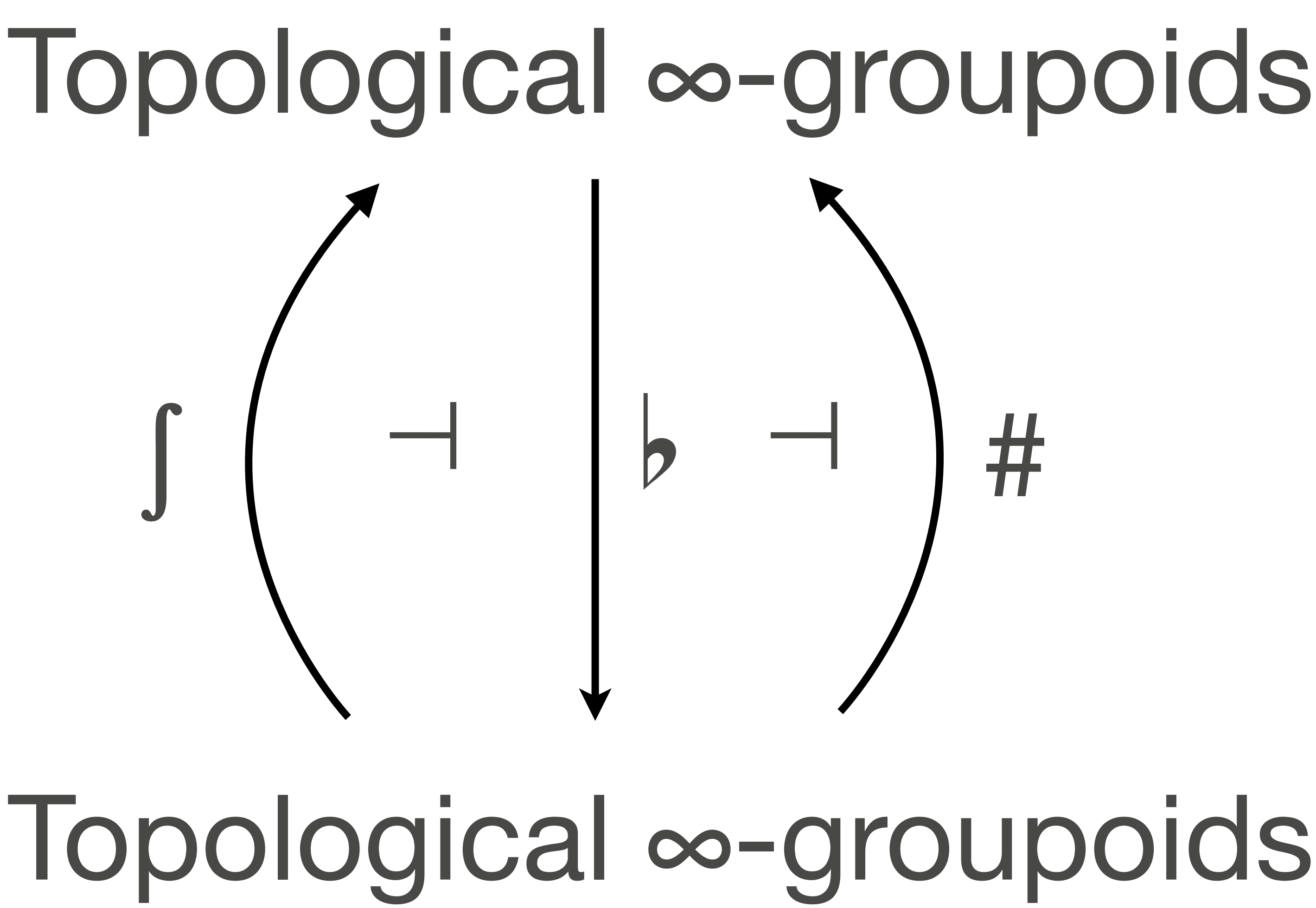

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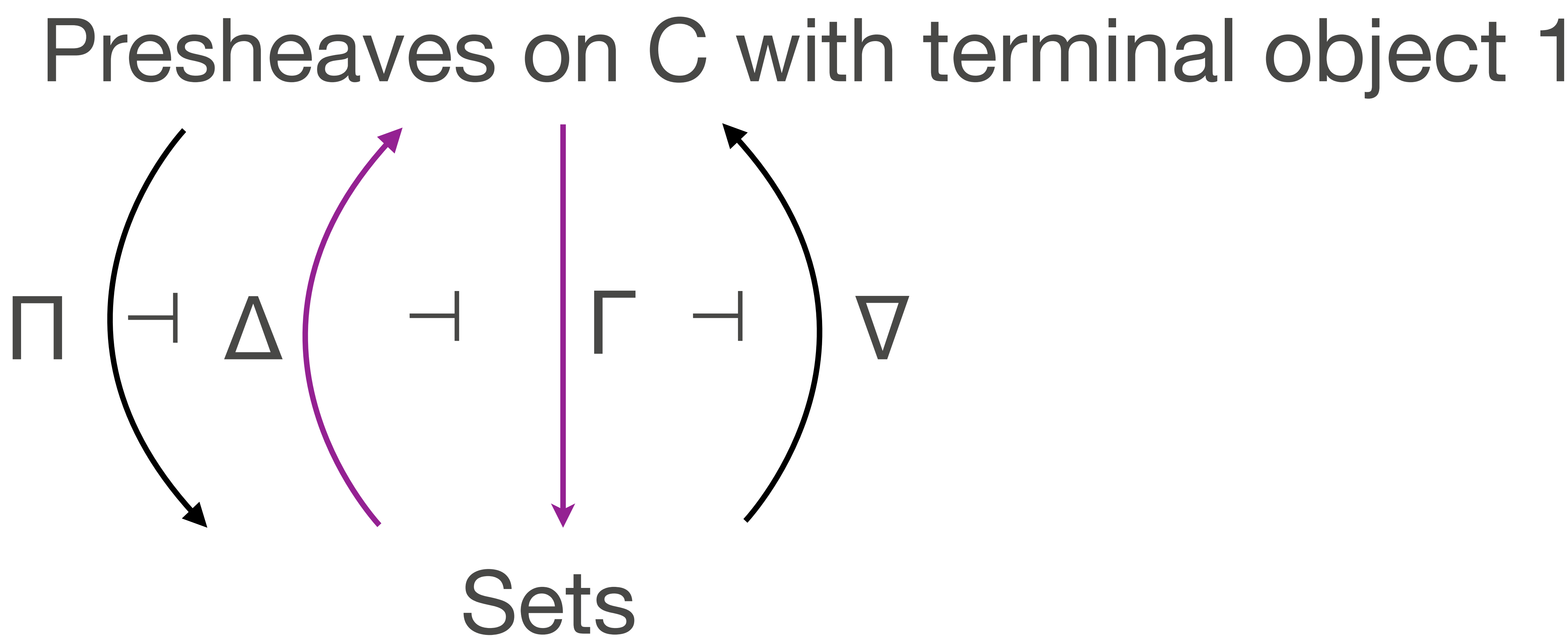
***Modality*: historically endofunctor on types/propositions**

$\Box A \quad \Diamond A \quad !A \quad ?A$

Cohesion in cubical models



$\Gamma(A)$ = set of objects (A_1) /global sections
$\Delta(X)$ = constant presheaf on X

Internal Universes in Cubical Models

[L.,Orton, Pitts,**Spitters**,‘18]

Thursday!

$$\Gamma \to \mathbb{U}_{\text{fib}} \quad \cong \quad \sum (A : \Gamma \to \mathbb{U}).\ \text{IsFib}\ A$$

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wrong: gives A(x) fibrant for all x:Γ implies A fibrant over Γ

Internal Universes in Cubical Models

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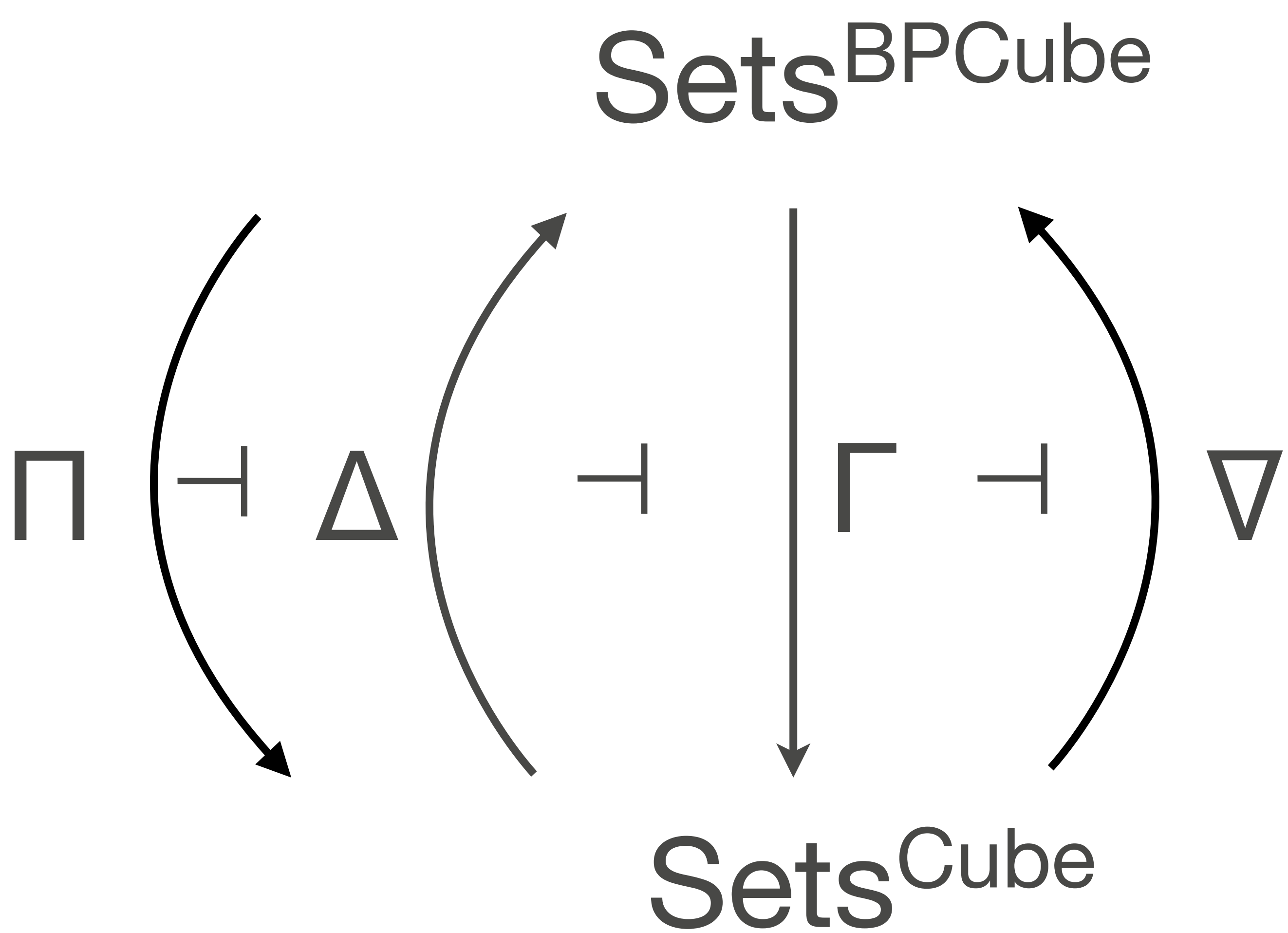
$$\flat(\Gamma \to \mathbb{U}_{\mathrm{fib}}) \cong \flat(\textstyle\sum (A : \Gamma \to \mathbb{U}).\ \mathrm{IsFib}\ A)$$

access to external statements in internal language of topos

Parametricity [Nuyts,Vezzosi,Devriese]



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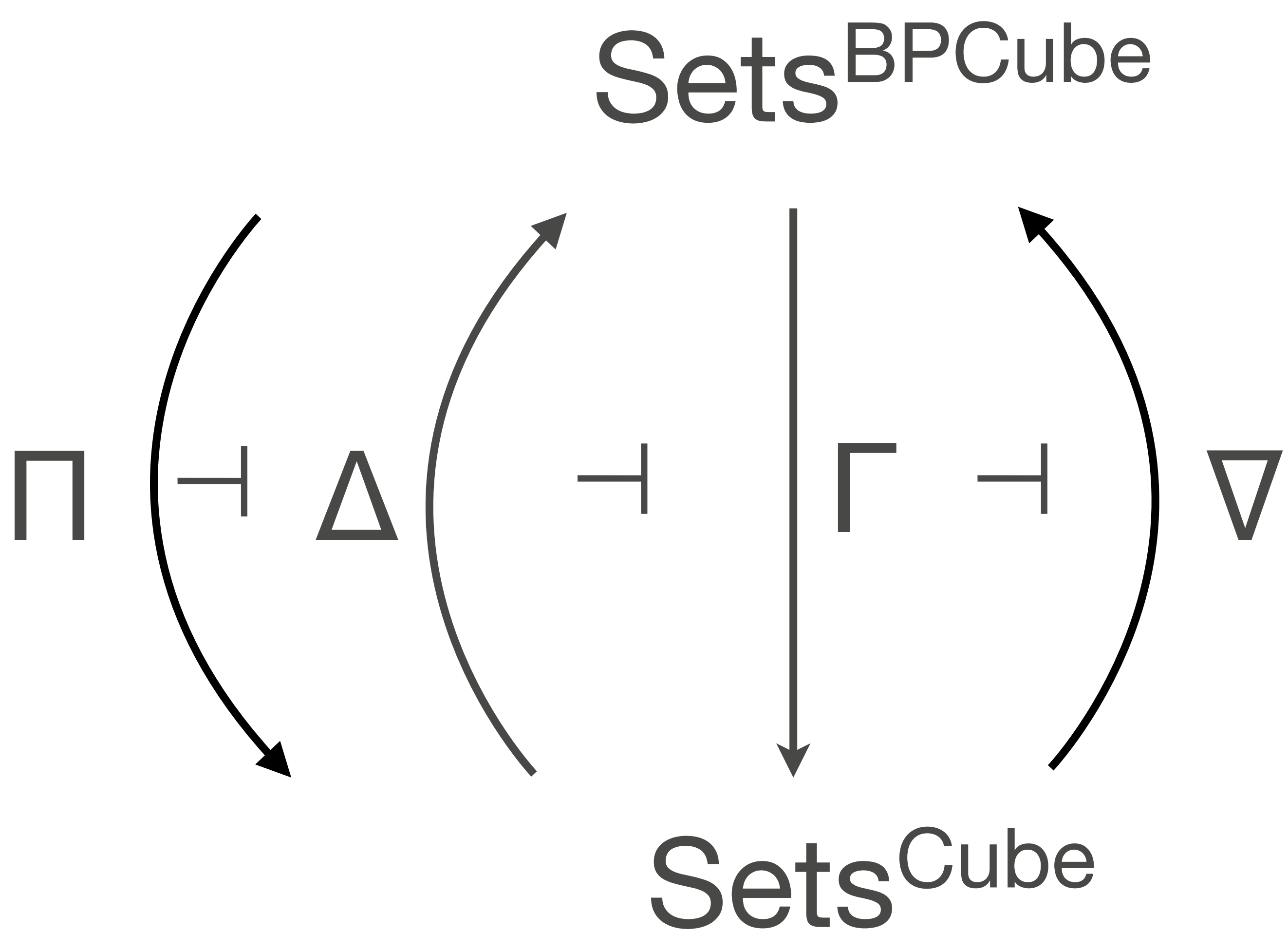

two kinds of intervals, paths and “bridges”/relations

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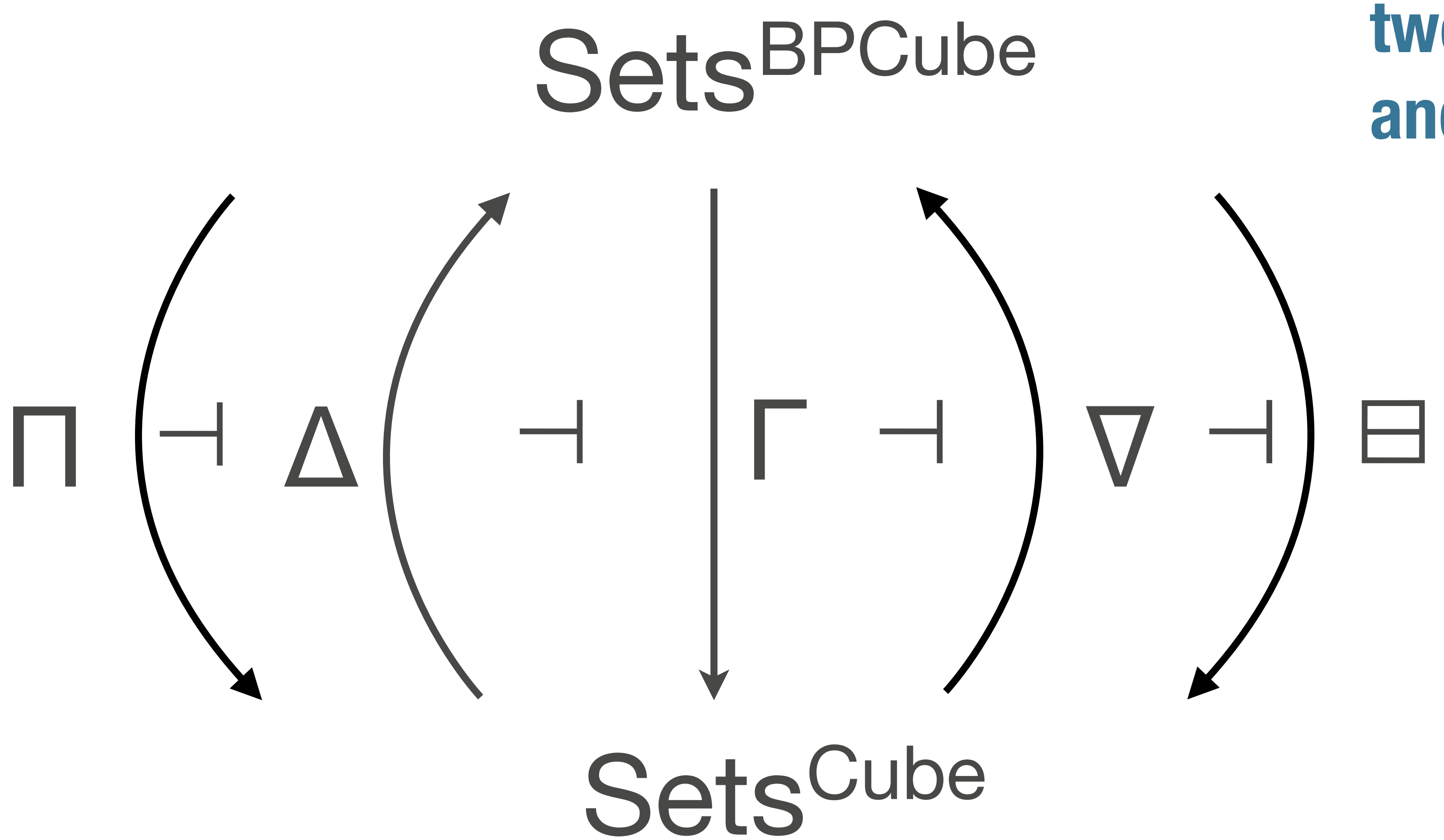
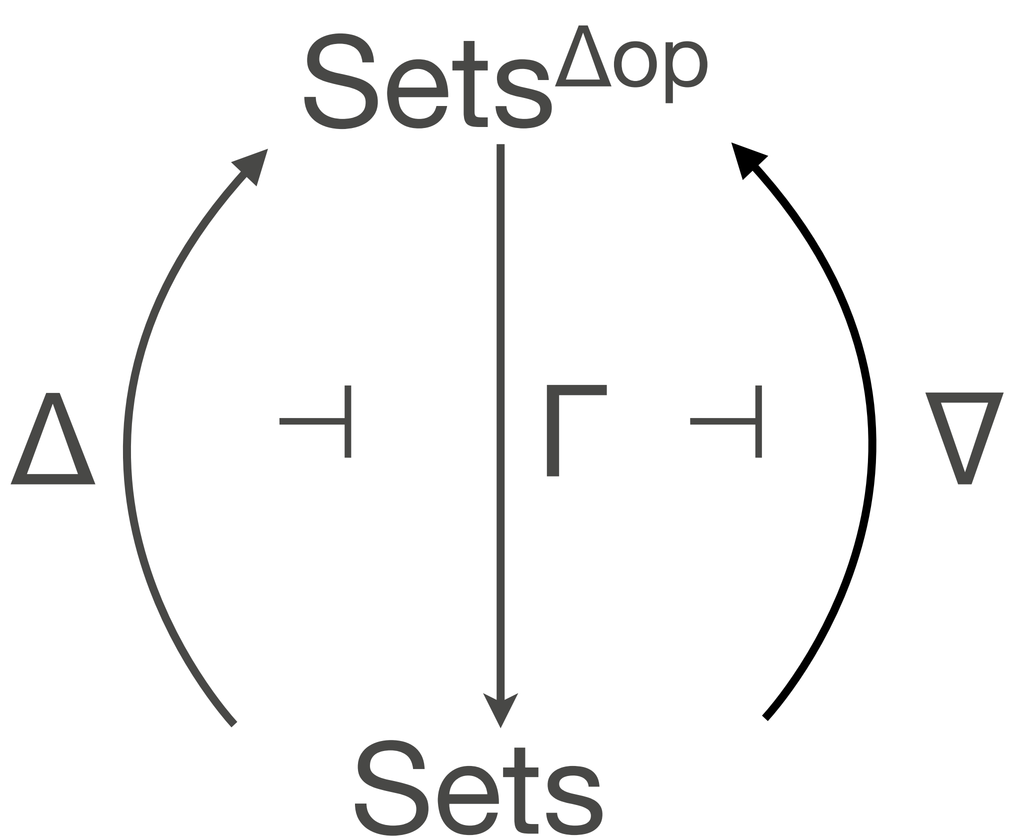


Bi-simplicial/cubical DirTT

[Riehl,Shulman;
Riehl,Sattler;
L.-Weaver]



forget paths

Bi-simplicial/cubical DirTT

$Sets^{\Delta op \times \Delta op}$

[Riehl,Shulman;
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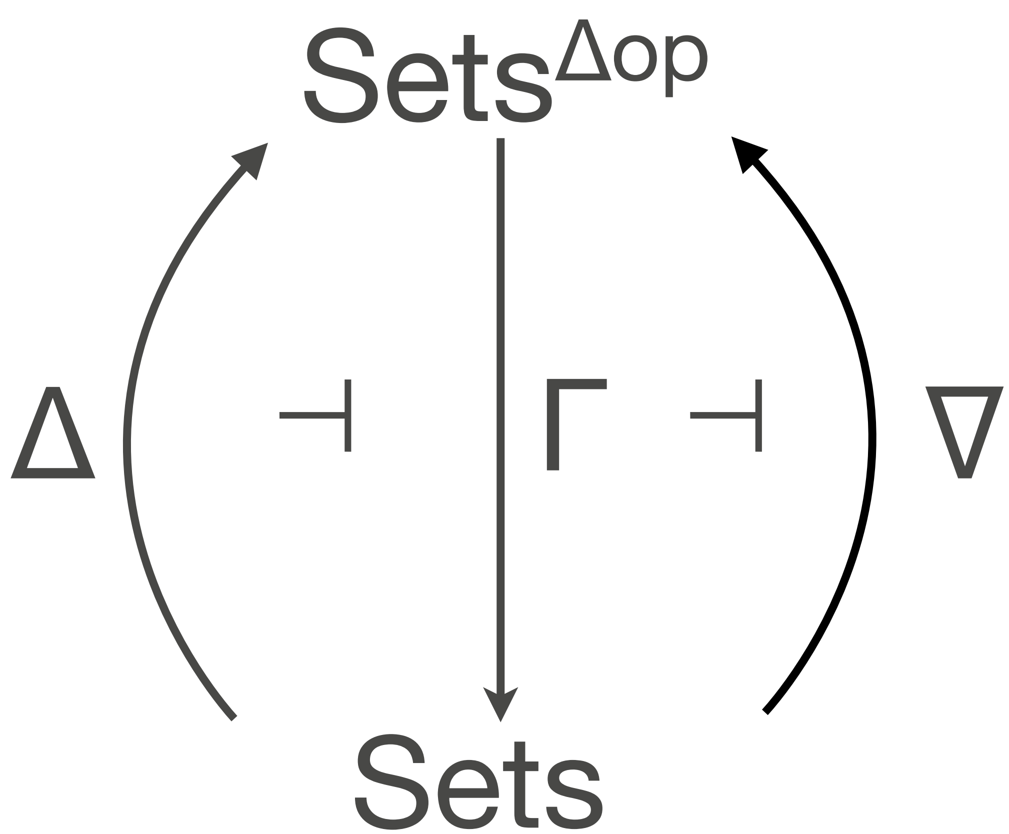

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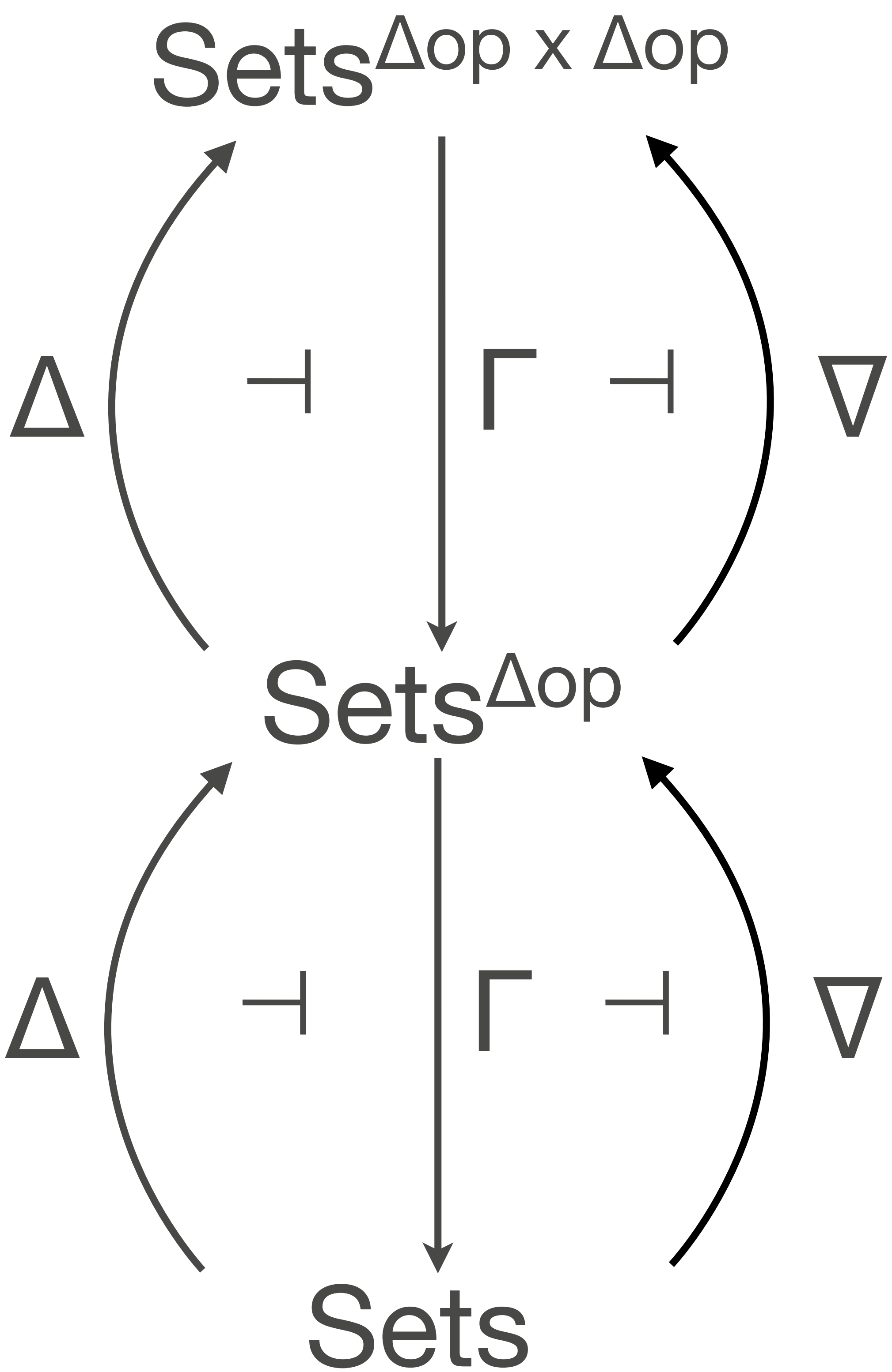

forget morphisms

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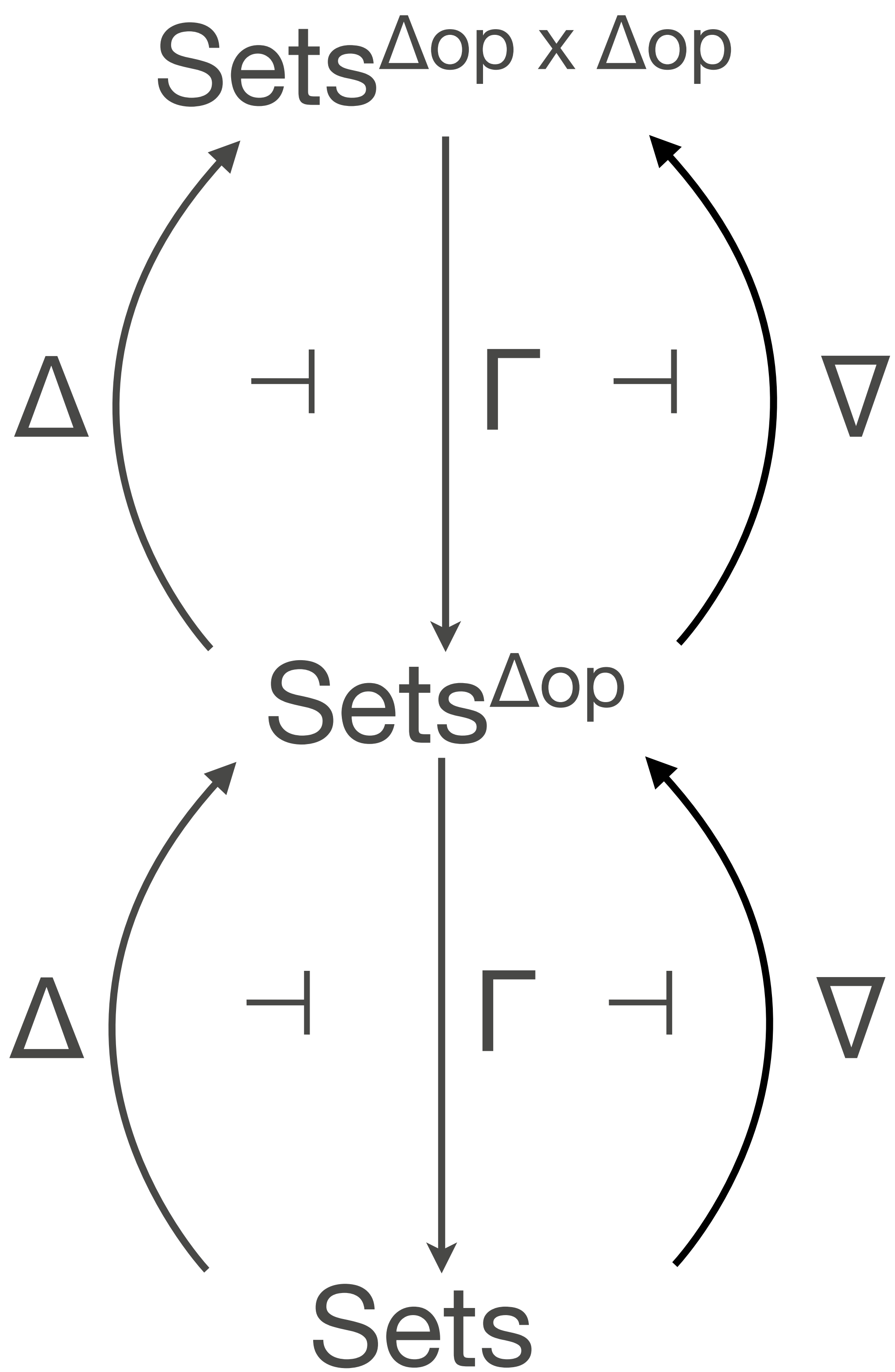

[Riehl,Shulman;
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forget morphisms

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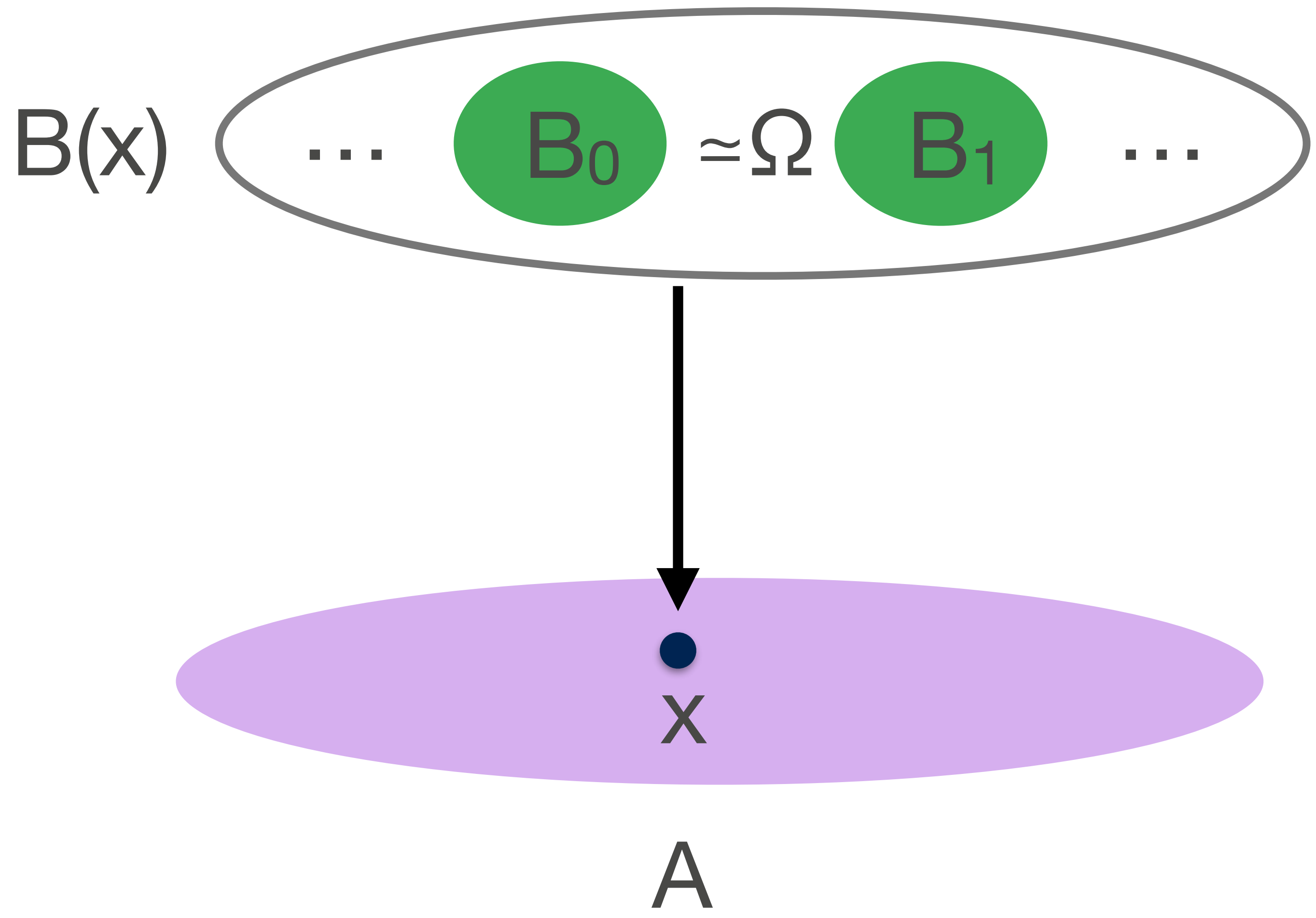
also core, opposites (self-adjoint)? [Nuyts'15]

Parametrized spectra

[Finster,L.,Morehouse,Riley]



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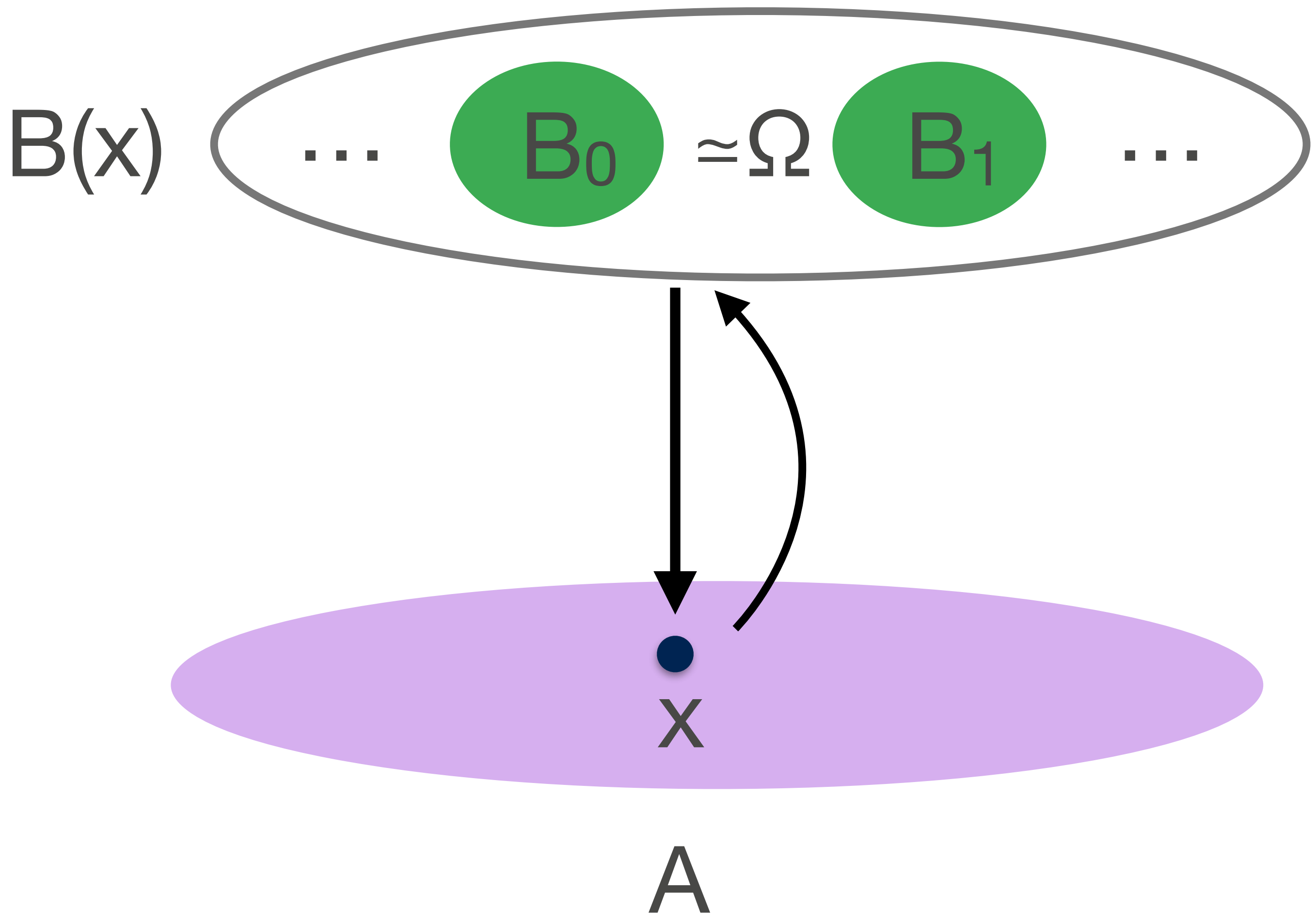

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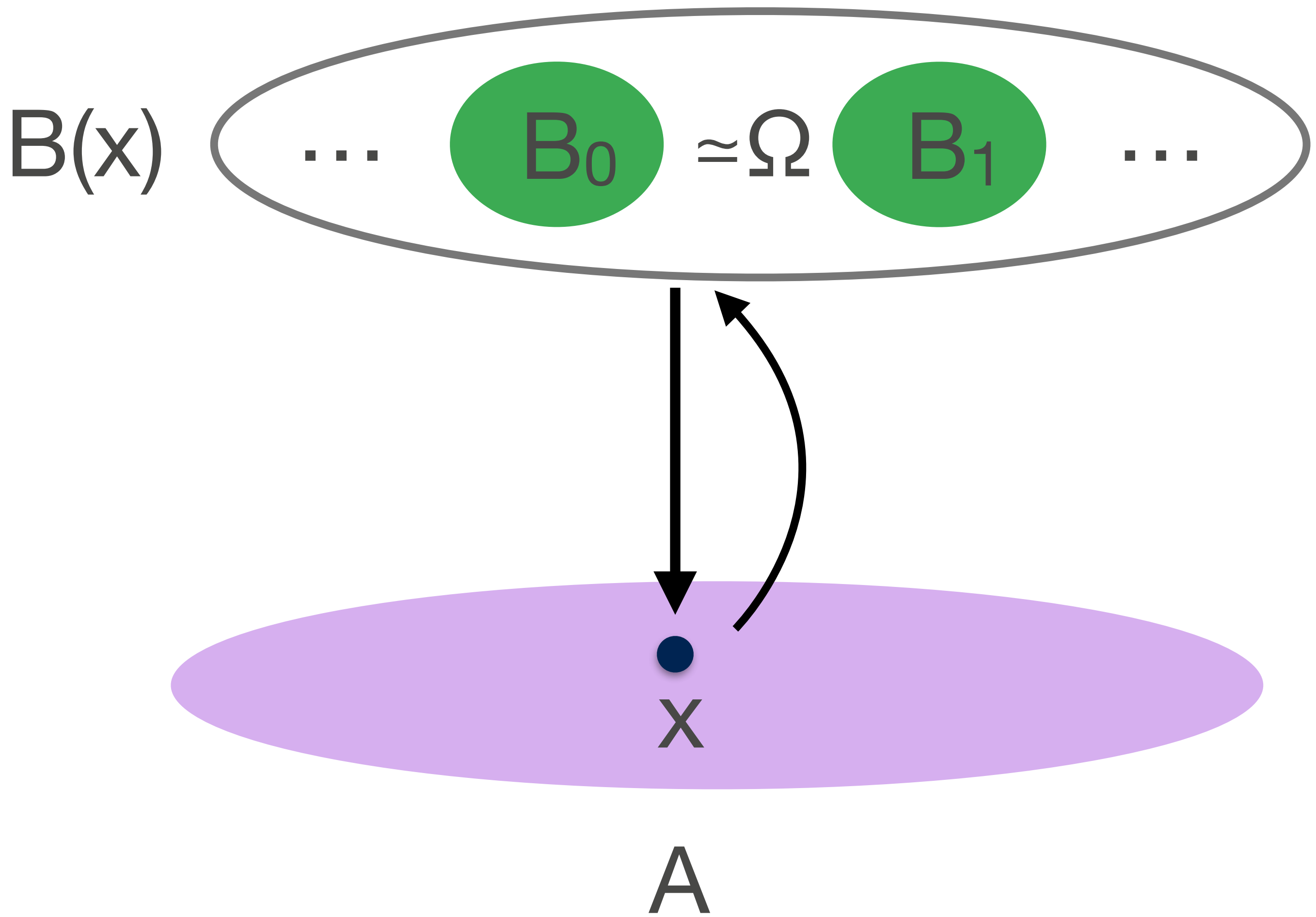

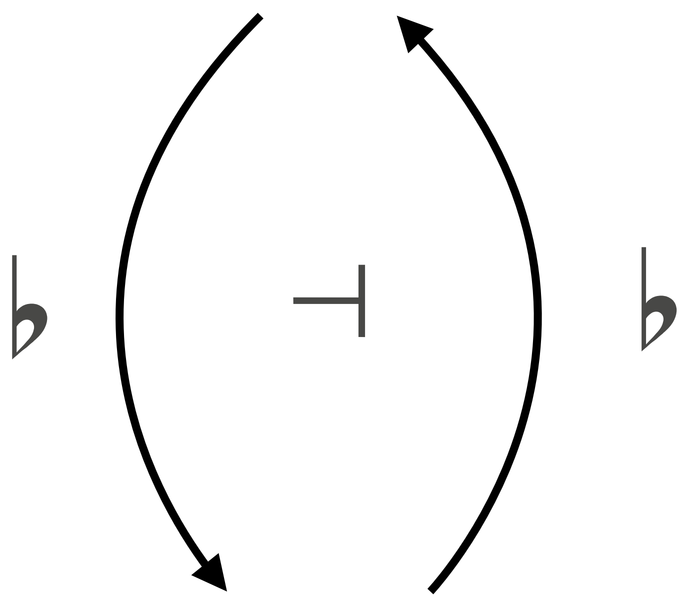

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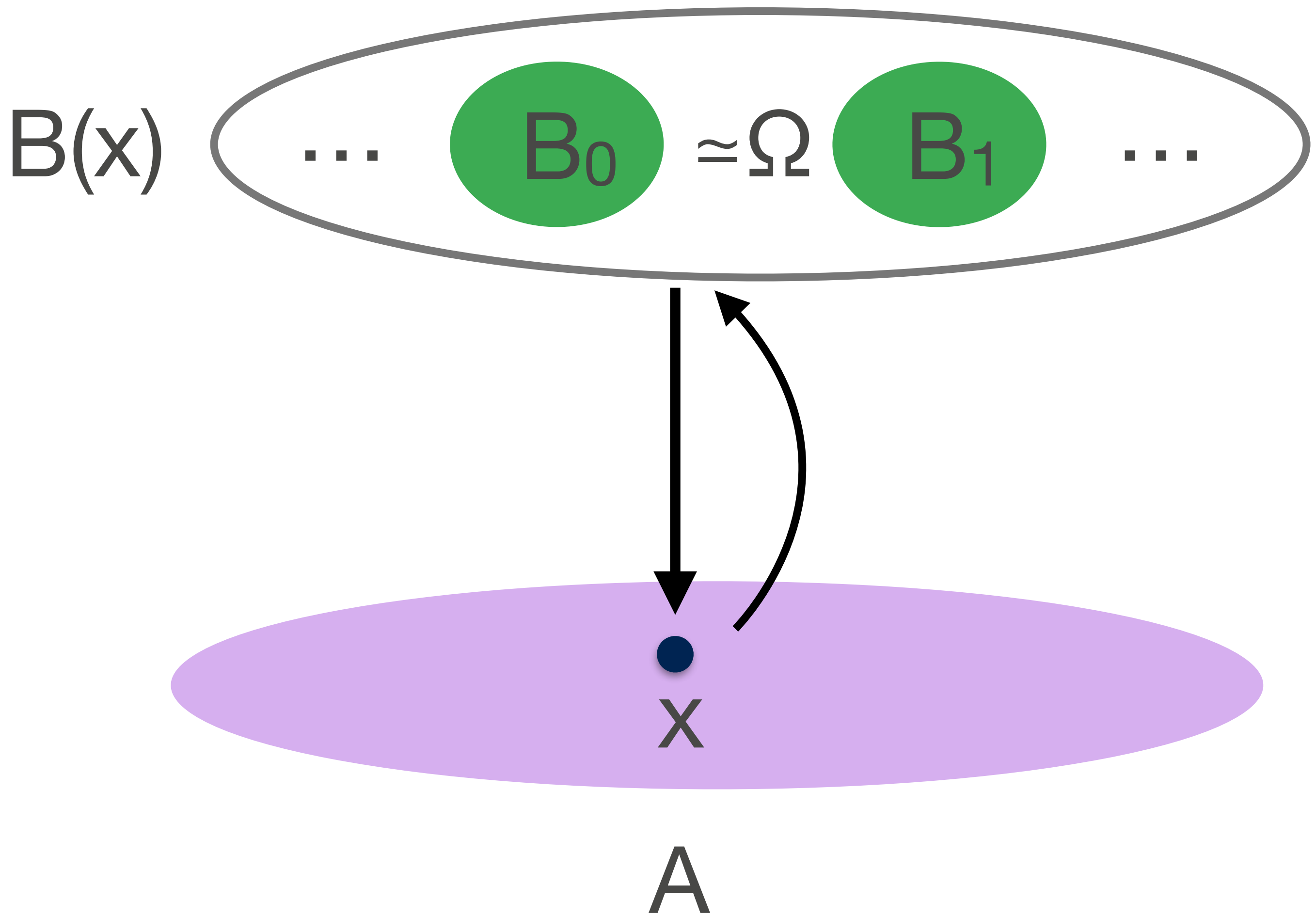

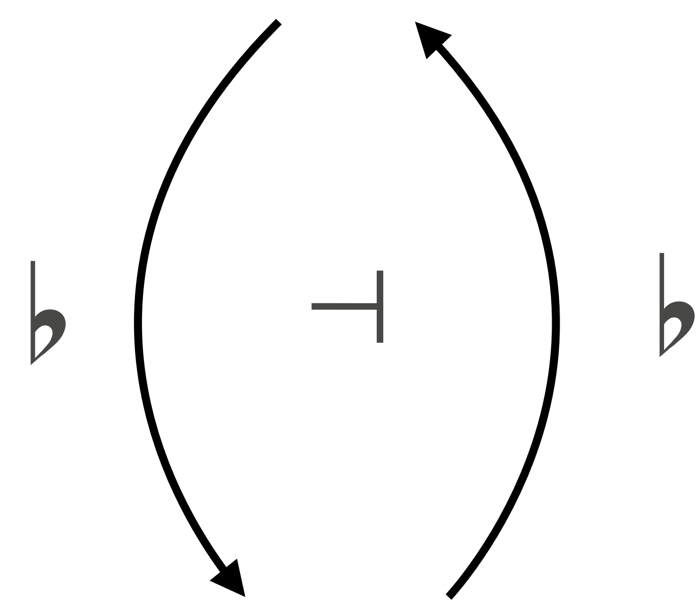

self-adjoint, idempotent
monad and comonad

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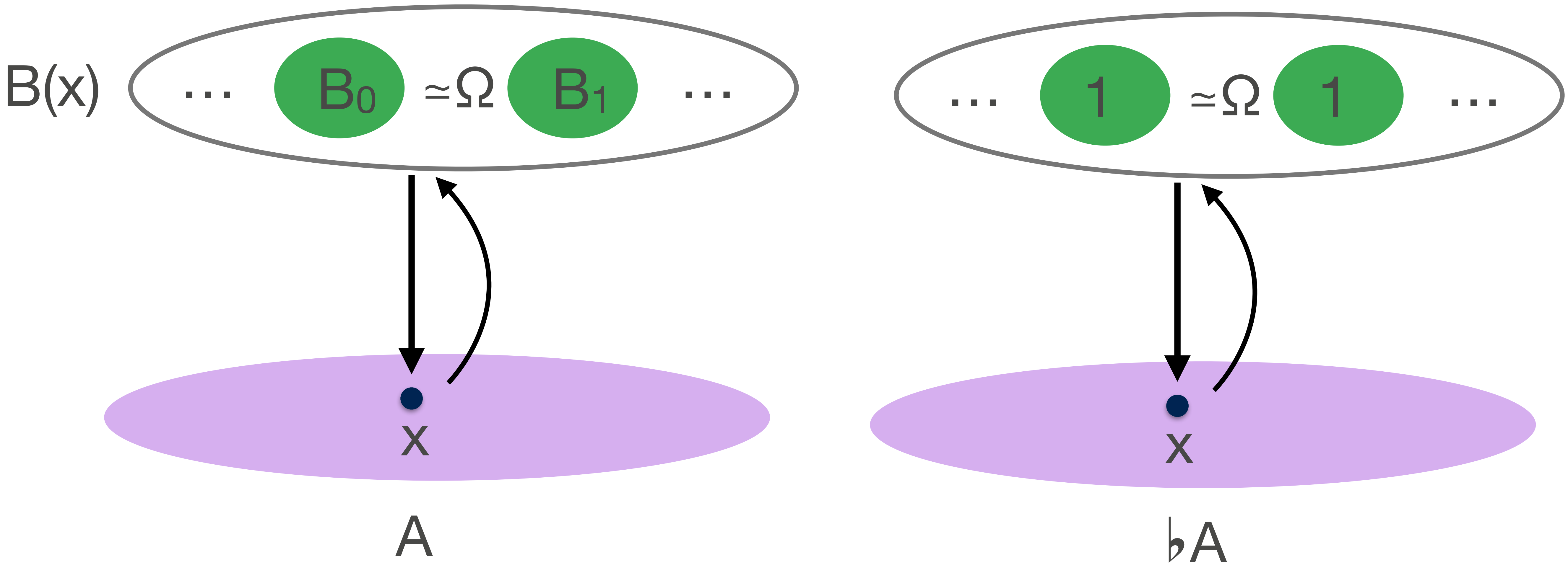


Differential cohesion

[Friday!]

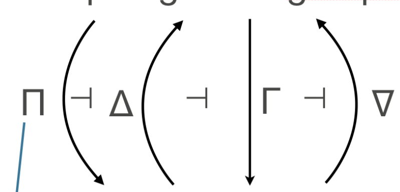
[Scheiber; W.; Gross,L.,New,Paykin,Riley,Shulman,W.]

$$
\begin{array}{ccccccccc}
\Re & \dashv & \Im & \dashv & \& & & \\
 & & \cup & & \cup & & \\
 & & \int & \dashv & \flat & \dashv & \sharp
\end{array}
$$

∞-categorical Cohesion

[Schreiber,Shulman]

“Topological ∞-groupoids”



∞-groupoids

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Δ and ∇ full and faithful…

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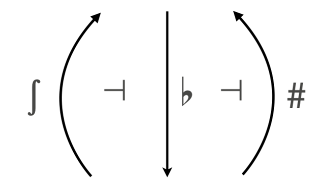


“Topological ∞-groupoids”

∫ = ΔΠ

♭ = ΔΓ comonad

= ∇Γ monad

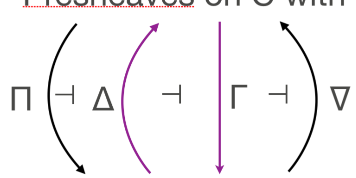
idempotent

Modality: historically endofunctor on types/propositions

□A ◇A !A ?A

Cohesion in cubical models

Presheaves on C with terminal object 1



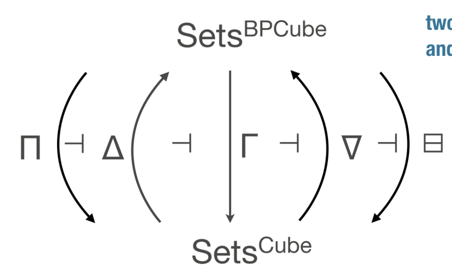
Sets

Γ(A) = set of objects (A_1)

Δ(X) = constant presheaf on X

Parametricity

[Nuyts,Vezzosi,Devriese]



two kinds of intervals, paths and “bridges”/relations

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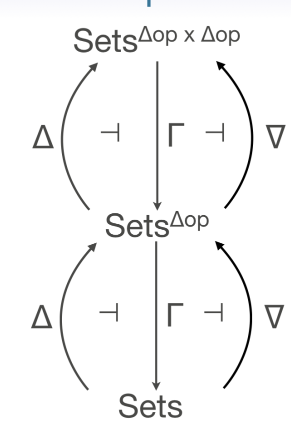

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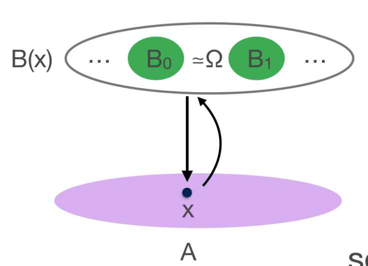

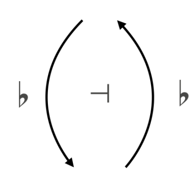

self-adjoint,
monad and comonad

Differential Cohesion

[Friday!]

[Scheiber; W.; Gross,L.,New,Paykin,Riley,Shulman,W.]

		ℜ	⊣	ℑ	⊣	&
				∪		∪
				∫	⊣	♭	⊣	♯

Questions

How do we extend type theory (MLTT, HoTT) to synthetically handle these situations?

How to add modalities like ΔA, ∇A, ∫A, ♭A, #A, …
representing adoint functors (full and faithful?), self-adoint functor, monads, comonads (idempotent?), both …

What can we do with them once we have them?

Frameworks, Doctrines, Theories, Models

[see Shulman n-Theory on n-Category Cafe, Type 2-Theories HoTTEST, April’18]

Doctrine, theory, model

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Doctrine:

- type constructors/logical connectives
- semantically specifies categorical structure of models (2-category where models are 1-morphisms)

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Theory in a Doctrine:

- axioms/signature using those connectives
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Model of a Theory (syntactic presentation):

- implementation of that signature by some other types/terms

Doctrine, theory, model

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Doctrine:

- simple type theory with $\times, 1$
- categories with finite products

Doctrine, theory, model

Doctrine:

- simple type theory with $\times, 1$
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Theory in a Doctrine:

- monoid:
  p type, $\odot : p \times p \to p$, $x \odot (y \odot z) = (x \odot y) \odot z$, …

Doctrine, theory, model

Doctrine:

- simple type theory with $\times$,1
- categories with finite products

Theory in a Doctrine:

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  p type, $\odot : p \times p \to p$, $x \odot (y \odot z) = (x \odot y) \odot z$, …

Model of a Theory (syntactic presentation):

- $(\mathbb{Z},+,0)$, $(\mathbb{Q},x,1)$, etc.

Theory of monadic modality in doctrine of Book HoTT

[Rijke,Shulman,Spitters]

Definition 7.7.5. A **modality** is an operation $\bigcirc : \mathcal{U} \to \mathcal{U}$ for which there are

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Theory of monadic modality [tomorrow!] in doctrine of Book HoTT

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Modalities as theory, models

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Theory:

- Let J be an unknown monadic modality…

Models (syntactically presented):

- $||A||_n$ defined as a HIT
- $\int A$ as nullification HIT making $\int A \cong (\mathbb{R} \to \int A)$

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assumption or definition! no new metatheory

Modalities in the doctrine

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  to the syntax of type theory (doctrine) itself
- changes the structure of categories
  considered in semantics

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- changes the structure of categories considered in semantics

Why?

- comonadic modalities of interest ($\flat$) are **not** internal functions $\mathbb{U} \to \mathbb{U}$ [Shulman'15]
- multiple categories (Δ, ∇ : Sets $\to$ Spaces) = multiple modes of types
- proof theory that's easier to use?

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- extend categorical semantics: internal language of several (∞-)categories with functors between them, rather than just one
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how to do this without fixing a doctrine?

Framework

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- clarify what is general to all modal type theories, what is specific to instances
- metatheory: prove initiality, cut elimination/ normalization once for all doctrines
- use framework as a bridge between semantics and more pleasant “surface” type theories by “WLOGing” framework rules

Related Work

[see L.,Shulman'16, L.,Shulman,Riley'17 bibliography]

- Multiple kinds of assumptions/multi-zoned contexts: Andreoli'92; Wadler'93; Plotkin'93;Barber'96; Benton'94; Pfenning,Davies'01
- Tree-structured contexts:
  Display logic: Belnap
  Bunched contexts: O'Hearn,Pym'99,
  Resource separation: **Atkey,'04**
- Multiple modes: Benton'94; Benton,Wadler'96, **Reed'09**
- Fibrational perspective: Melliès,Zeilberger'15

[Friday!]

Fibrational Frameworks for

- Today: functors in unary modal type theories
- Wed: adjunctions in unary modal type theories [L., Shulman, ’16]
- Wed: simple modal and substructural type theories [L., Shulman, Riley, ’17]
- Thurs: dependent modal and substructural type theories [L., Riley, Shulman, ongoing]

Being judgey about judgements

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- interpretable in intended semantics

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  (fails for inductives in MLTT: induction, universes)

Good doctrines

- interpretable in intended semantics
- cut elimination/normalization/operational semantics
- subformula property:
  proof of $A \vdash B$ only uses subformulas of A and B
  (fails for inductives in MLTT: induction, universes)
- judgemental: types given by
  intro&elim / universal properties
  relative to judgements;
  one type constructor per rule.

  [Martin-Löf, Pfenning]

cf. generalized-multicategorical perspective [Shulman]

Non-judgemental Monoid T.T.

$$\frac{}{A \vdash A} \qquad \frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

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$$\frac{}{A \vdash A} \qquad \frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

$$\frac{A \vdash A' \quad B \vdash B'}{A \otimes B \vdash A' \otimes B'}$$

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$$(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)$$

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[+ a lot of equations!]

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[+ βη!]

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[+ substitution equations]

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[+ βη!]

Weak from strict

$$\frac{}{A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C}$$

Weak from strict

$$\dfrac{\overline{A, (B \otimes C) \vdash (A \otimes B) \otimes C}}{A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C}$$

Weak from strict

$$\dfrac{\dfrac{\dfrac{}{A,B,C \vdash (A\otimes B)\otimes C}}{A,(B\otimes C) \vdash (A\otimes B)\otimes C}}{A\otimes(B\otimes C) \vdash (A\otimes B)\otimes C}$$

Weak from strict

$$\dfrac{\dfrac{\dfrac{A,B \vdash A \otimes B \qquad C \vdash C}{A,B,C \vdash (A\otimes B)\otimes C}}{A,(B\otimes C) \vdash (A\otimes B)\otimes C}}{A\otimes(B\otimes C) \vdash (A\otimes B)\otimes C}$$

Judgemental presentations

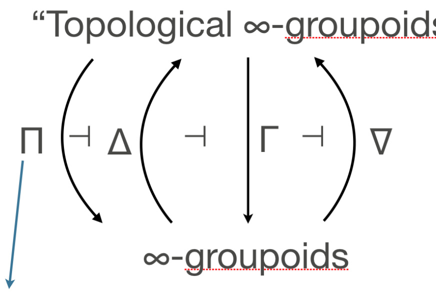
- modularity: add/remove types from doctrine without affecting others
- communication: judgement structure is a short-hand for the types
- easier to spot problems with cut elim/normalization/…
- manipulate weak structures by passing to stricter judgemental ones

Tutorial 3

Previously on Modal Dependent Type Theories…

∞-categorical Cohesion

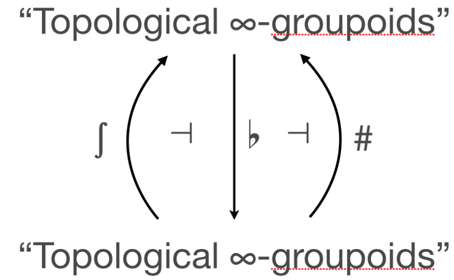
[Schreiber,Shulman]



fundamental ∞-groupoid! e.g. ΔΠ(topological $\mathbb{S}^1$) = HIT S^1

Δ and ∇ full and faithful…

∞-categorical Cohesion



∫ = ΔΠ

♭ = ΔΓ comonad

= ∇Γ monad

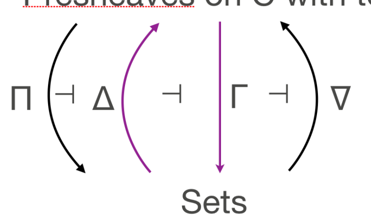
idempotent

Modality: historically endofunctor on types/propositions

□A ◇A !A ?A

Cohesion in cubical models

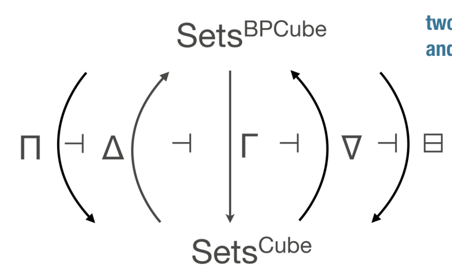
Presheaves on C with terminal object 1



Γ(A) = set of objects (A_1)

Δ(X) = constant presheaf on X

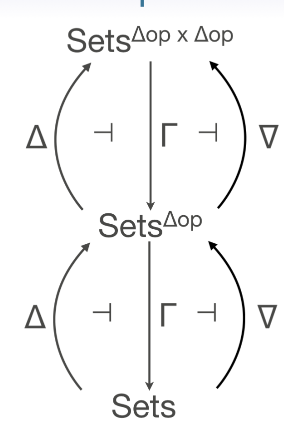
Parametricity [Nuyts,Vezzozi,Devriese]



two kinds of intervals, paths and "bridges"/relations

Bi-simplicial/cubical DirTT

[Riehl,Shulman; Riehl,Sattler; L.-Weaver]



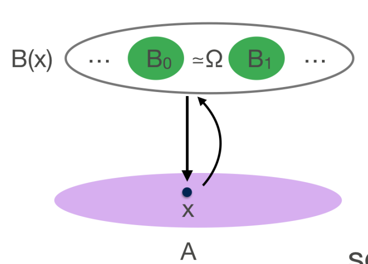
forget morphisms

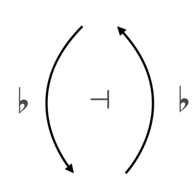
forget paths

also core, opposites? [Nuyts'15]

Parametrized Spectra

[Finster,L.,Morehouse,Riley]





self-adjoint, monad and comonad

Differential Cohesion [Friday!]

[Scheiber; W.; Gross,L.,New,Paykin,Riley,Shulman,W.]

$\Re$	⊣	$\Im$	⊣	&
		∪		∪
∫	⊣	♭	⊣	♯

Real-cohesion [Shulman]

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- Intended model: interpretation of every type A has both homotopy structure and topological structure on every level ("non-standard" model of Book HoTT)

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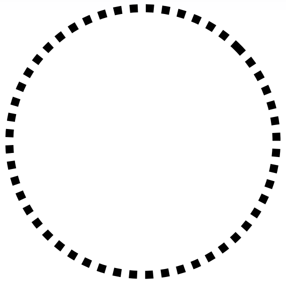
- topological structure detected by $\mathbb{R} \to A$, e.g.
- use type constructor (“modality”) $\int$ to relate the two

Real-cohesion



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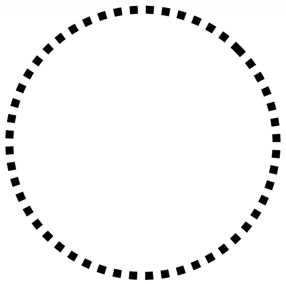

- $\mathbb{S}^1 := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ has topological paths but is an hset

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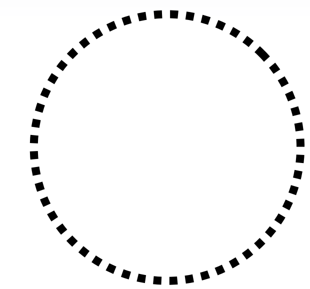
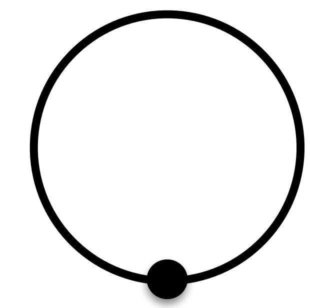


- $\mathbf{S}^1$:= HIT{base,loop} has homotopical paths but is topologically discrete



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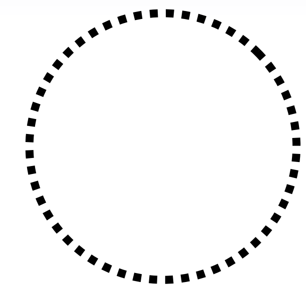

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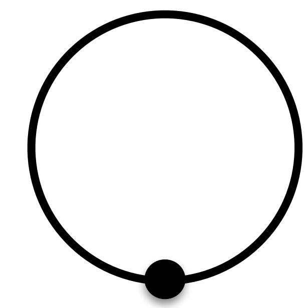

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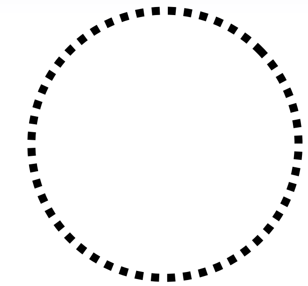

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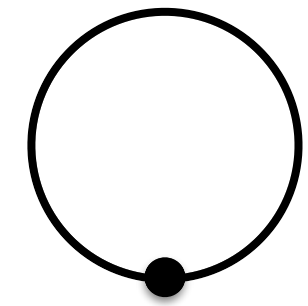

- $\int \mathbb{S}^1 \simeq \mathbf{S}^1$
- Yesterday: Felix used $\int$ to give a synthetic formulation of the relation between covering spaces and actions of the *topological* fundamental group

Shape

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  HIT making $\int A \simeq (\mathbb{R} \to \int A)$
  c.f. $\|A\|_0 \simeq (\mathbf{S}^1 \to \|A\|_0)$

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Monadic Modalities

Lots of lemmas can be proved for the theory of **any** monadic modality [Rijke,Shulman,Spitters]:

$$\bigcirc(\sum x{:}A.B(\eta(x))) = \sum x{:}\bigcirc A.\bigcirc B(x)$$

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Monadic Modalities

Felix and Egbert's covering space construction works for any monadic modality ○ :
one theorem interpreted in many settings!

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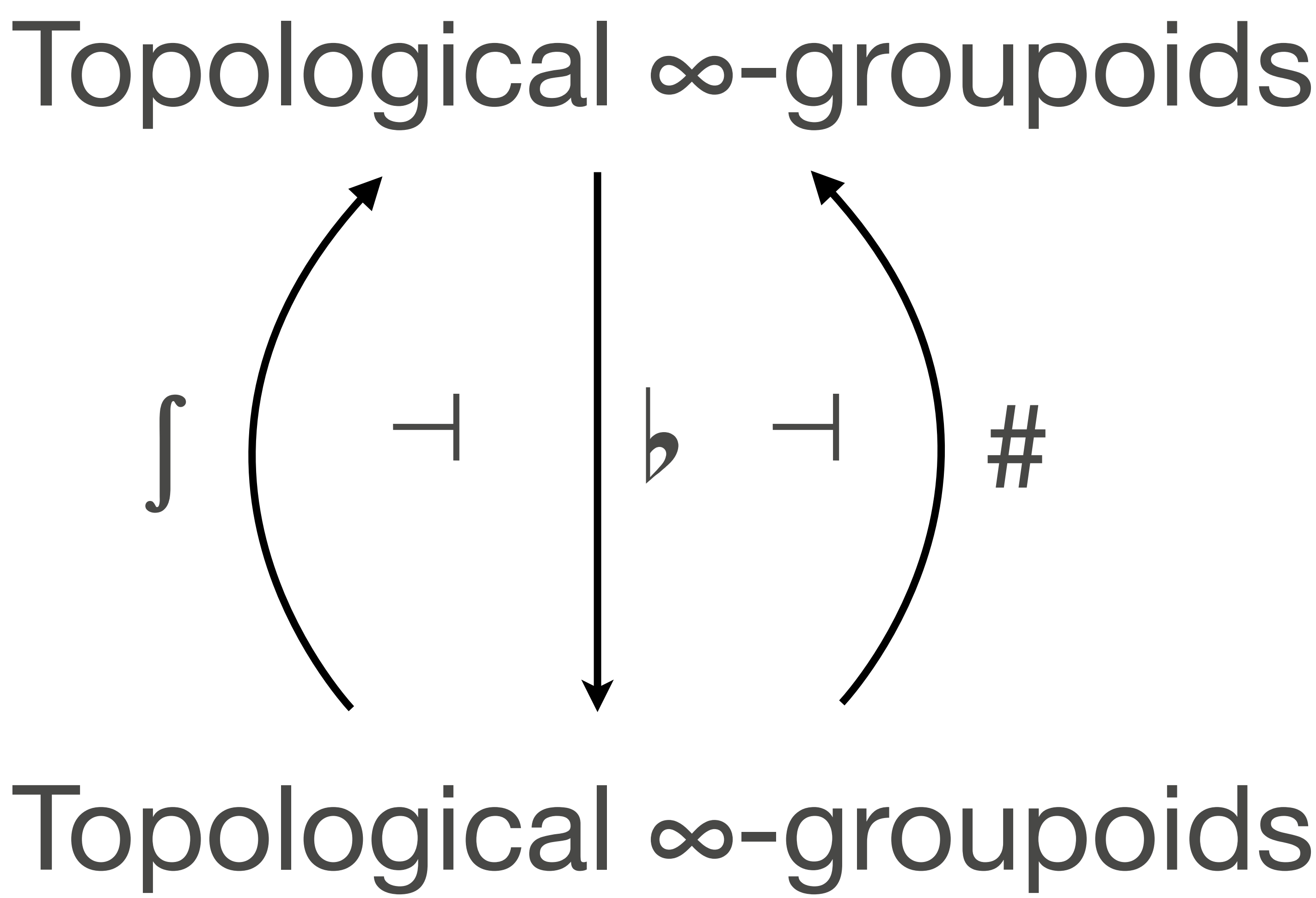
(iv) For any $z, z' : \bigcirc(A)$, the function $\eta^{\bigcirc}_{z=z'} : (z = z') \to \bigcirc(z = z')$ is an equivalence.

Real-cohesion



∫	**monad**
♭	**comonad**
#	**monad**

Theory of comonadic modality?

[Shulman]

Not what we want:

Theorem 4.1. *Suppose we have the following data:*

(1) *A predicate* $\mathsf{in}_\Box : \mathsf{Type} \to \mathsf{Prop}$ *that is invariant under equivalence, i.e.* $(A \simeq B) \to \mathsf{in}_\Box(A) \to \mathsf{in}_\Box(B)$. *(This condition is, of course, automatic with univalence.)*
(2) *An operation* $\Box : \mathsf{Type} \to \mathsf{Type}$, *such that* $\mathsf{in}_\Box(\Box(A))$ *for all* A.
(3) *For each* $A : \mathsf{Type}$, *a function* $\varepsilon_A : \Box A \to A$.
(4) *If* $\mathsf{in}_\Box(B)$, *then postcomposition with* ε_A *is an equivalence* $(B \to \Box A) \simeq (B \to A)$.

Then there exists $U : \mathsf{Prop}$ *such that for all* A *we have*

(a) $\mathsf{in}_\Box(A) \leftrightarrow (A \to U)$ *and*
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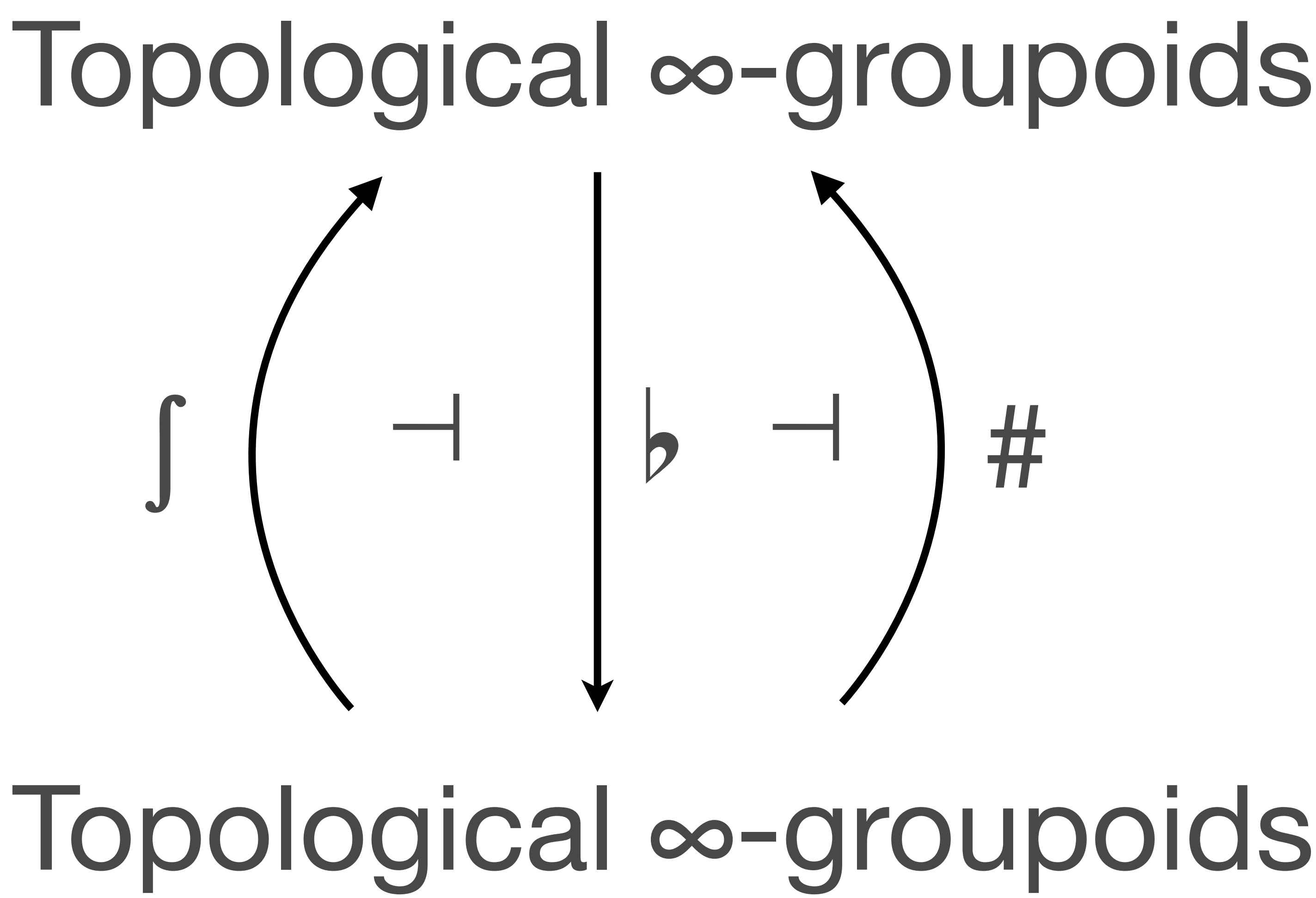
Idea: (4) can be applied in any context:

- ○A restricts all (one) conclusions to be modal
- □A doesn't restrict all assumptions

Real-cohesion



∫	**monad**
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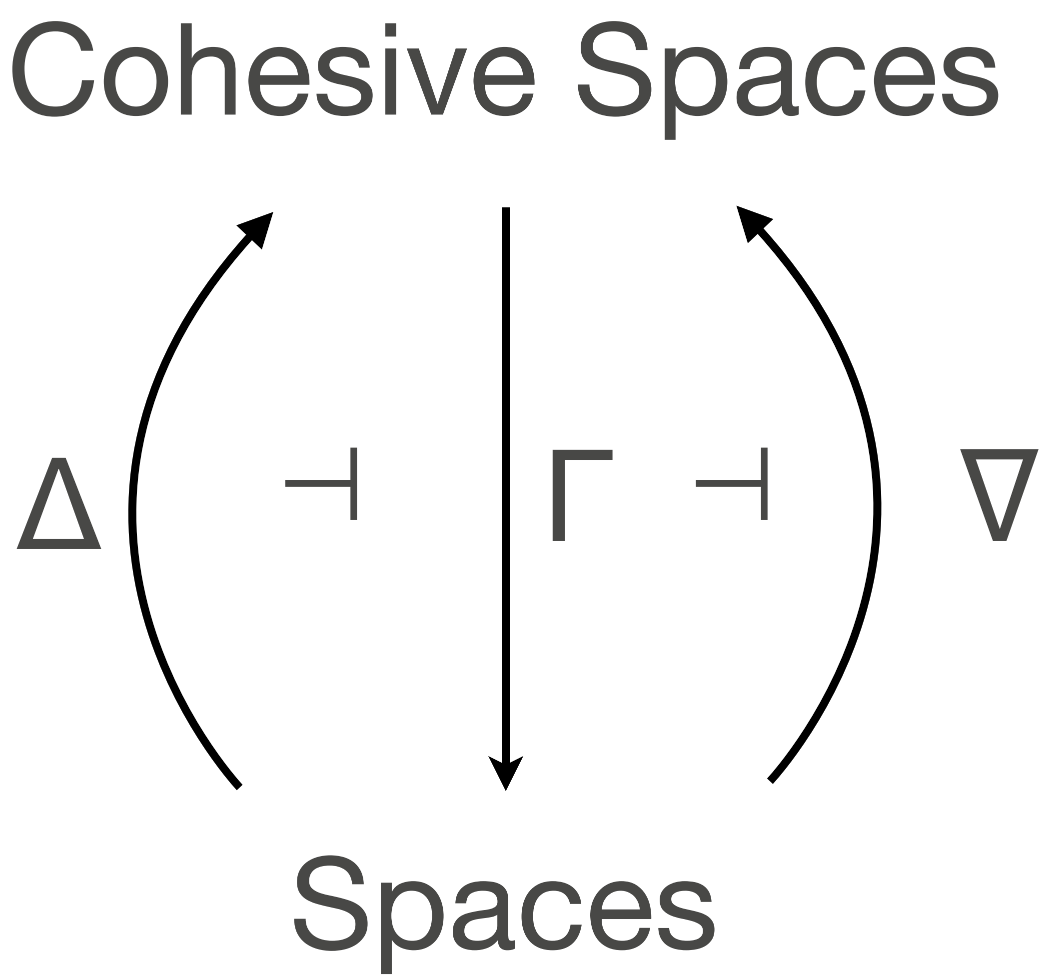
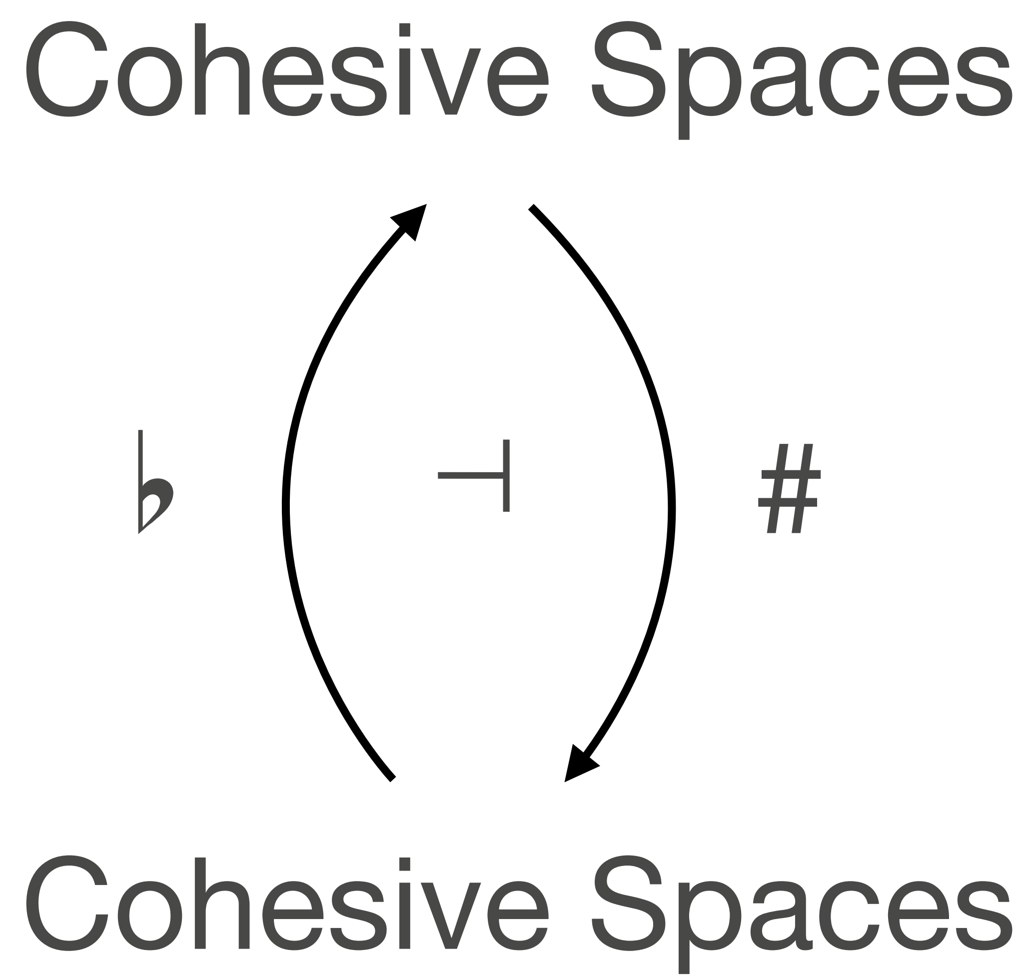
Today: how to add ♭ and # to the doctrine
what can we do with them?

A Framework for Functors in Unary Type Theory

Example instances (doctrines)





♭ idempotent comonad

idempotent monad

Mode theory

Theory in **framework**,
specifying a **doctrine**

Needs:

Mode theory

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Needs:

- multiple “modes” of types representing different categories, morphisms in each

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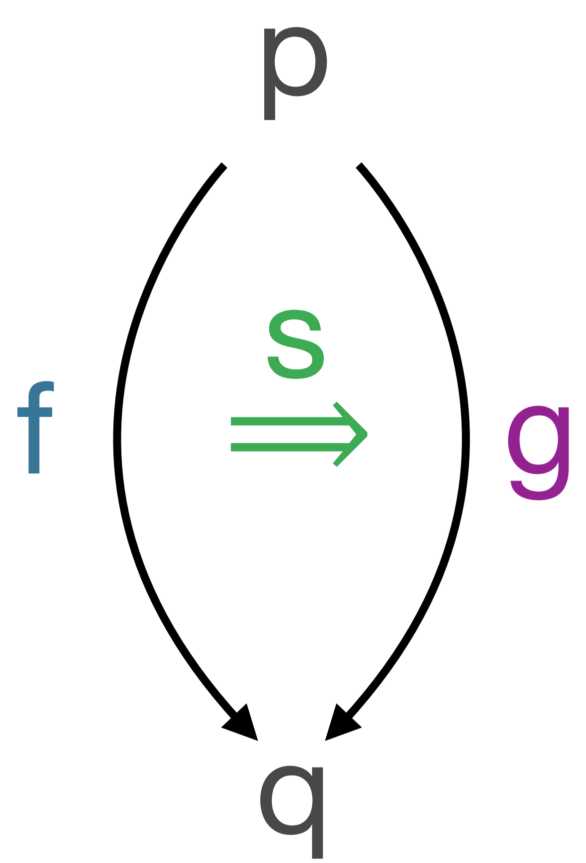
Mode theory

Theory in **framework**, specifying a **doctrine**

A mode theory $\mathcal{M}$ is a 2-category



Needs:

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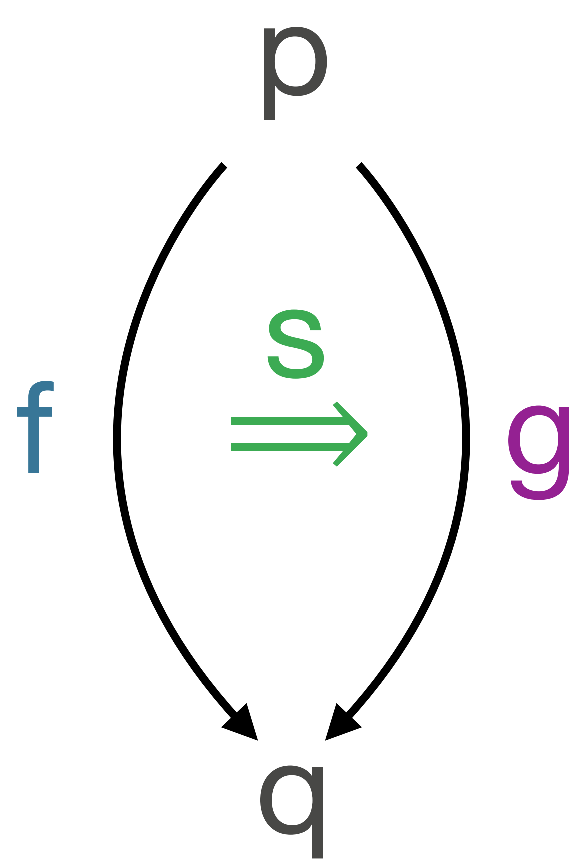
Doctrine of a mode theory

A mode theory $\mathcal{M}$ is a 2-category

Specifies doctrine of a pseudofunctor $\mathcal{M} \to \mathbf{Cat}$



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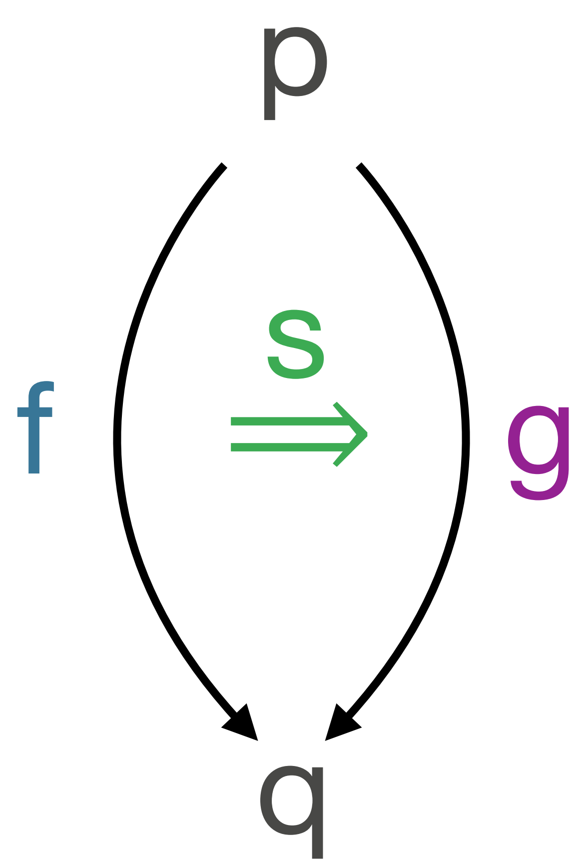

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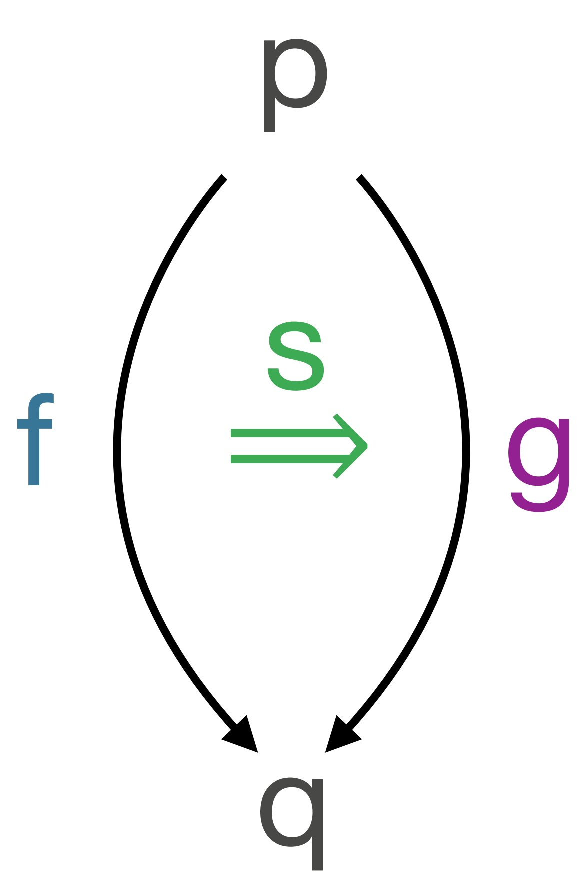

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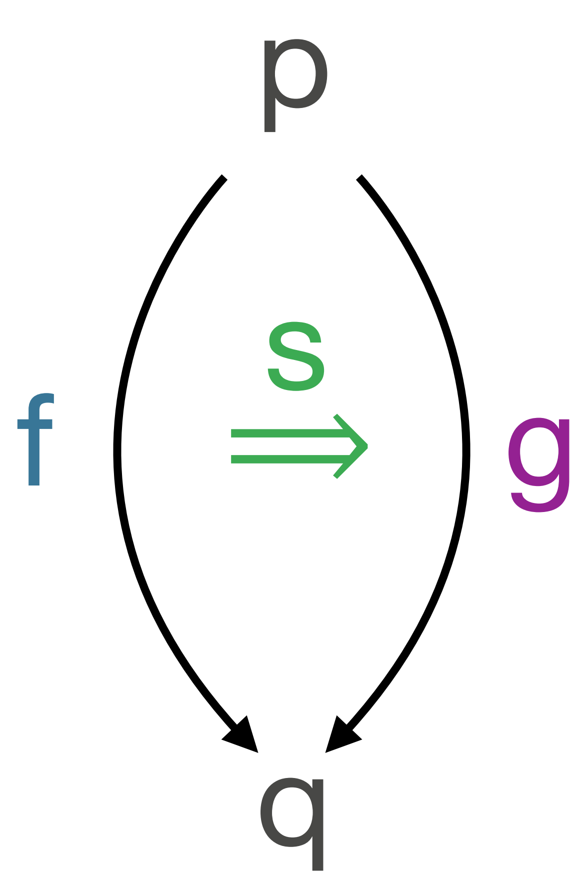

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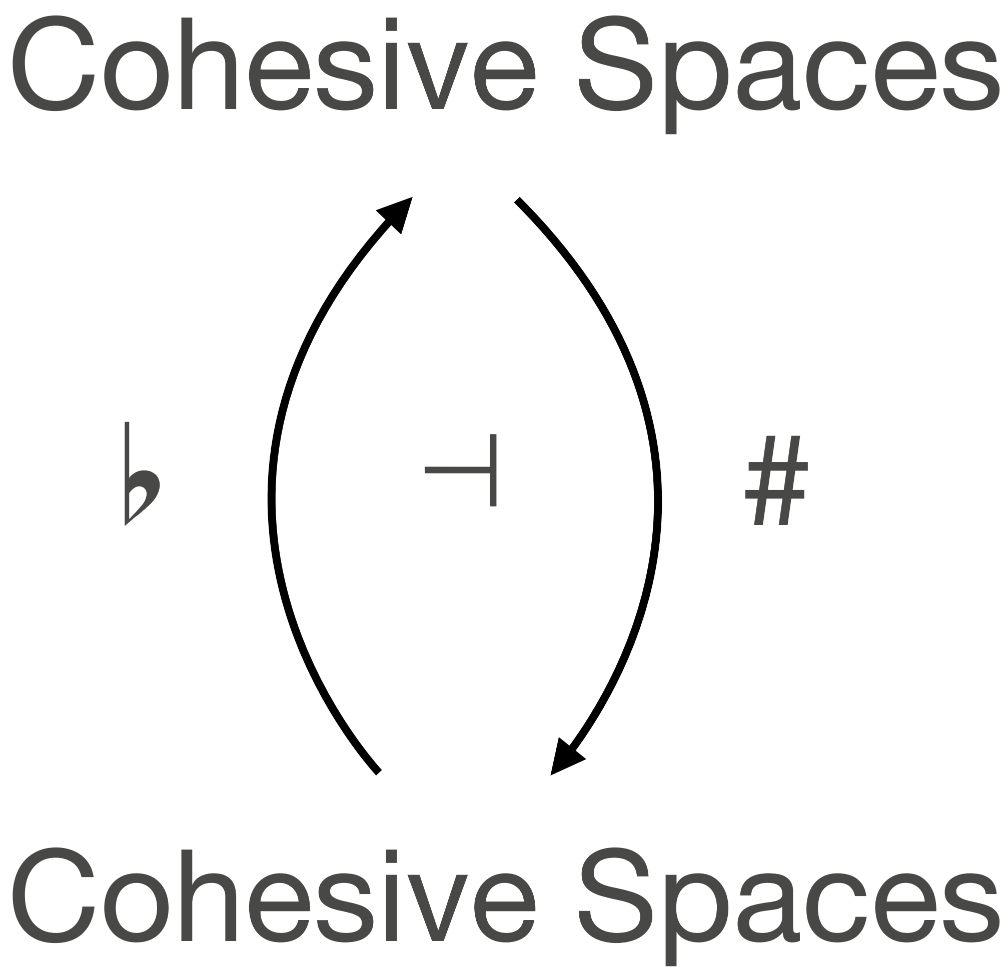
- each 1-cell f is a functor $\mathbf{F}_f : p \to q$
- each 2-cell $s : f \Rightarrow g$ is a nat. trans. $\mathbf{F}_f \Rightarrow \mathbf{F}_g$

Example mode theory 1



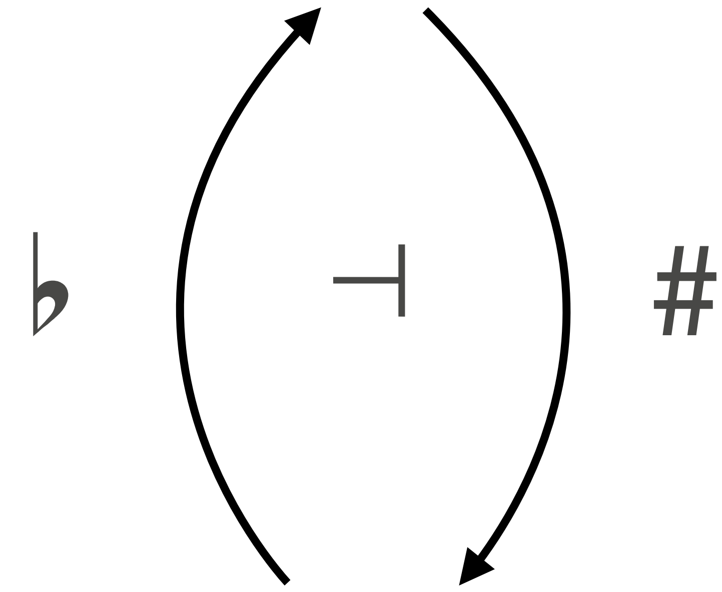
♭ idempotent comonad

idempotent monad

Example mode theory 1

c mode

Cohesive Spaces



Cohesive Spaces

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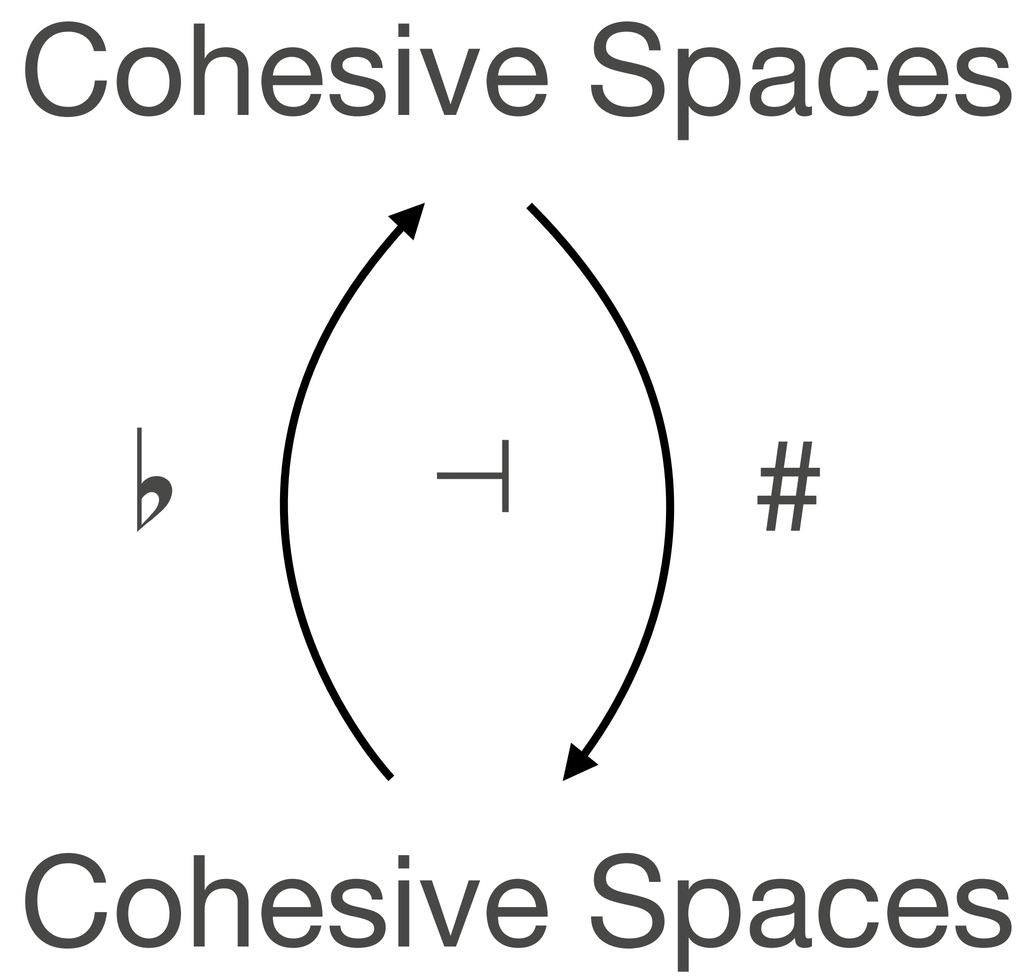
$\flat, \# : c \to c$



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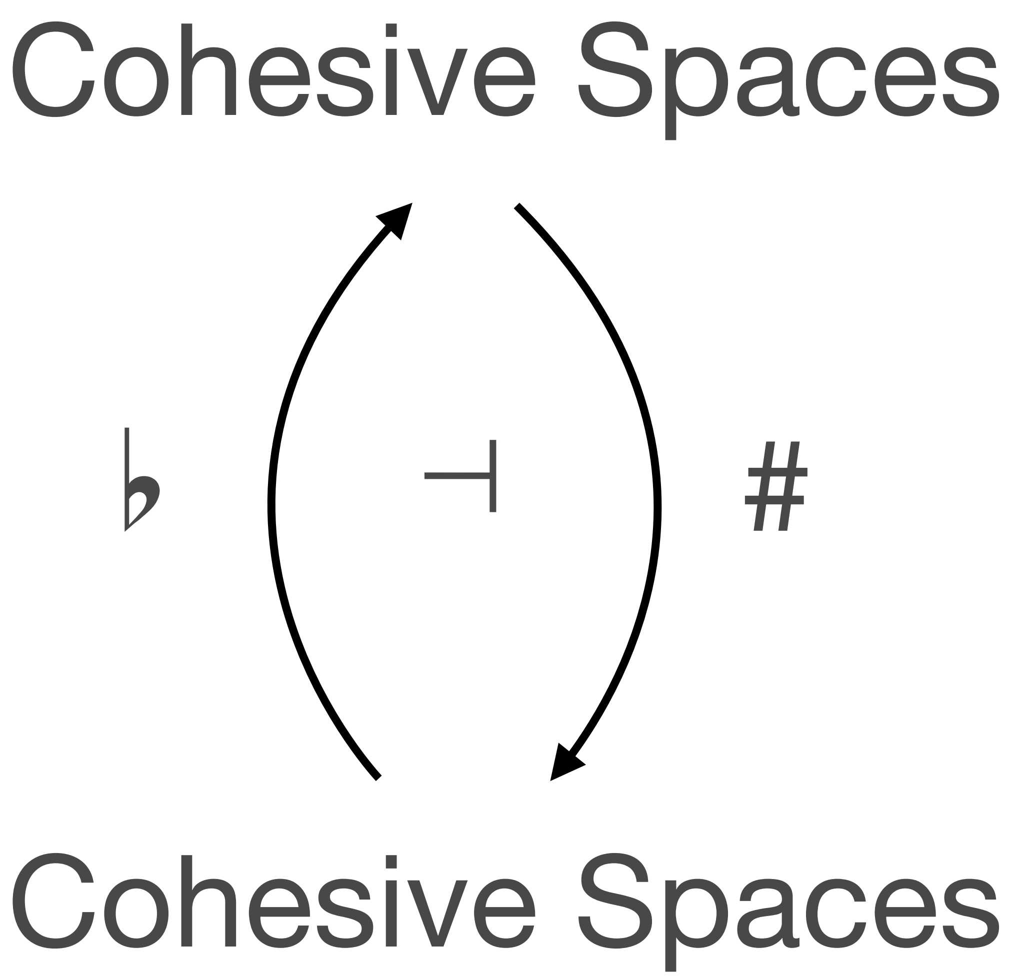

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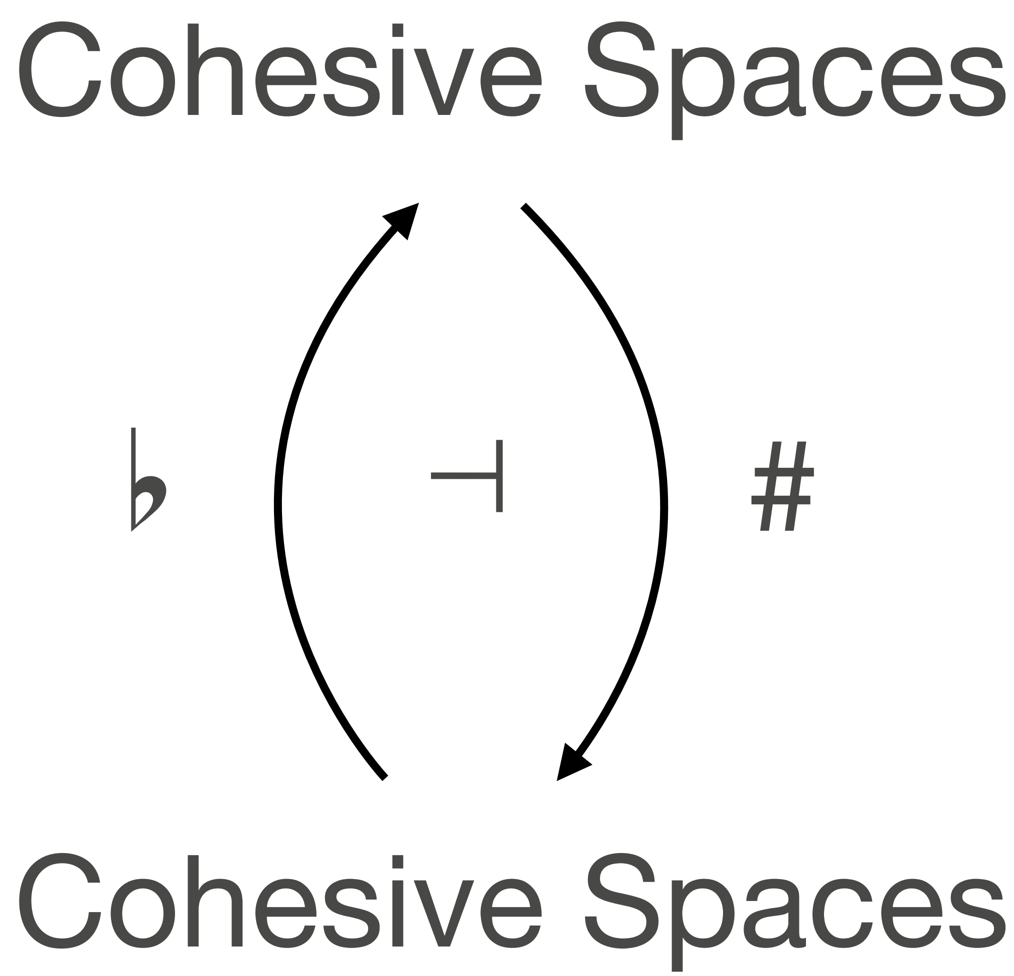

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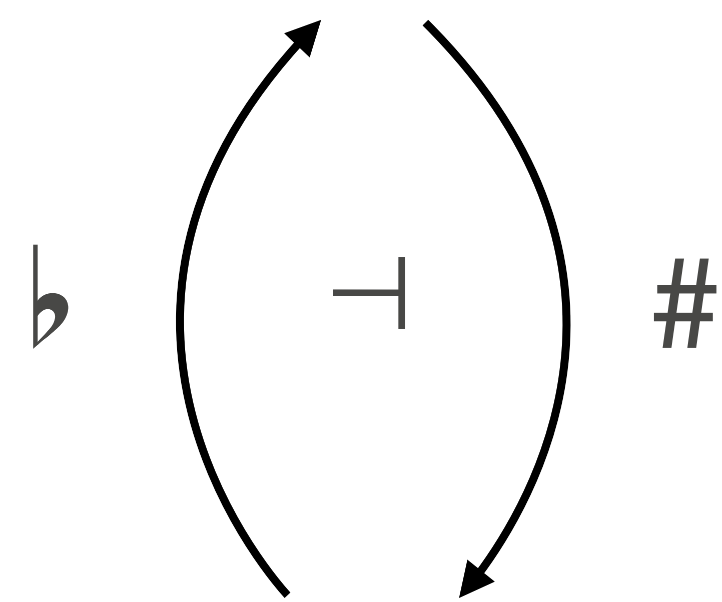

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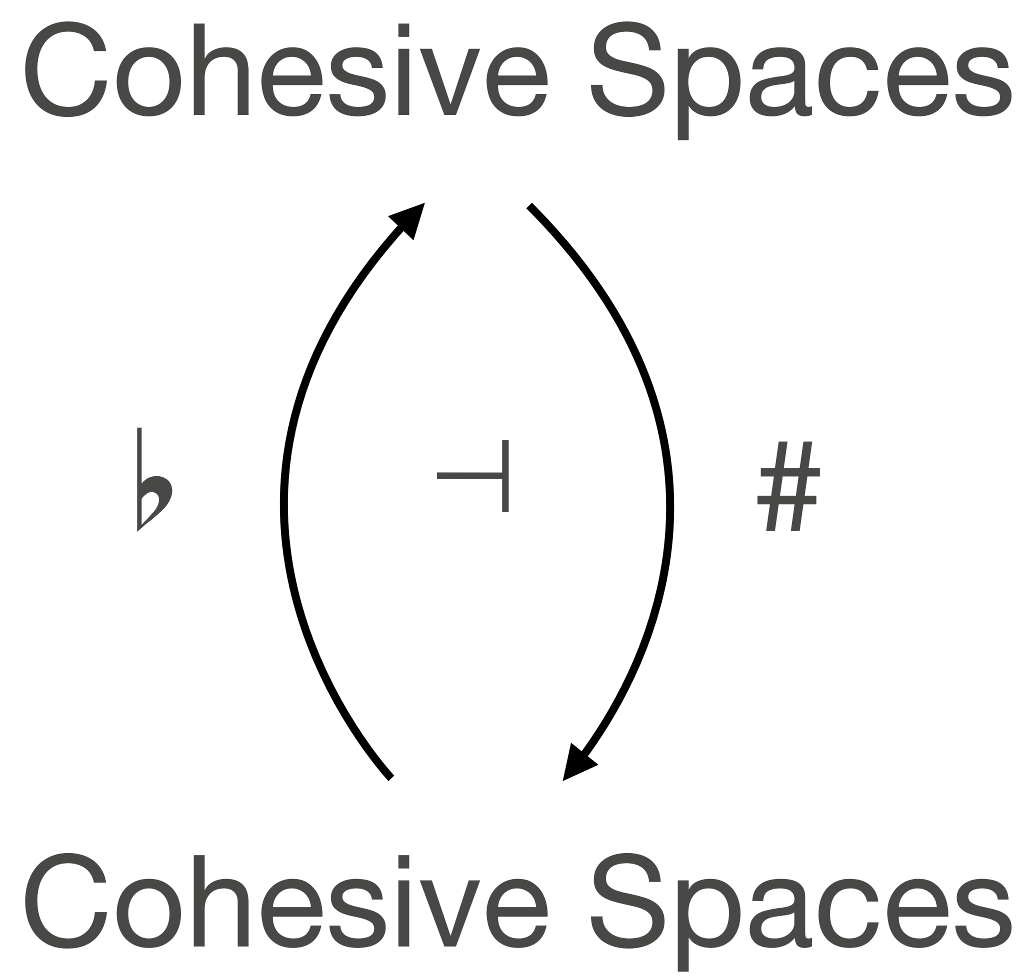

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$\#\# = \#$

Cohesive Spaces

$\flat \;\dashv\; \#$

Cohesive Spaces

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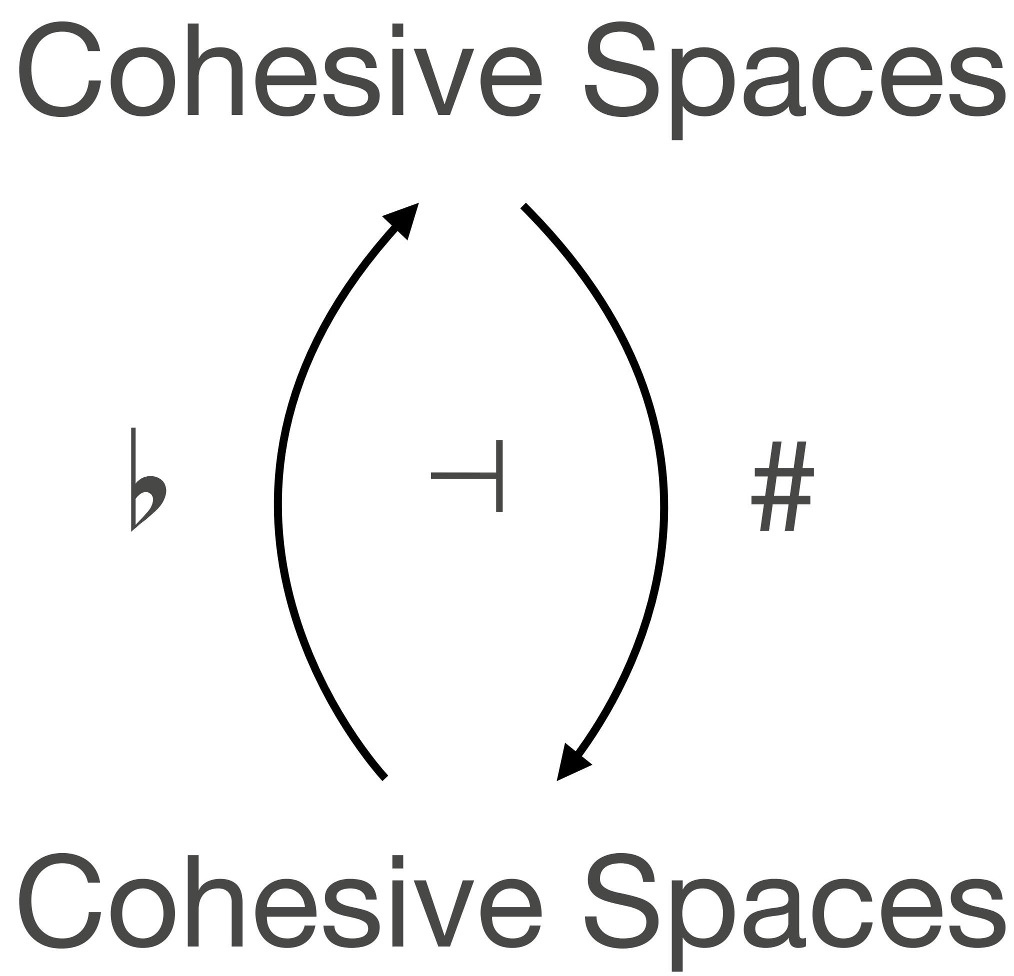

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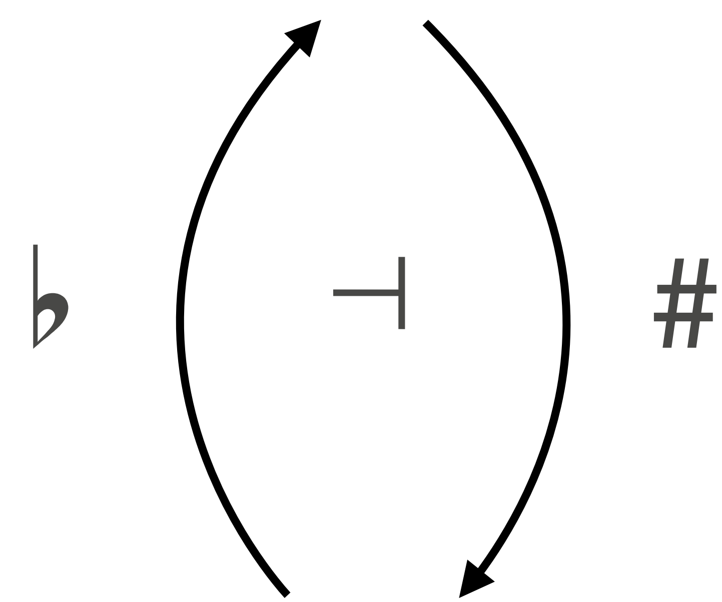
[+ triangle]

Cohesive Spaces



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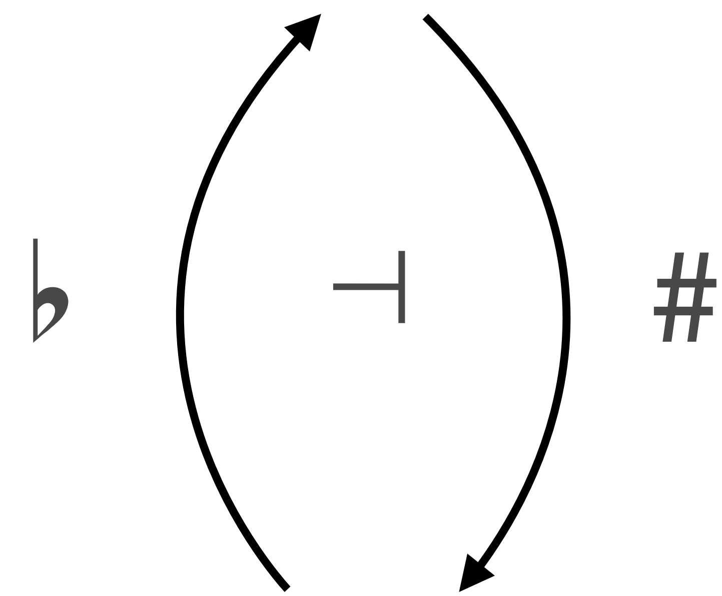
[+ triangle]

contexts stricter than types!

Cohesive Spaces



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Example mode theory 2

c,s mode

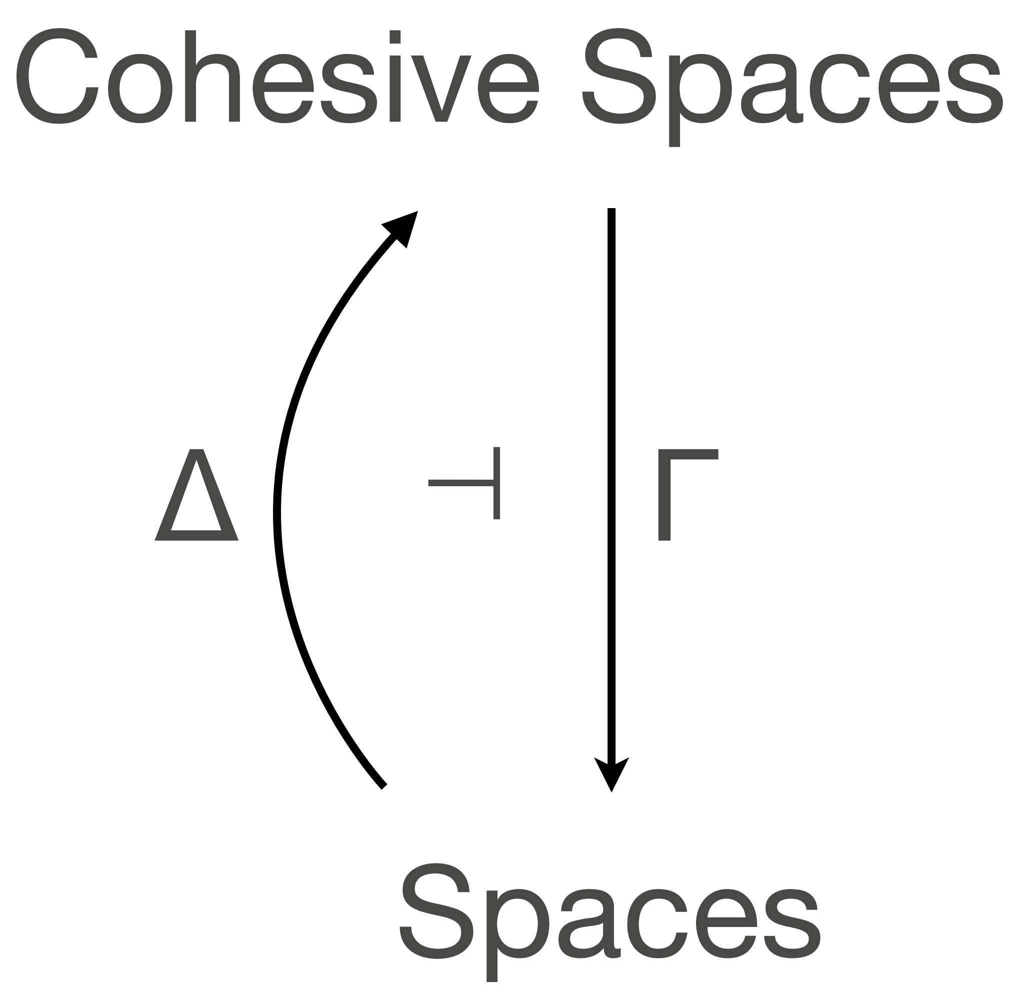
$\Delta : s \to c$

$\Gamma : c \to s$

$\text{counit} : \Delta\Gamma \Rightarrow 1_c$

$\text{unit} : 1_s \Rightarrow \Gamma\Delta$

[+ triangle equations]



Framework (non-judgemental)

$$\frac{}{A \vdash_p A} \qquad \frac{A \vdash_p B \quad B \vdash_p C}{A \vdash_p C}$$

$$\frac{A \text{ type}_p \qquad f : p \to q}{F_f\, A \text{ type}_q} \qquad \frac{A \vdash_p A'}{F_f\, A \vdash_q F_f\, A'}$$

$$\frac{}{F_1\, A \vdash A} \qquad \frac{}{A \vdash F_1\, A} \qquad \frac{}{F_{g \circ f}\, A \vdash F_g\, F_f\, A} \qquad \frac{}{F_g\, F_f\, A \vdash F_{g \circ f}\, A}$$

$$\frac{f \Longrightarrow g}{F_f\, A \vdash F_g\, A}$$

[+ a lot of equations!]

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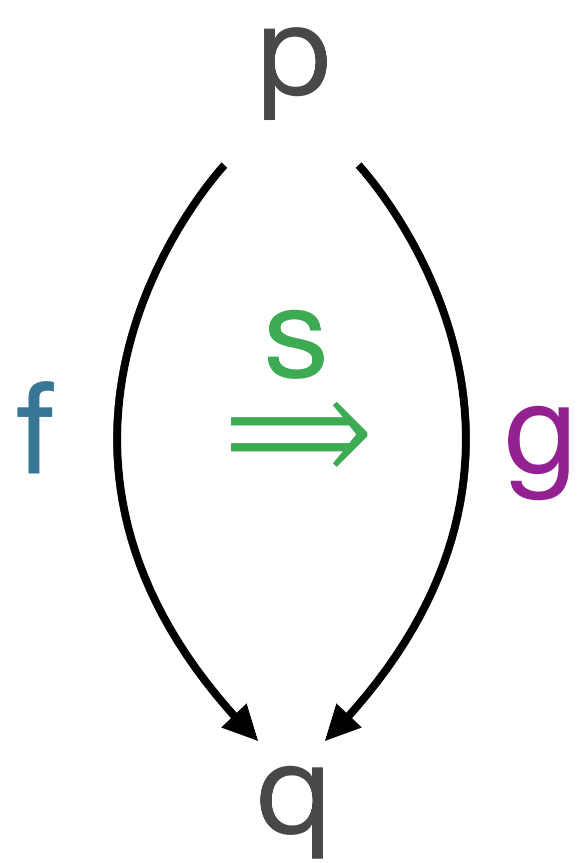
- not judgemental: hard to “predict” **F** types from judgements, lots of equations

Fibrational Framework

A mode theory $\mathcal{M}$ is a 2-category



Specifies doctrine of a local discrete (1-op)fibration $\pi : \mathcal{D} \to \mathcal{M}$

$\mathcal{D}$ is Groth. construction of previous pseudofunctor $\mathcal{M} \to$ Cat

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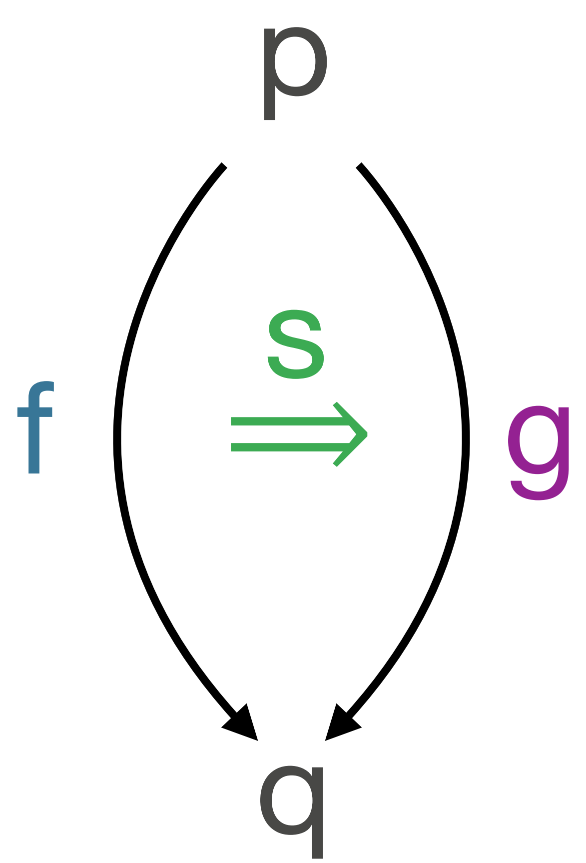

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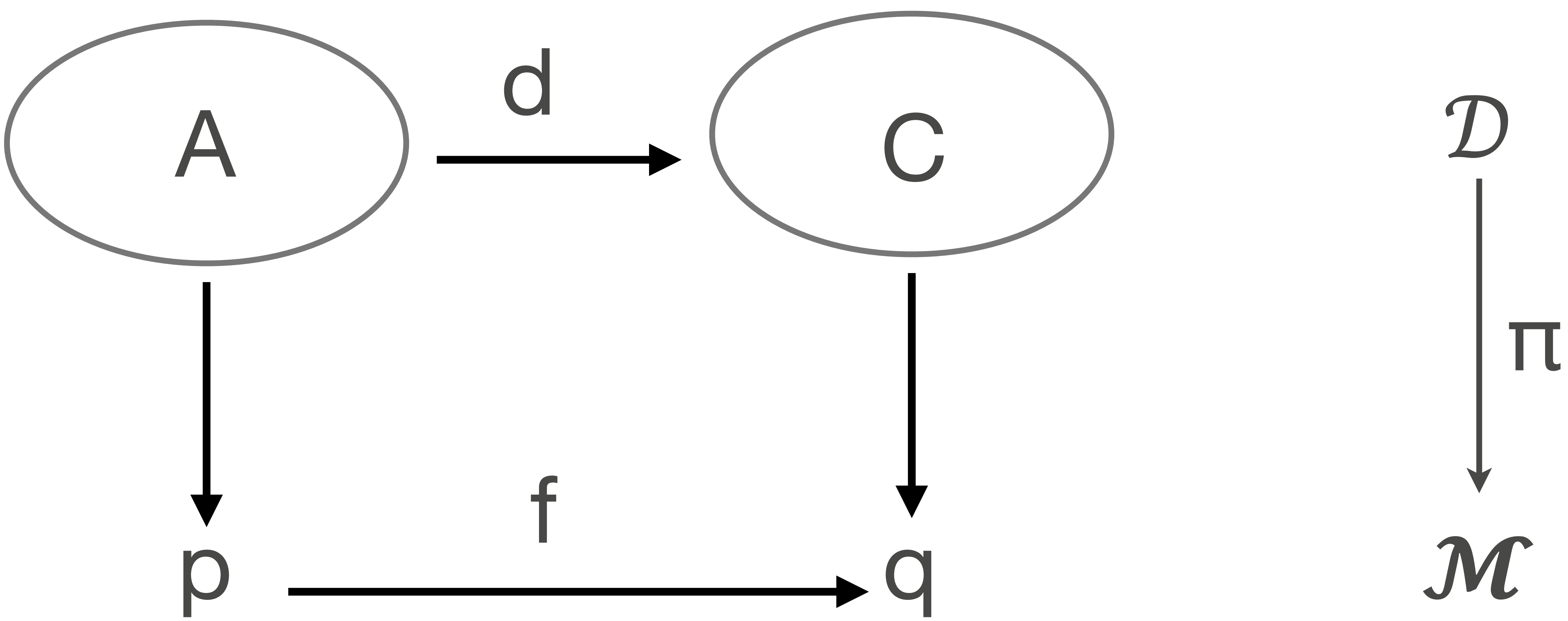
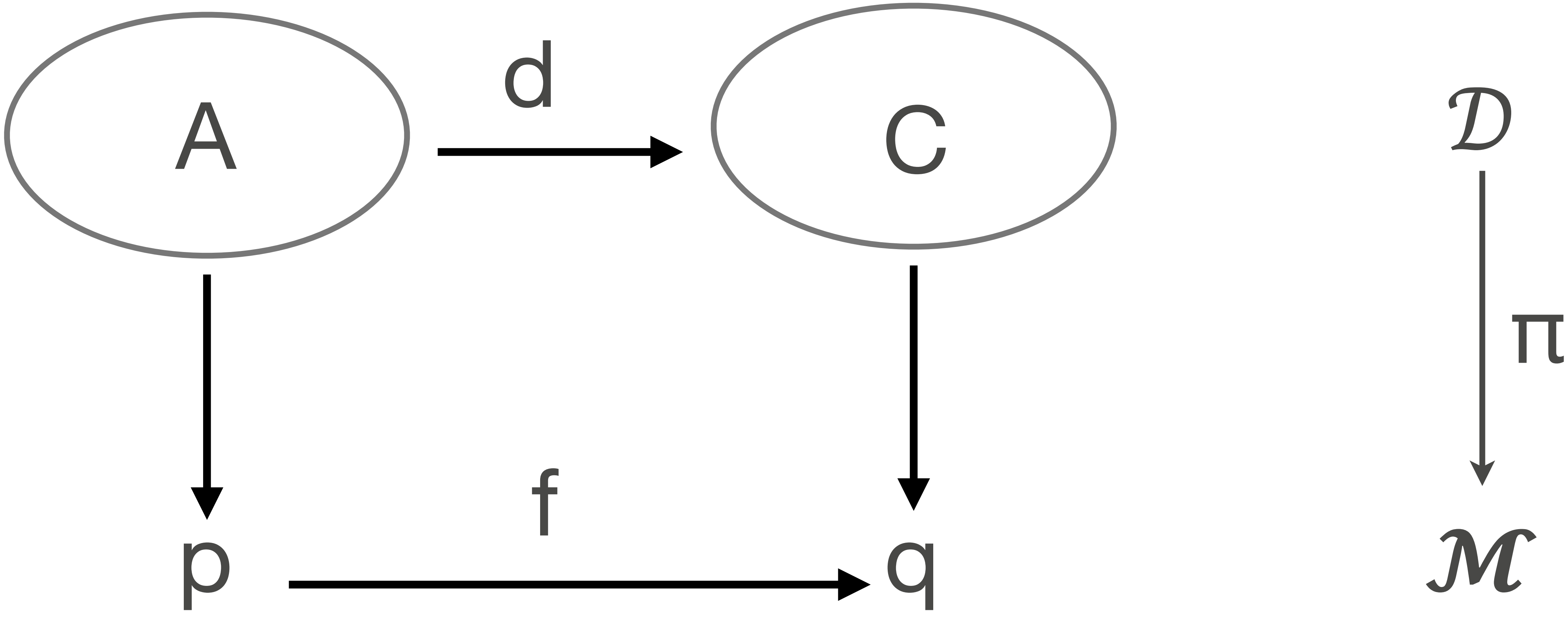
[Hermida,Buckley]

Fibrational Framework



Fibrational Framework

$d : A \vdash_f C$ means d in $\mathcal{D}(A,C)$
with $\pi(d) = f$



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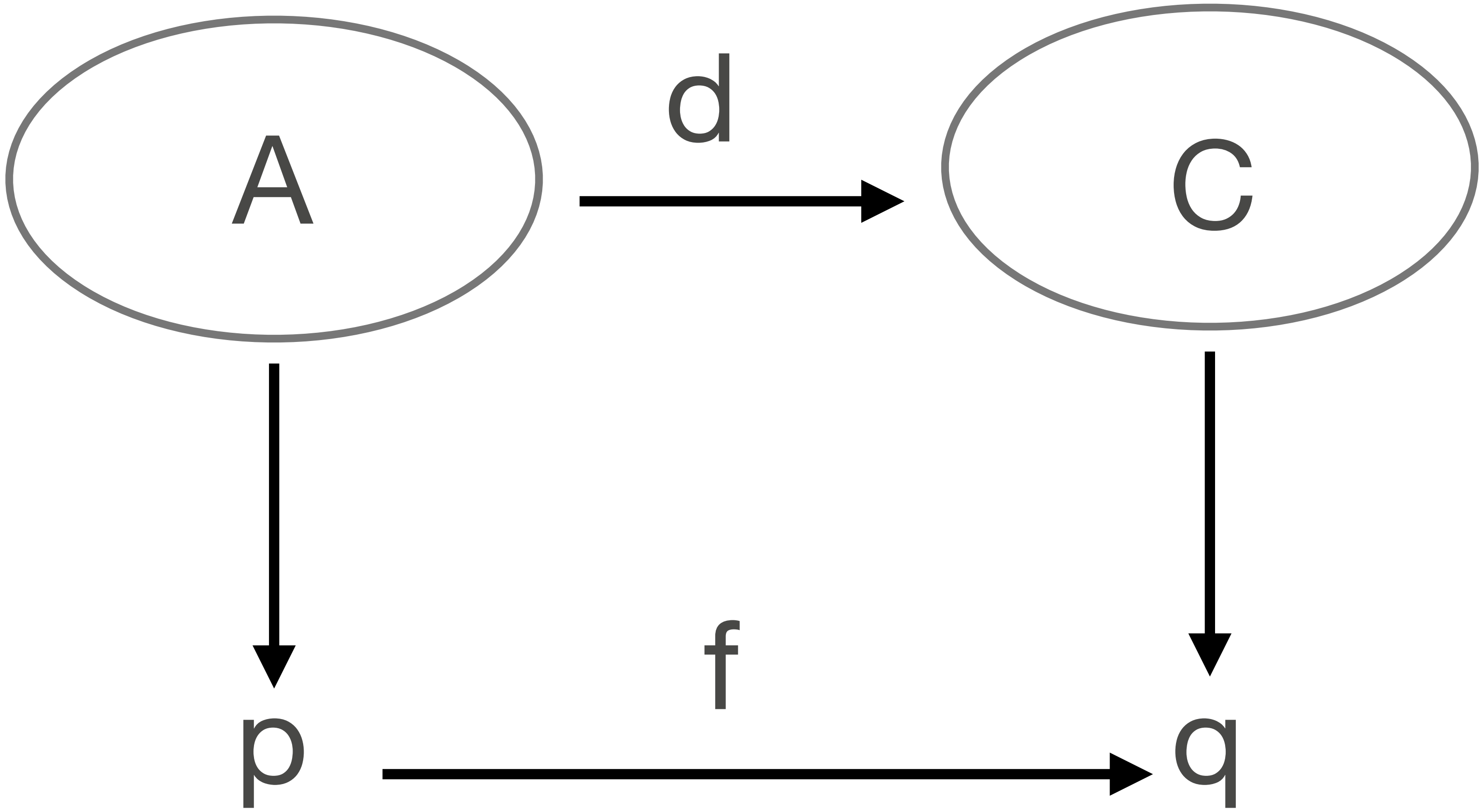
morphism over a morphism; c.f. pathovers and Melliès,Zeilberger'15



Identity and Cut/Composition

$$\frac{}{A \vdash_1 A} \qquad \frac{A \vdash_f B \quad B \vdash_g C}{A \vdash_{g \circ f} C}$$

(strict) functoriality of

$$\begin{array}{c} \mathcal{D} \\ \downarrow \pi \\ \mathcal{M} \end{array}$$

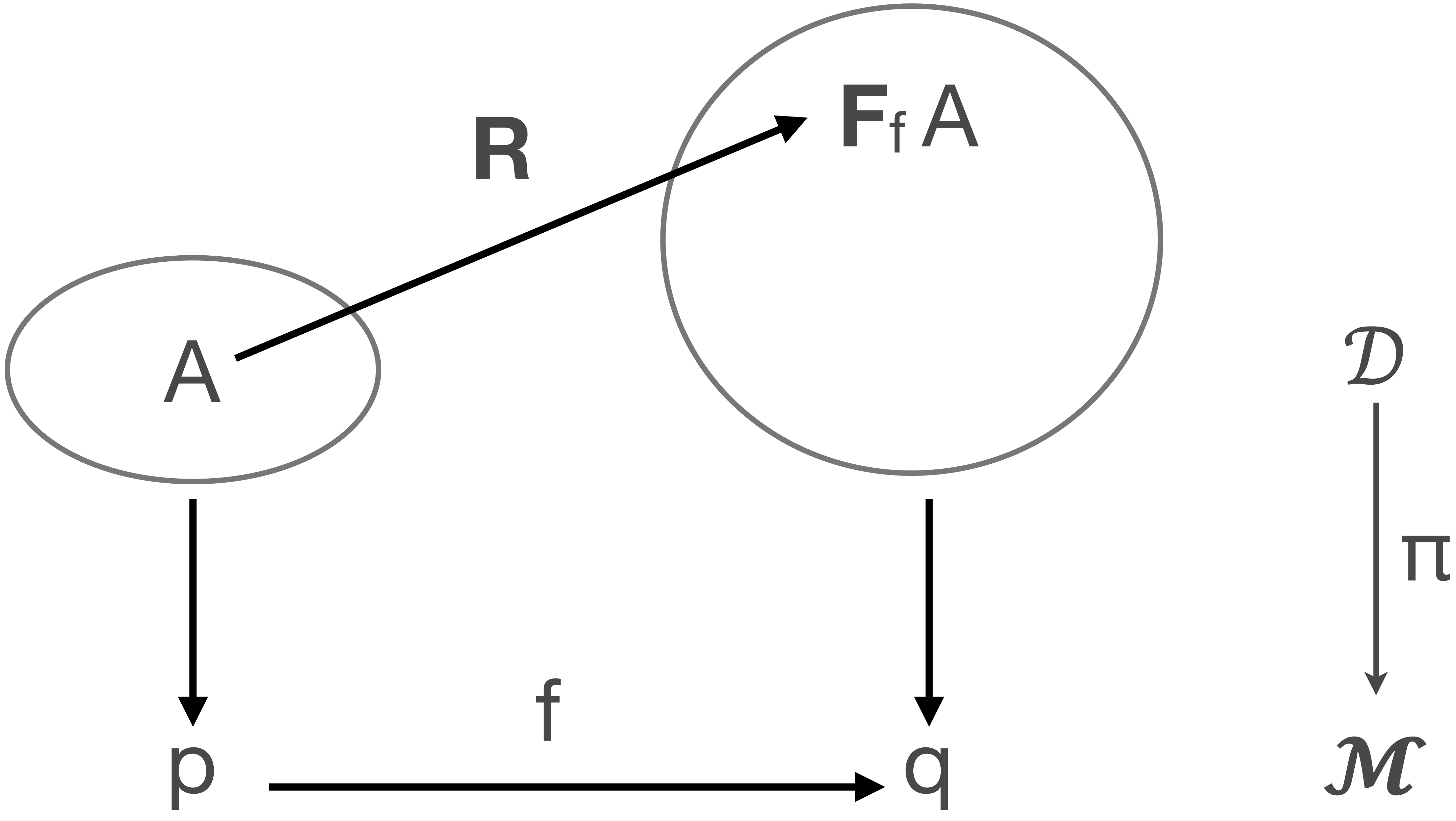
Action of mode 2-cells

$$\frac{A \vdash_g C \quad s : f \Rightarrow g}{A \vdash_f C}$$

$$\mathcal{D}(A,B) \xrightarrow{\pi} \mathcal{M}(p,q)$$

discrete fibration of 1-categories

F types: opfibration



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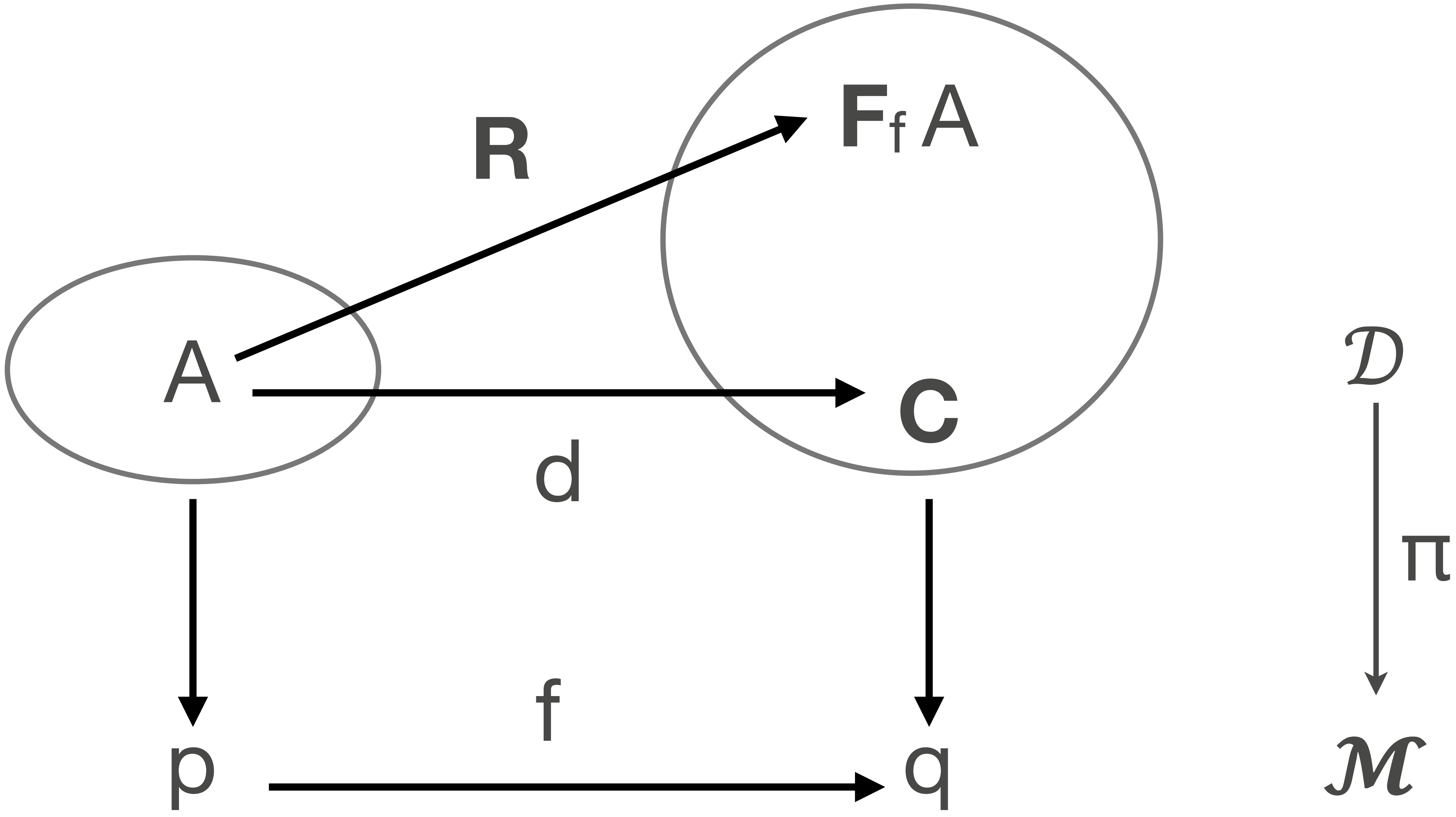

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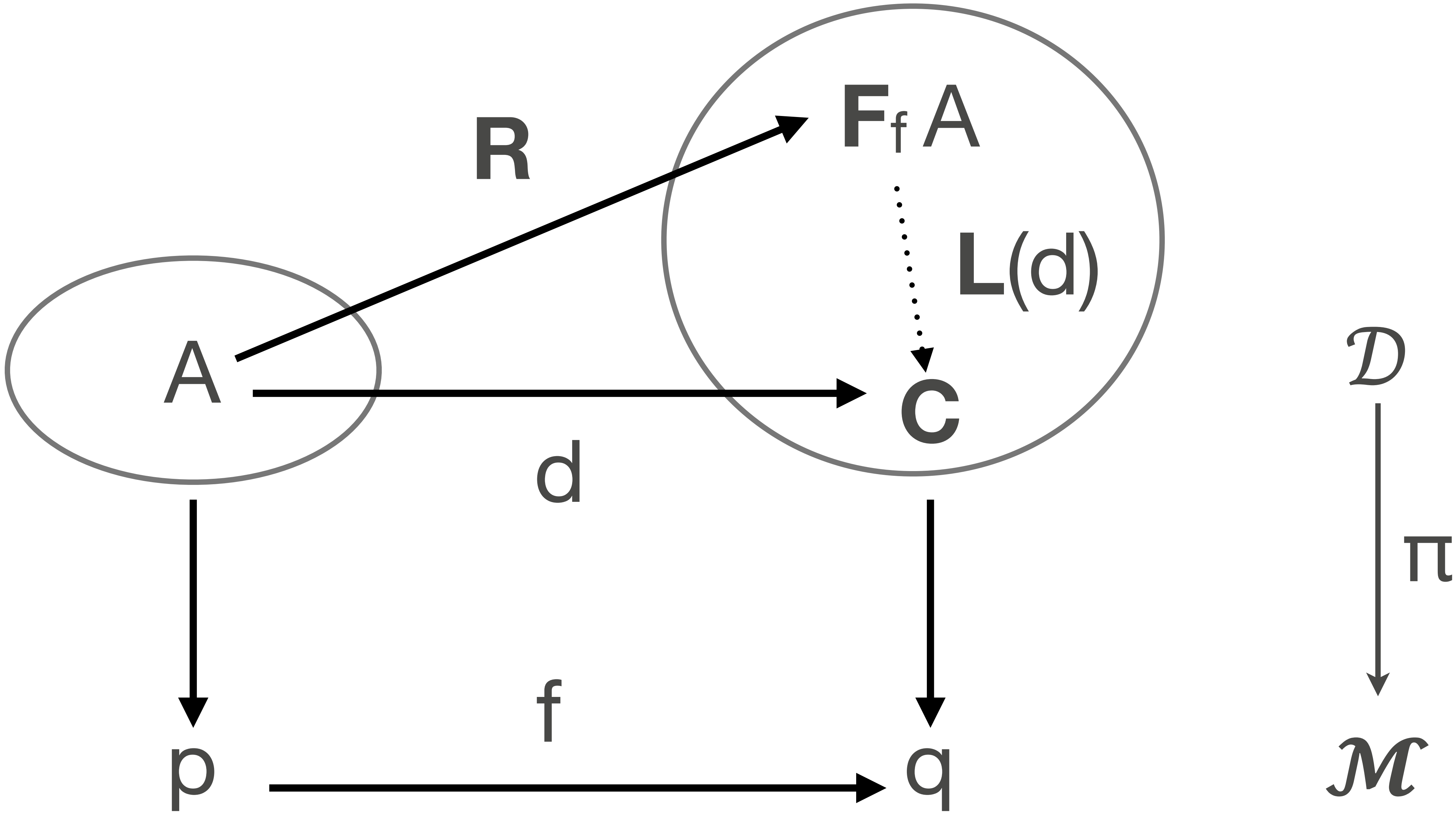

$$p \xrightarrow[\mathcal{M}]{\mathrm{f}} q$$

$$p \xrightarrow[\mathcal{M}]{f} q$$

$$\frac{A \text{ type}_p}{\mathbf{F}_f\, A \text{ type}_q}$$

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$$\frac{A\ \mathrm{type}_p}{\mathbf{F}_f\, A\ \mathrm{type}_q}$$

$$\pi(\mathbf{F}_f\, A) = q$$

$$p \xrightarrow{f} q \quad \mathcal{M}$$

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$$\frac{}{A \vdash_f \mathbf{F}_f\, A}\ \mathbf{R}$$

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$$\mathcal{D}_f(A, \mathbf{F}_f A)$$

$$p \xrightarrow{\;f\;} q \quad \mathcal{M}$$

$$\frac{A\ \mathrm{type}_p}{\mathbf{F}_f\, A\ \mathrm{type}_q}$$

$$\pi(\mathbf{F}_f\, A) = q$$

$$\frac{}{A \vdash_f \mathbf{F}_f\, A}\ \mathbf{R}$$

$$\mathcal{D}_f(A, \mathbf{F}_f A)$$

$$\frac{A \vdash_{g \circ f} C}{\mathbf{F}_f\, A \vdash_g C}\ \mathbf{L}$$

$$p \xrightarrow{\;f\;}_{\mathcal{M}} q$$

$$\frac{A\ \mathrm{type}_p}{\mathbf{F}_f\, A\ \mathrm{type}_q} \qquad\qquad \pi(\mathbf{F}_f\, A) = q$$

$$\frac{}{A \vdash_f \mathbf{F}_f\, A}\ \mathbf{R} \qquad\qquad \mathcal{D}_f(A, \mathbf{F}_f A)$$

$$\frac{A \vdash_{g \circ f} C}{\mathbf{F}_f\, A \vdash_g C}\ \mathbf{L}$$

$$\begin{array}{ccc} \mathcal{D}(\mathbf{F}_f A, C) & \xrightarrow{\;-\circ R\;} & \mathcal{D}(A, C) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{M}(q, r) & \xrightarrow{\;-\circ f\;} & \mathcal{M}(p, r) \end{array}$$

(of 1-cats)

$$\frac{}{A \vdash_1 A} \qquad \frac{A \vdash_f B \quad B \vdash_g C}{A \vdash_{g \circ f} C}$$

a;id = a = id;a

(a;b);c = (a;b);c

$$\frac{A \vdash_g C \qquad f \Longrightarrow g}{A \vdash_f C}$$

1*(a) = a

(s;t)*(a) = s*(t*a)

(s[t])*(a;b) = t*a;s*b

$$\frac{}{A \vdash_f \mathbf{F}_f A}\ \mathbf{R}$$

$$\frac{A \vdash_{g \circ f} C}{\mathbf{F}_f A \vdash_g C}\ \mathbf{L}$$

R;L(d) = d β

d = **L**(**R**;d) η

Theorems

$$\frac{A \vdash_{g \circ f} C}{\mathbf{F}_f\, A \vdash_g C} \qquad \frac{A \vdash_p A'}{F_f\, A \vdash_q F_f\, A'} \qquad \frac{f \Longrightarrow g}{F_f\, A \vdash F_g\, A}$$

(the first rule is drawn with a double line)

$$\frac{}{F_1\, A \vdash A} \qquad \frac{}{A \vdash F_1\, A} \qquad \frac{}{F_{g \circ f}\, A \vdash F_g\, F_f\, A} \qquad \frac{}{F_g\, F_f\, A \vdash F_{g \circ f}\, A}$$

[+ a lot of equations!]

Example mode theory

c mode

$\flat, \sharp : c \to c$

counit : $\flat \Rightarrow 1_c$

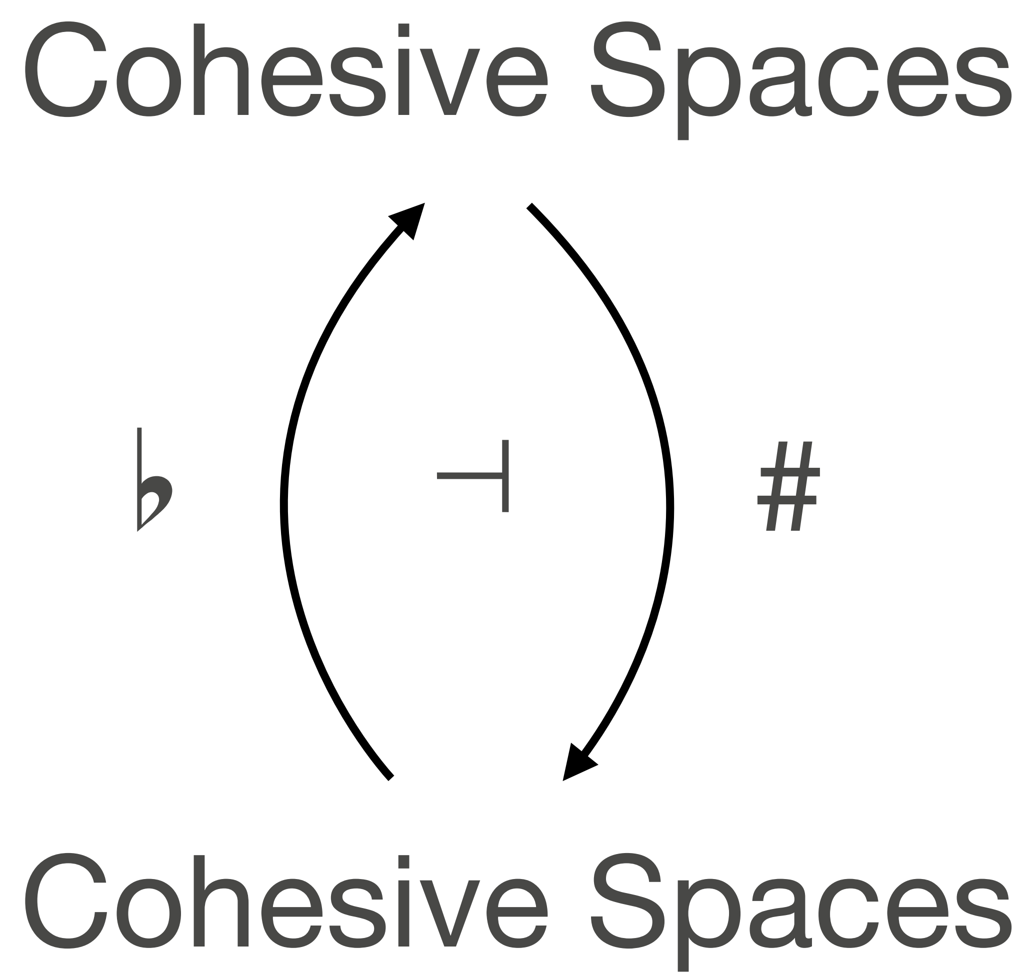
unit : $1_c \Rightarrow \sharp$

$\flat\flat = \flat \quad \flat\sharp = \flat$

$\sharp\sharp = \sharp \quad \sharp\flat = \sharp$

[+ triangle]

weak types from strict contexts!



$\flat$ idempotent comonad

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Examples of derivations

$$\frac{}{\mathbf{F}\flat\mathbf{F}\flat A \vdash_1 \mathbf{F}\flat\, A}$$

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$$\frac{\mathbf{F}_\flat\, A \vdash_\flat \mathbf{F}_\flat\, A}{\mathbf{F}_\flat \mathbf{F}_\flat A \vdash_1 \mathbf{F}_\flat\, A}$$

Examples of derivations

$$\cfrac{\cfrac{\mathbf{F}_\flat\, A \vdash_1 \mathbf{F}_\flat\, A \quad \text{counit} : \flat \Longrightarrow 1}{\mathbf{F}_\flat\, A \vdash_\flat \mathbf{F}_\flat\, A}}{\mathbf{F}_\flat \mathbf{F}_\flat A \vdash_1 \mathbf{F}_\flat\, A}$$

Examples of derivations

$$\dfrac{\dfrac{\mathbf{F}_\flat\, A \vdash_1 \mathbf{F}_\flat\, A \quad \text{counit} : \flat \Longrightarrow 1}{\mathbf{F}_\flat\, A \vdash_\flat \mathbf{F}_\flat\, A}}{\mathbf{F}_\flat \mathbf{F}_\flat A \vdash_1 \mathbf{F}_\flat\, A}$$

$$\dfrac{\dfrac{}{\mathbf{F}_\flat\, A \vdash_\flat \mathbf{F}_\flat\, A}}{\mathbf{F}_\flat \mathbf{F}_\flat A \vdash_1 \mathbf{F}_\flat\, A}$$

Examples of derivations

$$\dfrac{\dfrac{\mathbf{F}_\flat\, A \vdash_1 \mathbf{F}_\flat\, A \qquad \text{counit} : \flat \Longrightarrow 1}{\mathbf{F}_\flat\, A \vdash_\flat \mathbf{F}_\flat\, A}}{\mathbf{F}_\flat \mathbf{F}_\flat A \vdash_1 \mathbf{F}_\flat\, A}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash_{\flat\flat = \flat} \mathbf{F}_\flat\, A}}{\mathbf{F}_\flat\, A \vdash_\flat \mathbf{F}_\flat\, A}}{\mathbf{F}_\flat \mathbf{F}_\flat A \vdash_1 \mathbf{F}_\flat\, A}$$

Examples of derivations

$$\dfrac{\dfrac{\mathbf{F}_\flat A \vdash_1 \mathbf{F}_\flat A \quad \text{counit} : \flat \Rightarrow 1}{\mathbf{F}_\flat A \vdash_\flat \mathbf{F}_\flat A}}{\mathbf{F}_\flat \mathbf{F}_\flat A \vdash_1 \mathbf{F}_\flat A}$$

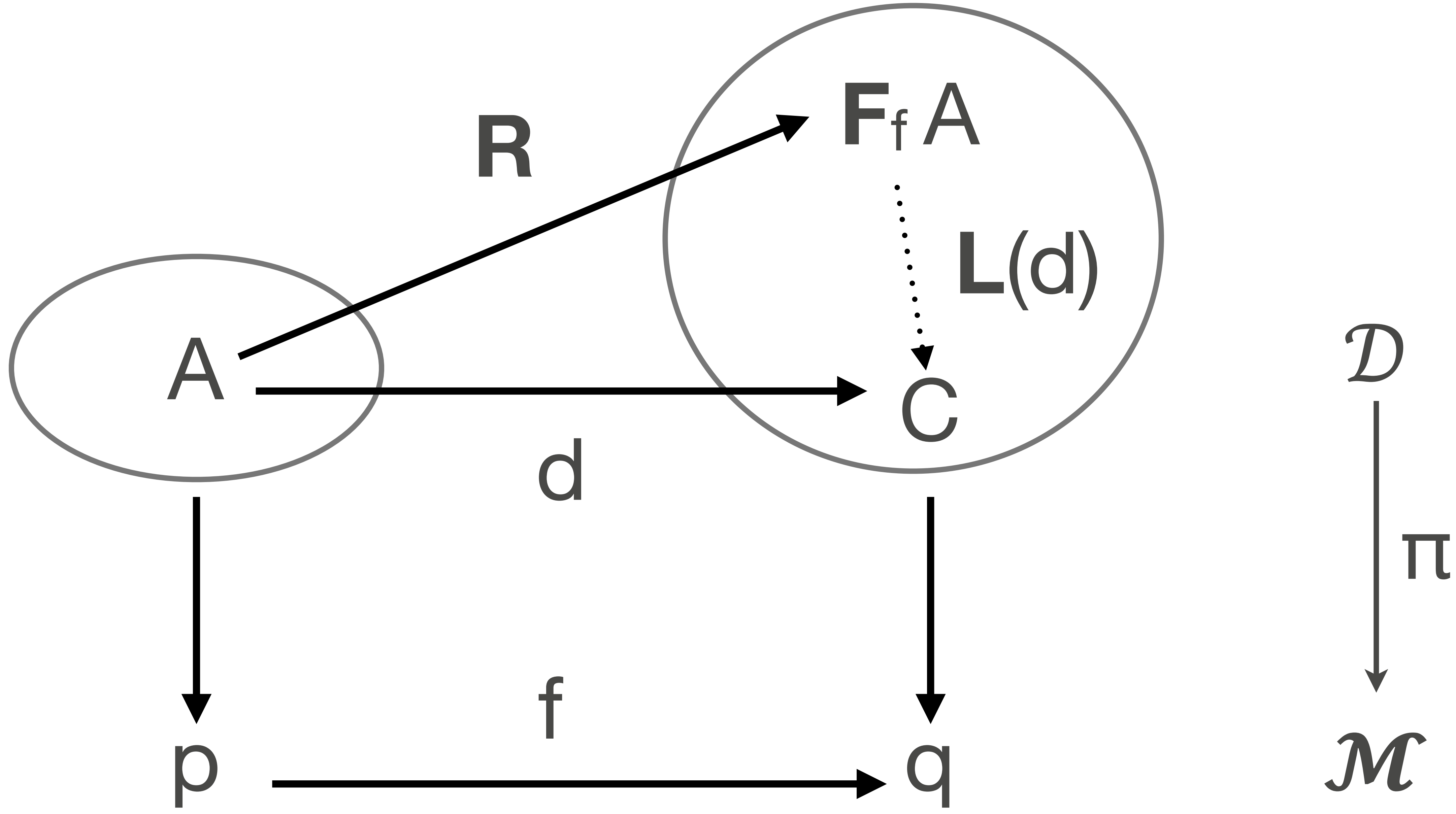
equal in equational theory using triangle law

$$\dfrac{\dfrac{\dfrac{}{A \vdash_{\flat\flat = \flat} \mathbf{F}_\flat A}}{\mathbf{F}_\flat A \vdash_\flat \mathbf{F}_\flat A}}{\mathbf{F}_\flat \mathbf{F}_\flat A \vdash_1 \mathbf{F}_\flat A}$$

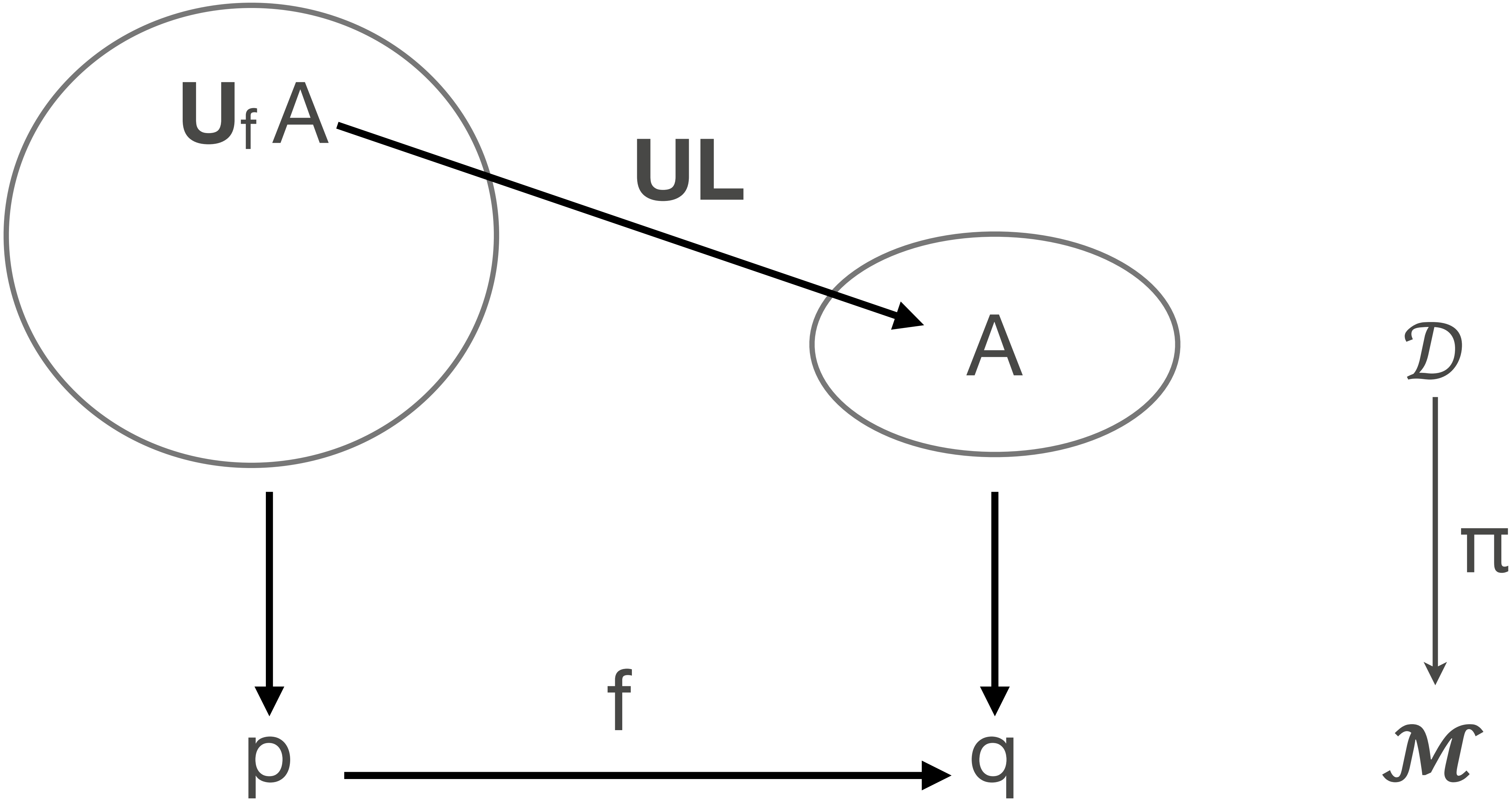
A Framework for Adjunctions in Unary Type Theory

[L., Shulman, ’16,
2-categorification of Reed’09]

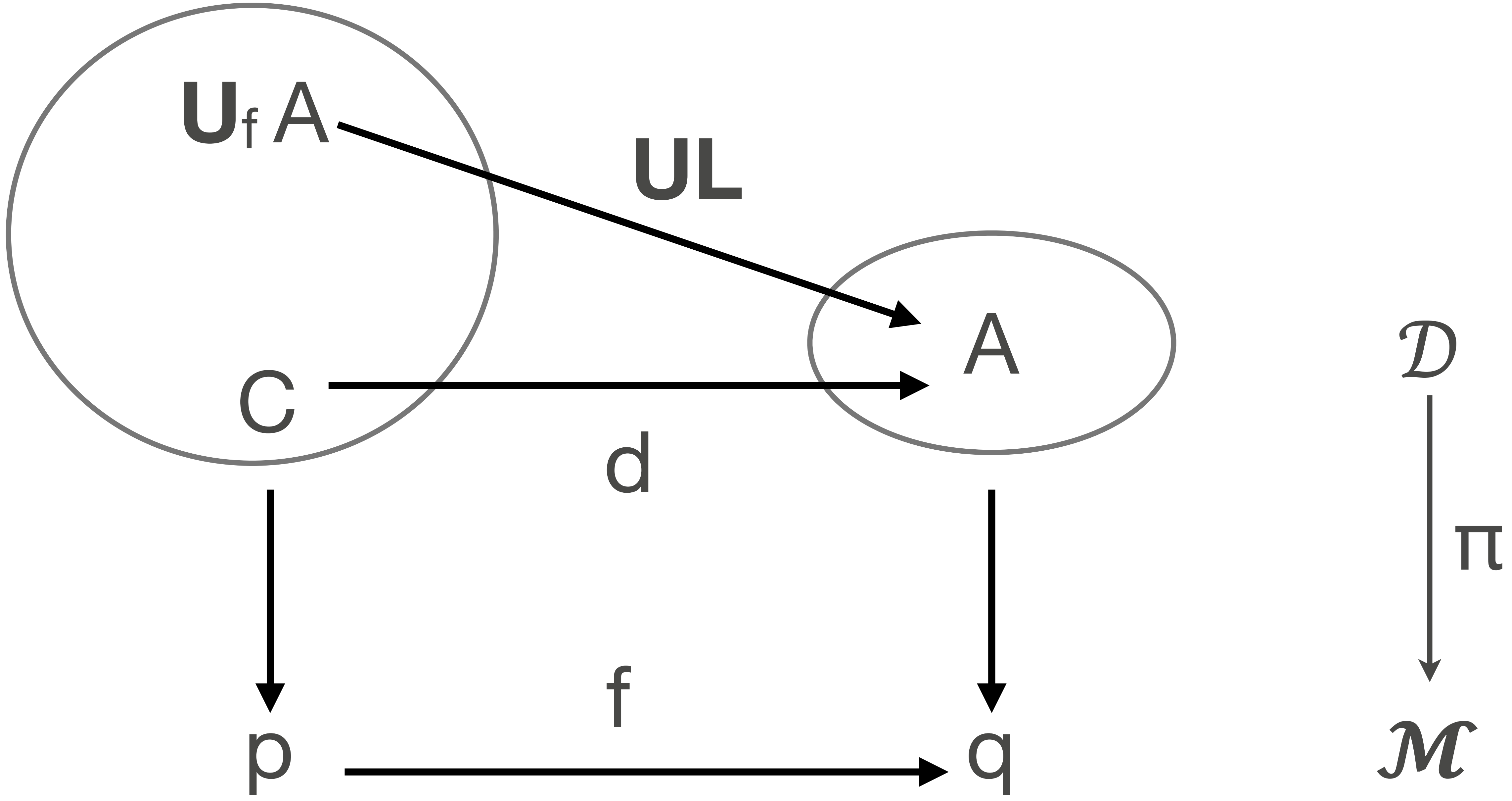
F types: opfibration



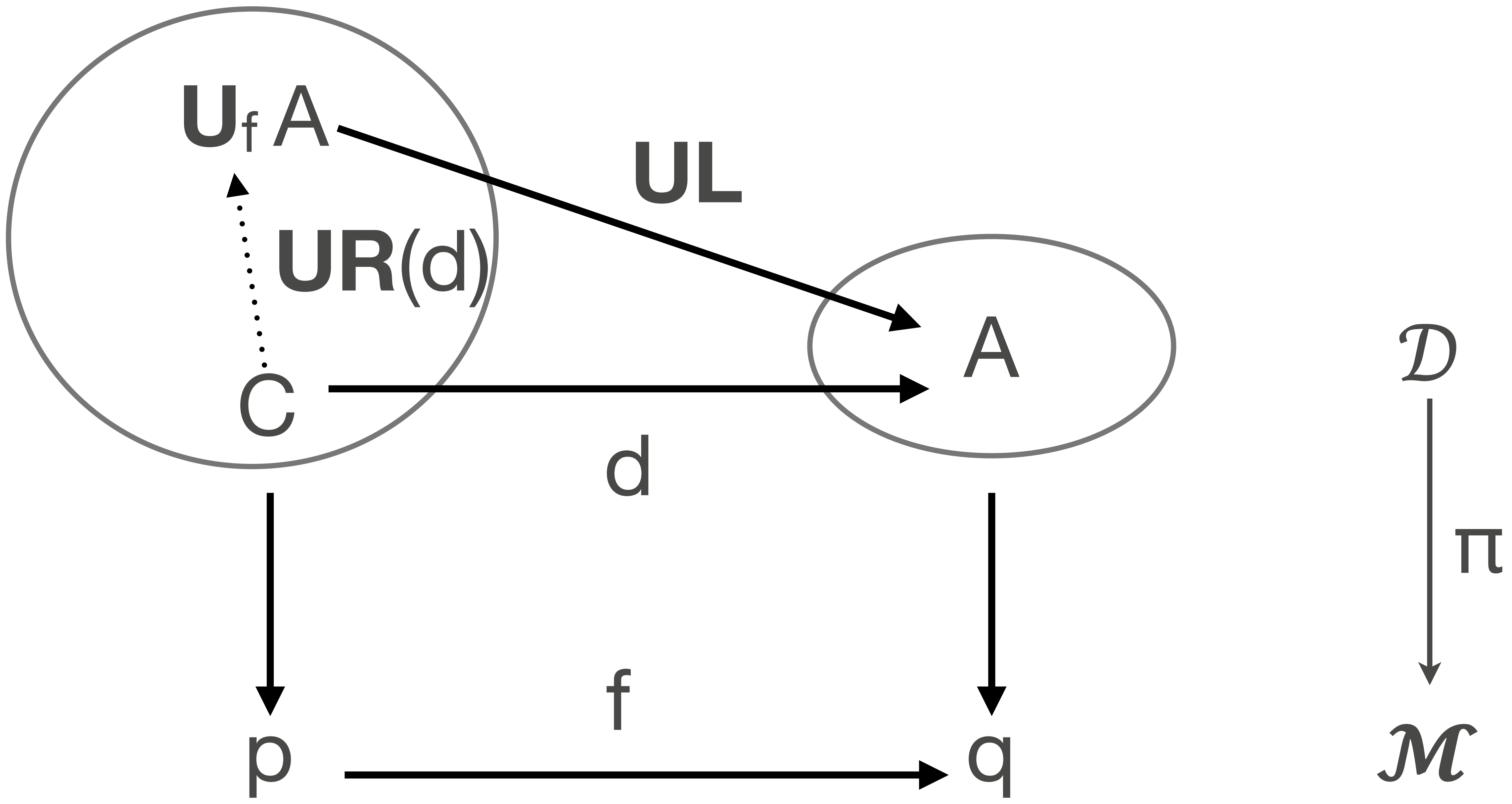
U types: fibration



U types: fibration



U types: fibration



U types: fibration

$$\frac{}{A \vdash_f \mathbf{F}_f\, A}\ \mathbf{FR}$$

$$\frac{A \vdash_{g \circ f} C}{\mathbf{F}_f\, A \vdash_g C}\ \mathbf{FL}$$

$$\mathbf{FR;FL}(d) = d \qquad \beta$$

$$d = \mathbf{FL}(\mathbf{FR};d) \qquad \eta$$

U types: fibration

$$\frac{}{\mathbf{U}_f A \vdash_f A}\ \mathbf{UL}$$

$$\frac{C \vdash_{f \circ g} A}{C \vdash_g \mathbf{U}_f A}\ \mathbf{UR}$$

$$\mathbf{UR}(d);\mathbf{UL} = d \quad \beta$$

$$d = \mathbf{UR}(d;\mathbf{UL}) \quad \eta$$

$$\frac{}{A \vdash_f \mathbf{F}_f A}\ \mathbf{FR}$$

$$\frac{A \vdash_{g \circ f} C}{\mathbf{F}_f A \vdash_g C}\ \mathbf{FL}$$

$$\mathbf{FR};\mathbf{FL}(d) = d \quad \beta$$

$$d = \mathbf{FL}(\mathbf{FR};d) \quad \eta$$

U types: fibration

"Fitch-style" — see Bas's talk on Thursday

$$\frac{}{\mathbf{U}_f A \vdash_f A}\ \mathbf{UL}$$

$$\frac{C \vdash_{f \circ g} A}{C \vdash_g \mathbf{U}_f A}\ \mathbf{UR}$$

$$\mathbf{UR}(d);\mathbf{UL} = d \quad \beta$$

$$d = \mathbf{UR}(d;\mathbf{UL}) \quad \eta$$

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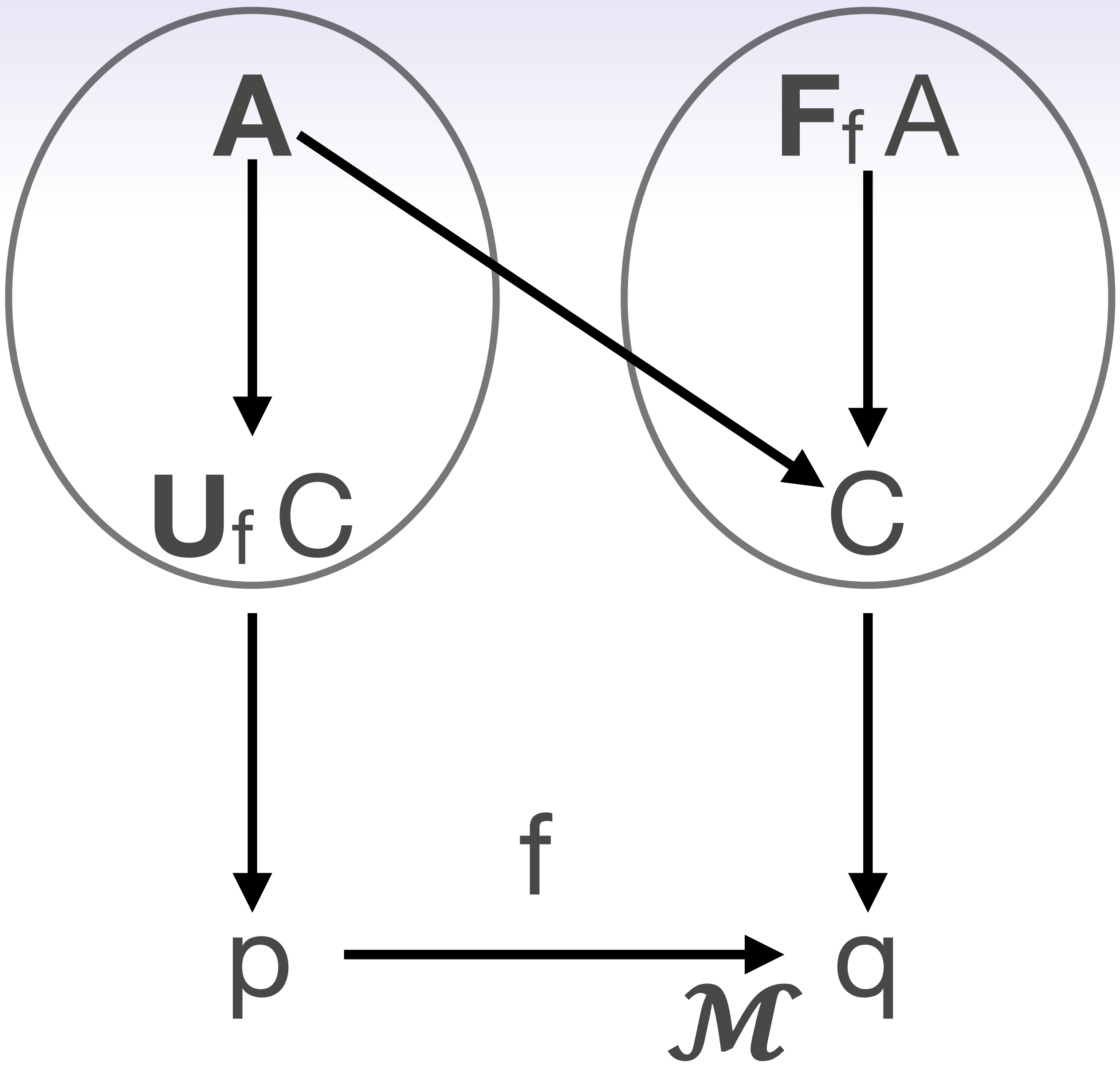
$$\frac{A \vdash_{g \circ f} C}{\mathbf{F}_f A \vdash_g C}\ \mathbf{FL}$$

$$\mathbf{FR};\mathbf{FL}(d) = d \quad \beta$$

$$d = \mathbf{FL}(\mathbf{FR};d) \quad \eta$$

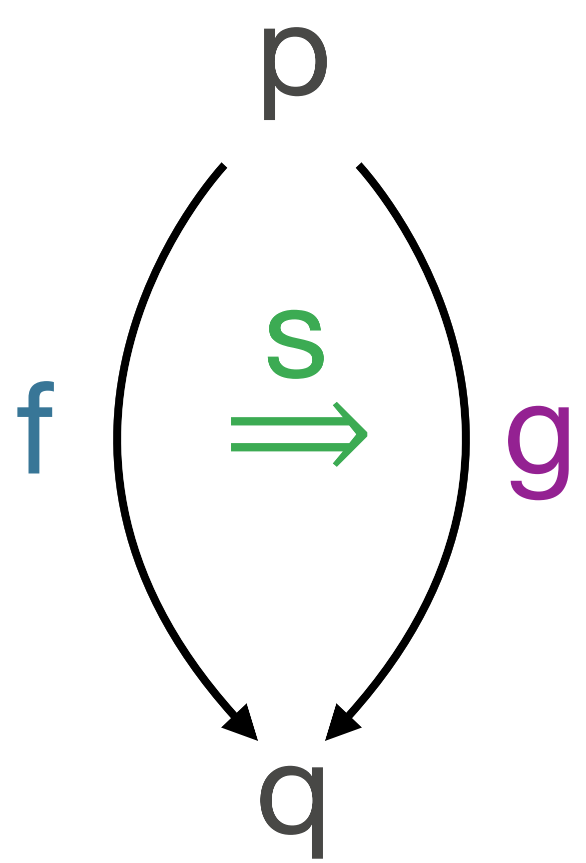
Adjoint

$$\dfrac{\dfrac{A \vdash_p \mathbf{U}_f\, C}{A \vdash_f C}}{\mathbf{F}_f\, A \vdash_q C}$$



Fibrational Framework

A mode theory $\mathcal{M}$ is a 2-category



Specifies doctrine of a local discrete bifibration $\pi : \mathcal{D} \to \mathcal{M}$

or a pseudofunctor $\mathcal{M} \to$ Adj

Theorems

$$\frac{A \vdash_p A'}{F_f\, A \vdash_q F_f\, A'} \qquad \frac{f \Longrightarrow g}{F_f\, A \vdash F_g\, A}$$

$$\frac{}{F_1\, A \vdash A} \qquad \frac{}{A \vdash F_1\, A} \qquad \frac{}{F_{g \circ f}\, A \vdash F_g\, F_f\, A} \qquad \frac{}{F_g\, F_f\, A \vdash F_{g \circ f}\, A}$$

$$\frac{A \vdash_p A'}{U_f\, A \vdash_q U_f\, A'} \qquad \frac{f \Longrightarrow g}{U_g\, A \vdash U_f\, A}$$

$$\frac{}{U_1\, A \vdash A} \qquad \frac{}{A \vdash U_1\, A} \qquad \frac{}{U_{g \circ f}\, A \vdash U_f\, Ug\, A} \qquad \frac{}{U_g\, U_f\, A \vdash U_{g \circ f}\, A}$$

[+ 2x a lot of equations!]

U types: fibration

$$\dfrac{}{\mathbf{U}_f A \vdash_f A}\ \mathbf{UL}$$

$$\dfrac{C \vdash_{f \circ g} A}{C \vdash_g \mathbf{U}_f A}\ \mathbf{UR}$$

$$\mathbf{UR}(d);\mathbf{UL} = d \quad \beta$$

$$d = \mathbf{UR}(d;\mathbf{UL}) \quad \eta$$

$$\dfrac{}{A \vdash_f \mathbf{F}_f A}\ \mathbf{FR}$$

$$\dfrac{A \vdash_{g \circ f} C}{\mathbf{F}_f A \vdash_g C}\ \mathbf{FL}$$

$$\mathbf{FR};\mathbf{FL}(d) = d \quad \beta$$

$$d = \mathbf{FL}(\mathbf{FR};d) \quad \eta$$

Functor framework

$\flat \dashv \#$

$\flat$ idem comonad

$\#$ idem monad

c mode

$\flat, \# : c \to c$

$\text{counit} : \flat \Longrightarrow 1_c$

$\text{unit} : 1_c \Longrightarrow \#$

$\flat\flat = \flat \quad \flat\# = \flat$

$\#\# = \# \quad \#\flat = \#$

[+ triangle]

$$\flat \dashv \sharp$$

$\flat$ idem comonad

$\sharp$ idem monad

Functor framework

c mode

$\flat, \sharp : c \to c$

$\text{counit} : \flat \Rightarrow 1_c$

$\text{unit} : 1_c \Rightarrow \sharp$

$\flat\flat = \flat \quad \flat\sharp = \flat$

$\sharp\sharp = \sharp \quad \sharp\flat = \sharp$

[+ triangle]

Adjunction framework

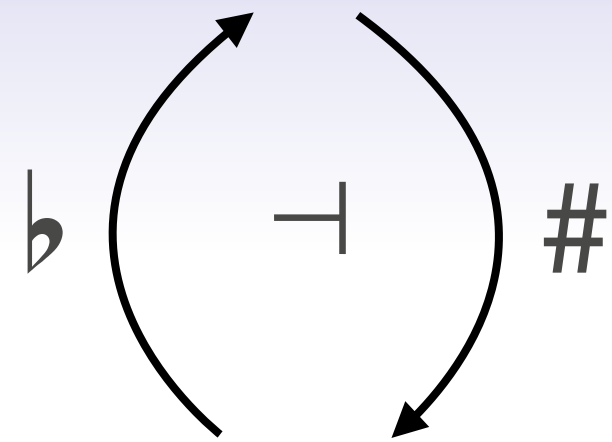
c mode

$\flat : c \to c$

$\text{counit} : \flat \Rightarrow 1_c$

$\flat\flat = \flat$

[+ triangle]



$\flat$ idem comonad

$\sharp$ idem monad

Adjunction framework

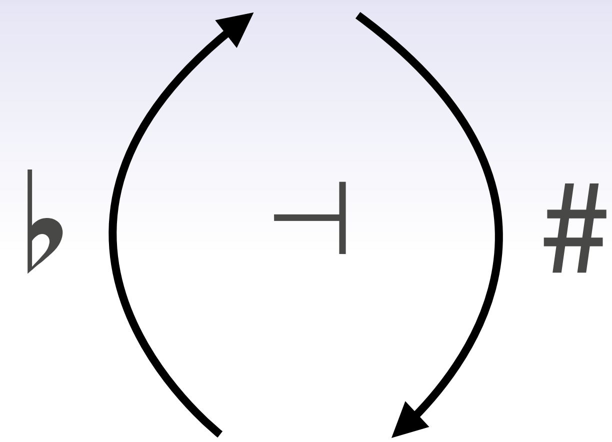
c mode

$\flat : c \to c$

$\text{counit} : \flat \Rightarrow 1_c$

$\flat\flat = \flat$

[+ triangle]



$\flat$ idem comonad
$\sharp$ idem monad

$\flat A := \mathbf{F}_\flat A$
$\sharp A := \mathbf{U}_\flat A$

Adjunction framework

c mode

$\flat : c \to c$

$\text{counit} : \flat \Rightarrow 1_c$

$\flat\flat = \flat$

[+ triangle]

$\flat \dashv \sharp$

$\flat$ idem comonad

$\sharp$ idem monad

$\flat A := \mathbf{F}_\flat A$

$\sharp A := \mathbf{U}_\flat A$

Adjunction framework

c mode

$\flat : c \to c$

$\flat A \vdash A$

$A \vdash \sharp A$

counit : $\flat \Longrightarrow 1_c$

$\flat\flat = \flat$

[+ triangle]

WLOG

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- For a particular mode theory, can often prove “without loss of generality” simplified rules are sound and complete (for equivalence classes)

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- Eagerly reduce mode morphisms using =

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- Eagerly reduce mode morphisms using =
- Only allow certain cuts into

$$\overline{A \vdash_f \mathbf{F}_f A} \qquad \overline{\mathbf{U}_f A \vdash_f A}$$

WLOG

- For a particular mode theory, can often prove "without loss of generality" simplified rules are sound and complete (for equivalence classes)
- Eagerly reduce mode morphisms using =
- Only allow certain cuts into

$$\frac{}{A \vdash_f \mathbf{F}_f\, A} \qquad \frac{}{\mathbf{U}_f\, A \vdash_f A}$$

- Restrict use of n.t.'s to certain points in a term

$$\frac{A \vdash_g B \qquad f \Longrightarrow g}{A \vdash_f C}$$

Variable rules

$$\frac{}{x : A \vdash_x \; A}$$

can use either kind of variable
(projection, or projection + counit)

Variable rules

$$\frac{}{x : A \vdash_{x} \; A} \qquad \frac{}{x : A \vdash_{\flat(x)} \; A}$$

can use either kind of variable
(projection, or projection + counit)

Variable rules

$$\frac{}{x : A \vdash_x A}$$

$$\frac{\flat \Longrightarrow 1_c}{x : A \vdash_{\flat(x)} A}$$

can use either kind of variable
(projection, or projection + counit)

Variable rules

$$\frac{}{x : A \vdash_{x} A} \qquad \frac{x : A \vdash_{x} A \quad \flat \Longrightarrow 1_c}{x : A \vdash_{\flat(x)} A}$$

can use either kind of variable
(projection, or projection + counit)

Variable rules

$$\frac{}{x : A \vdash_x A}$$

$$\frac{x : A \vdash_x A \quad \flat \Longrightarrow 1_c}{x : A \vdash_{\flat(x)} A}$$

$$\frac{}{\Delta \mid \Gamma, x{:}A, \Gamma' \vdash x : A}$$

$\flat$ **variables**   **non-** $\flat$ **variables**

can use either kind of variable
(projection, or projection + counit)

Variable rules

$$\frac{}{x : A \vdash_{x} A} \qquad \frac{x : A \vdash_{x} A \quad \flat \Longrightarrow 1_{c}}{x : A \vdash_{\flat(x)} A}$$

$$\frac{}{\Delta \mid \Gamma, x{:}A, \Gamma' \vdash x : A} \qquad \frac{}{\Delta, x{:}A, \Delta' \mid \Gamma \vdash x : A}$$

$\flat$ variables **non-$\flat$ variables**

can use either kind of variable
(projection, or projection + counit)

♭-intro

$$\frac{}{x : C \vdash_x \mathbf{F}_\flat A}$$

$\flat$ -intro

$$\frac{}{x : C \vdash_x \mathbf{F}_\flat A}$$

$$\frac{x : C \vdash_x A \qquad A \vdash_\flat \mathbf{F}_\flat A}{x : C \vdash_{\flat(x)} \mathbf{F}_\flat A}$$

♭ -intro

$$\frac{x : C \vdash_{-} \; A}{x : C \vdash_{x} \; \mathbf{F}_{\flat} A}$$

$$\frac{x : C \vdash_{x} \; A \qquad A \vdash_{\flat} \mathbf{F}_{\flat} A}{x : C \vdash_{\flat(x)} \mathbf{F}_{\flat} A}$$

♭-intro

$$\frac{}{x : C \vdash_{\flat(x)} \mathbf{F}_\flat\, A}$$

$$\frac{x : C \vdash_x A \qquad A \vdash_\flat \mathbf{F}_\flat\, A}{x : C \vdash_{\flat(x)} \mathbf{F}_\flat\, A}$$

$\flat$ -intro

$$\frac{x : C \vdash_{x} A}{x : C \vdash_{\flat(x)} \mathbf{F}_{\flat} A}$$

$$\frac{x : C \vdash_{x} A \qquad A \vdash_{\flat} \mathbf{F}_{\flat} A}{x : C \vdash_{\flat(x)} \mathbf{F}_{\flat} A}$$

$\flat$ -intro

$$\frac{}{x : C \vdash_{\flat(x)} \mathbf{F}_\flat A}$$

$$\frac{x : C \vdash_x A \quad A \vdash_\flat \mathbf{F}_\flat A}{x : C \vdash_{\flat(x)} \mathbf{F}_\flat A}$$

counit : $\flat \Rightarrow 1_C$ means $\flat$ stronger than 1

$\flat$ -intro

$$\frac{x : C \vdash_{\flat(x)} A}{x : C \vdash_{\flat(x)} \mathbf{F}_{\flat} A}$$

$$\frac{x : C \vdash_{x} A \qquad A \vdash_{\flat} \mathbf{F}_{\flat} A}{x : C \vdash_{\flat(x)} \mathbf{F}_{\flat} A}$$

counit : $\flat \Rightarrow 1_C$ means $\flat$ stronger than 1

$\flat$ -intro

$$\frac{x : C \vdash_{\flat(x)} A}{x : C \vdash_{\flat(x)} \mathbf{F}_{\flat}\, A}$$

$$\frac{x : C \vdash_{\flat(x)} A \qquad A \vdash_{\flat} \mathbf{F}_{\flat}\, A}{x : C \vdash_{\flat\ (x)} \mathbf{F}_{\flat}\, A}$$

counit : $\flat \Longrightarrow 1_c$ means $\flat$ stronger than 1

$\flat$ -intro

$$\frac{x : C \vdash_{\flat(x)} A}{x : C \vdash_{\flat(x)} \mathbf{F}_\flat A}$$

$$\frac{x : C \vdash_{\flat(x)} A \qquad A \vdash_\flat \mathbf{F}_\flat A}{x : C \vdash_{\flat\flat(x)} \mathbf{F}_\flat A}$$

counit : $\flat \Rightarrow 1_c$ means $\flat$ stronger than 1

$\flat$ -intro

$$\frac{x : C \vdash_{\flat(x)} A}{x : C \vdash_{\flat(x)} \mathbf{F}_\flat A}$$

$$\frac{x : C \vdash_{\flat(x)} A \qquad A \vdash_{\flat} \mathbf{F}_\flat A}{x : C \vdash_{\flat\flat(x)} \mathbf{F}_\flat A}$$

counit : $\flat \Rightarrow 1_c$ means $\flat$ stronger than 1

$$\flat\flat = \flat$$

♭ -intro

$$\frac{\Delta \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^{\flat} : \flat A}$$

[Pfenning-Davies]

♭ variables (Δ) **non- ♭ variables** (Γ)

make a map into ♭A
from a map into A that uses only ♭ variables
(and they stay ♭ in the premise)

♭ -intro

$$\frac{\Delta \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^{\flat} : \flat A}$$

[Pfenning-Davies]

♭ variables **non- ♭ variables**

make a map into ♭A
from a map into A that uses only ♭ variables
(and they stay ♭ in the premise)

(assumes ♭ preserves products — tomorrow!)

$\flat$ -induction

$$\frac{x : A \vdash_{\flat(x)} C}{x : \mathbf{F}_{\flat} A \vdash_{x} C}$$

make a map from the $\flat A$ type
from a map that uses x flatly/“crisply”

♭ -induction

$$\frac{x : A \vdash_{\flat(x)} C}{x : \mathbf{F}_\flat A \vdash_x C}$$

make a map from the ♭A type
from a map that uses x flatly/“crisply”

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash C : \mathsf{Type} \qquad \Delta \mid \Gamma \vdash M : \flat A \qquad \Delta, u :: A \mid \Gamma \vdash N : C[u^\flat / x]}{\Delta \mid \Gamma \vdash (\mathsf{let}\ u^\flat := M\ \mathsf{in}\ N) : C[M/x]}$$

intro

$$\frac{}{x : A \vdash_x \ \mathbf{U}_\flat\, C}$$

♯ intro

$$\frac{x : A \vdash_{\flat(x)} C}{x : A \vdash_{x} \mathbf{U}_{\flat} C}$$

♯ intro

$$\frac{x : A \vdash_{\flat(x)} C}{x : A \vdash_{x} \mathbf{U}_{\flat} C}$$

$$\frac{}{x : A \vdash_{\flat(x)} \mathbf{U}_{\flat} C}$$

intro

$$\frac{x : A \vdash_{\flat(x)} C}{x : A \vdash_{x} \mathbf{U}_{\flat} C} \qquad \frac{x : A \vdash_{\flat\flat(x)} C}{x : A \vdash_{\flat(x)} \mathbf{U}_{\flat} C}$$

♯ intro

$$\frac{x : A \vdash_{\flat(x)} C}{x : A \vdash_{x} \mathbf{U}_{\flat} C} \qquad \frac{x : A \vdash_{\flat\flat(x)=\flat(x)} C}{x : A \vdash_{\flat(x)} \mathbf{U}_{\flat} C}$$

intro

$$\frac{x : A \vdash_{\flat(x)} C}{x : A \vdash_{x} \mathbf{U}_{\flat} C} \qquad \frac{x : A \vdash_{\flat\flat(x)=\flat(x)} C}{x : A \vdash_{\flat(x)} \mathbf{U}_{\flat} C}$$

$$\frac{\Delta, \Gamma \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^{\sharp} : \sharp A}$$

a map into the #A type can use **all** variables flatly

♯ elim

$$\frac{x{:}C \vdash_{\flat(x)} \mathbf{U}_\flat\, A \qquad \mathbf{U}_\flat\, A \vdash_\flat A}{x{:}C \vdash_{\flat\flat(x)} A}$$

$$\frac{\Delta \mid \cdot \vdash M : \sharp A}{\Delta \mid \Gamma \vdash M_\sharp : A}$$

elim

$$\frac{x{:}C \vdash_{\flat(x)} \mathbf{U}_\flat\, A \qquad \mathbf{U}_\flat\, A \vdash_\flat A}{x{:}C \vdash_{\flat\flat(x) = \flat(x)} A}$$

$$\frac{\Delta \mid \cdot \vdash M : \sharp A}{\Delta \mid \Gamma \vdash M_\sharp : A}$$

make a map into A
from a map into #A that
uses each variable flatly

Dependency

$$\frac{\Delta \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^{\flat} : \flat A}$$

$$\frac{\Delta, \Gamma \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^{\sharp} : \sharp A}$$

Dependency

$$\frac{\Delta \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \flat A : \mathsf{Type}}$$

$$\frac{\Delta, \Gamma \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \sharp A : \mathsf{Type}}$$

$$\frac{\Delta \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^{\flat} : \flat A}$$

$$\frac{\Delta, \Gamma \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^{\sharp} : \sharp A}$$

Main ideas

Main ideas

- Methodology:
  1. intended semantics → mode theory
  2. that instance of framework is *a* calculus
  3. simplify by WLOG reasoning

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- Methodology:
  1. intended semantics → mode theory
  2. that instance of framework is *a* calculus
  3. simplify by WLOG reasoning
- Individual modal type theories look weird, but *only step 3 is ad-hoc*!

Main ideas

- Methodology:
  1. intended semantics → mode theory
  2. that instance of framework is *a* calculus
  3. simplify by WLOG reasoning

- Individual modal type theories look weird, but *only step 3 is ad-hoc*!

- Fibrational/judgemental nicer than pseudofunctorial/combinator-logic

Next

- How to use $\flat$ and # in real-cohesive HoTT

Tomorrow

- Interaction between modalities and other connectives
- Unary to simple types (multiple assumptions)
- Dependent types

Tutorial 5

A Framework for Adjunctions in Simple Type Theory

[L., Shulman, Riley, '17]

Analogy

$$\frac{}{A \vdash_{\flat} \mathbf{F}_{\flat} A} \qquad \frac{x : A \vdash_{\flat(x)} C}{x : \mathbf{F}_{\flat} A \vdash_{x} C}$$

Analogy

$$\frac{}{A \vdash_{\flat} \mathbf{F}_{\flat} A} \qquad \frac{x : A \vdash_{\flat(x)} C}{x : \mathbf{F}_{\flat} A \vdash_{x} C}$$

$$\frac{}{A,B \vdash A \otimes B} \qquad \frac{\Gamma,A,B \vdash C}{\Gamma,A \otimes B \vdash C}$$

Unary type theory

local discrete bifibration of 2-categories

$$\begin{array}{ccl} \mathcal{D} & & \text{derivations} \\ \downarrow \pi & & \\ \mathcal{M} & & \text{diagram of adjunctions} \end{array}$$

Simple type theory

local discrete bifibration of cartesian 2-multicategories

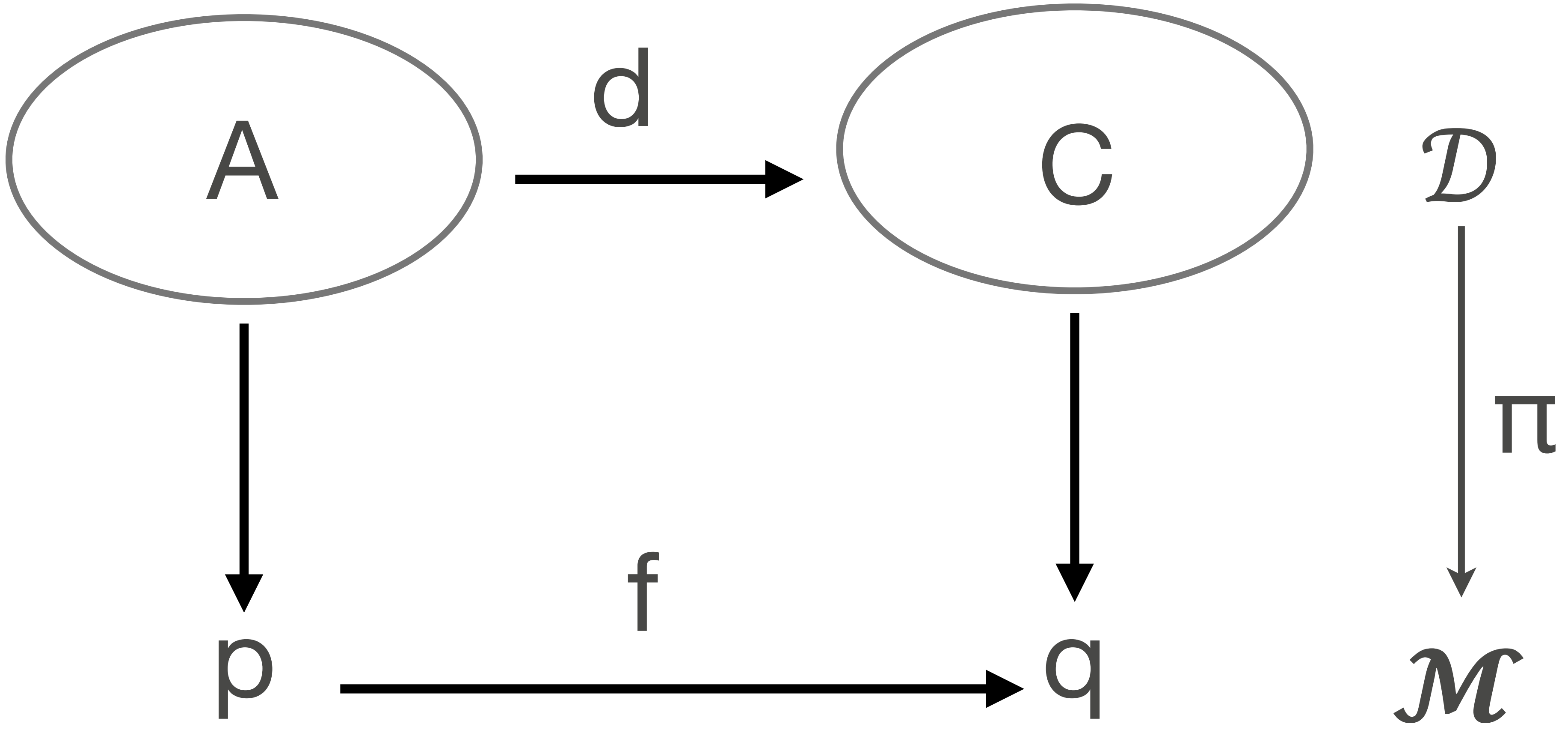
$$\begin{array}{c} \mathcal{D} \\ \downarrow \pi \\ \mathcal{M} \end{array}$$

$\mathcal{D}$ derivations

$\mathcal{M}$ diagram of multi-variable adjunctions

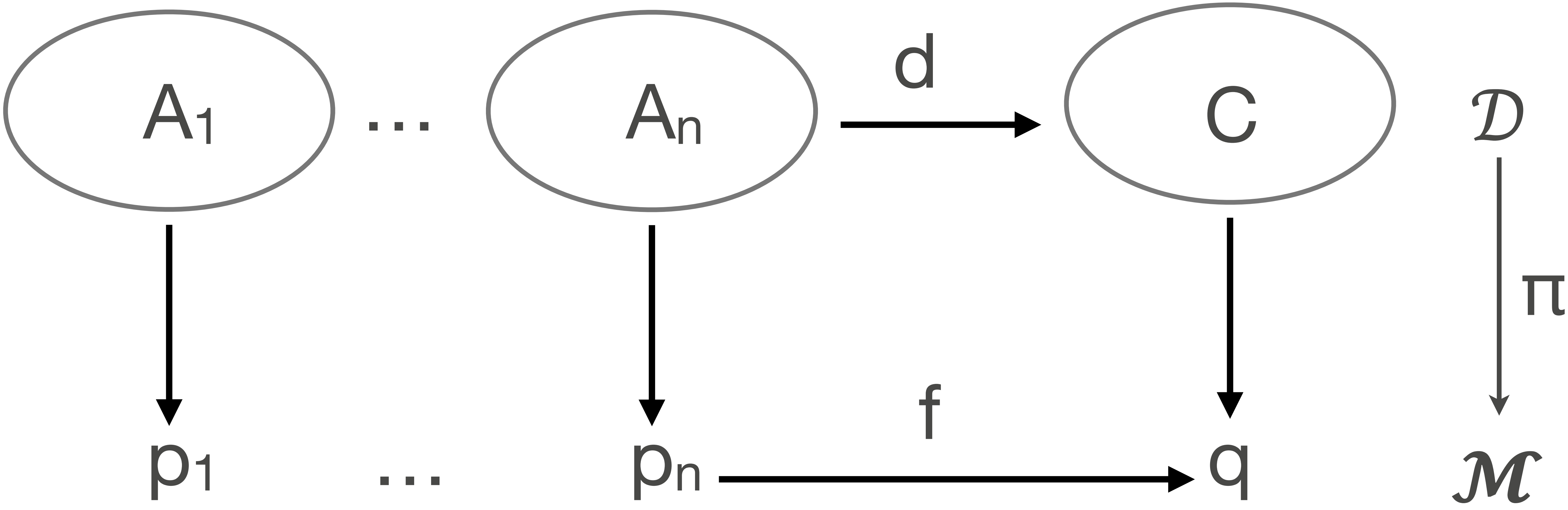
Unary type theory

$A \vdash_f C$



Simple type theory

$$d : x_1{:}A_1, \ldots, x_n{:}A_n \vdash_f C$$



Structural Rules

Projection

$$\frac{}{\Gamma, x{:}A, \Gamma' \vdash_{x} A}$$

Composition

$$\frac{\Gamma \vdash_{f} A \qquad \Gamma, x{:}A \vdash_{g} C}{\Gamma \vdash_{g[f/x]} C}$$

Structural Rules

Weakening

$$\frac{\Gamma \vdash_f B}{\Gamma, x{:}A \vdash_f B}$$

Exchange

$$\frac{\Gamma, y{:}B, x{:}A \vdash_f C}{\Gamma, x{:}A, y{:}B \vdash_f C}$$

Contraction

$$\frac{\Gamma, x{:}A, y{:}A \vdash_f B}{\Gamma, x{:}A \vdash_{f[y/x]} B}$$

Mode theories

Mode theories

$x{:}p,\ y{:}p \vdash x \otimes y : p$ **magma**

Mode theories

$x{:}p, y{:}p \vdash x \otimes y : p$ **magma**

$\cdot \vdash 1 : p$ $\qquad x \otimes 1 = x = 1 \otimes x$ **with unit**

Mode theories

$x{:}p,\ y{:}p \vdash x \otimes y : p$ **magma**

$\cdot \vdash 1 : p$ $\quad x \otimes 1 = x = 1 \otimes x$ **with unit**

$x \otimes (y \otimes z) = (x \otimes y) \otimes z$ **monoid**

Mode theories

$x{:}p,\ y{:}p \vdash x \otimes y : p$ **magma**

$\cdot \vdash 1 : p$ $\qquad x \otimes 1 = x = 1 \otimes x$ **with unit**

$x \otimes (y \otimes z) = (x \otimes y) \otimes z$ **monoid**

$x \otimes y = y \otimes x$ **commutative monoid**

Mode theories

$x{:}p,\ y{:}p \vdash x \otimes y : p$		**magma**
$\cdot \vdash 1 : p$	$x \otimes 1 = x = 1 \otimes x$	**with unit**
	$x \otimes (y \otimes z) = (x \otimes y) \otimes z$	**monoid**
	$x \otimes y = y \otimes x$	**commutative monoid**
	$x \Longrightarrow 1$	**semicartiesian monoid**

Mode theories

$x{:}p, y{:}p \vdash x \otimes y : p$		**magma**
$\cdot \vdash 1 : p$	$x \otimes 1 = x = 1 \otimes x$	**with unit**
	$x \otimes (y \otimes z) = (x \otimes y) \otimes z$	**monoid**
	$x \otimes y = y \otimes x$	**commutative monoid**
	$x \Longrightarrow 1$	**semicartiesian monoid**
	$x \Longrightarrow x \otimes x$	**cartesian monoid**

Mode theories

$x{:}p,\ y{:}p \vdash x \otimes y : p$ **magma**

$\cdot \vdash 1 : p$ $\quad x \otimes 1 = x = 1 \otimes x$ **with unit**

$x \otimes (y \otimes z) = (x \otimes y) \otimes z$ **monoid**

$x \otimes y = y \otimes x$ **commutative monoid**

$x \Longrightarrow 1$ **semicartiesian monoid**

$x \Longrightarrow x \otimes x$ **cartesian monoid**

microcosm:
using cartesianness
of the setting

$\mathrm{id} \Longrightarrow (\lambda\_.1) : p \to p$

Linear logic

Linear logic

Let (p,⊗,1) be a commutative monoid in $\mathcal{M}$

Linear logic

Let (p,⊗,1) be a commutative monoid in $\mathcal{M}$

$$x{:}A, y{:}B, z{:}C \vdash_{x \otimes (y \otimes z)} D$$

uses all three

Linear logic

Let $(p,\otimes,1)$ be a commutative monoid in $\mathcal{M}$

$$x{:}A, y{:}B, z{:}C \vdash_{x\otimes(y\otimes z)} D$$ **uses all three**

$$x{:}A, y{:}B, z{:}C \vdash_{(x\otimes y)\otimes z} D$$ **same derivations**

Linear logic

Let $(p,\otimes,1)$ be a commutative monoid in $\mathcal{M}$

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$x{:}A, y{:}B, z{:}C \vdash_{x\otimes y} D$ **uses x and y**

Linear logic

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$x{:}A, y{:}B, z{:}C \vdash_{x\otimes y} D$ **uses x and y**

$x{:}A, y{:}B, z{:}C \vdash_{y\otimes x} D$ **same derivations**

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$$x{:}A, y{:}B, z{:}C \vdash_{x\otimes y} D$$ **uses x and y**

$$x{:}A, y{:}B, z{:}C \vdash_{y\otimes x} D$$ **same derivations**

$$x{:}A, y{:}B, z{:}C \vdash_{1} D$$ **uses none**

Linear logic

Let $(p,\otimes,1)$ be a commutative monoid in $\mathcal{M}$

$x{:}A, y{:}B, z{:}C \vdash_{x\otimes(y\otimes z)} D$ **uses all three**

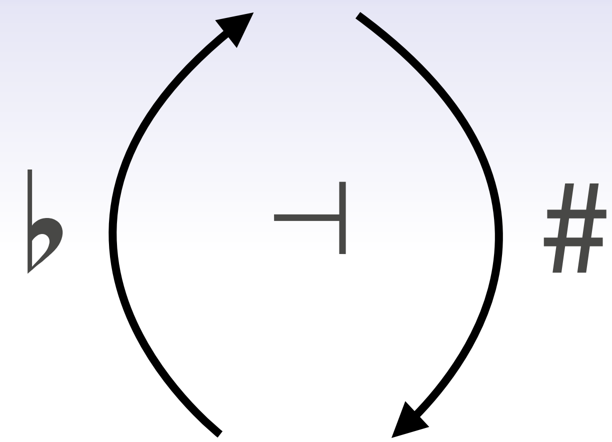
$x{:}A, y{:}B, z{:}C \vdash_{(x\otimes y)\otimes z} D$ **same derivations**

$x{:}A, y{:}B, z{:}C \vdash_{x\otimes y} D$ **uses x and y**

$x{:}A, y{:}B, z{:}C \vdash_{y\otimes x} D$ **same derivations**

$x{:}A, y{:}B, z{:}C \vdash_{1} D$ **uses none**

$x{:}A, y{:}B, z{:}C \vdash_{x\otimes x} D$ **uses x twice**



$\flat$ idem comonad

$\sharp$ idem monad

$\flat A := \mathbf{F}_\flat A$

$\sharp A := \mathbf{U}_\flat A$

Adjunction framework

c mode

$\flat : c \to c$

counit : $\flat \Longrightarrow 1_c$

$\flat\flat = \flat$

[+ triangle]

$\flat A \vdash A$

$A \vdash \sharp A$

Adjunction framework

c mode

$\otimes$: c,c $\rightarrow$ c

1 : $\cdot$ $\rightarrow$ c

… commutative
monoid laws …

$A \otimes B := \mathbf{F}_{\otimes}(A,B)$

$A \multimap B := \mathbf{U}_{\otimes}(A \mid B)$

Adjunction framework

c mode

$\otimes : c,c \to c$

$1 : \cdot \to c$

… commutative monoid laws …

$$A\otimes - \;\dashv\; A\multimap -$$

Adjunction framework

$A\otimes B := \mathbf{F}_{\otimes}(A,B)$

$A\multimap B := \mathbf{U}_{\otimes}(A|B)$

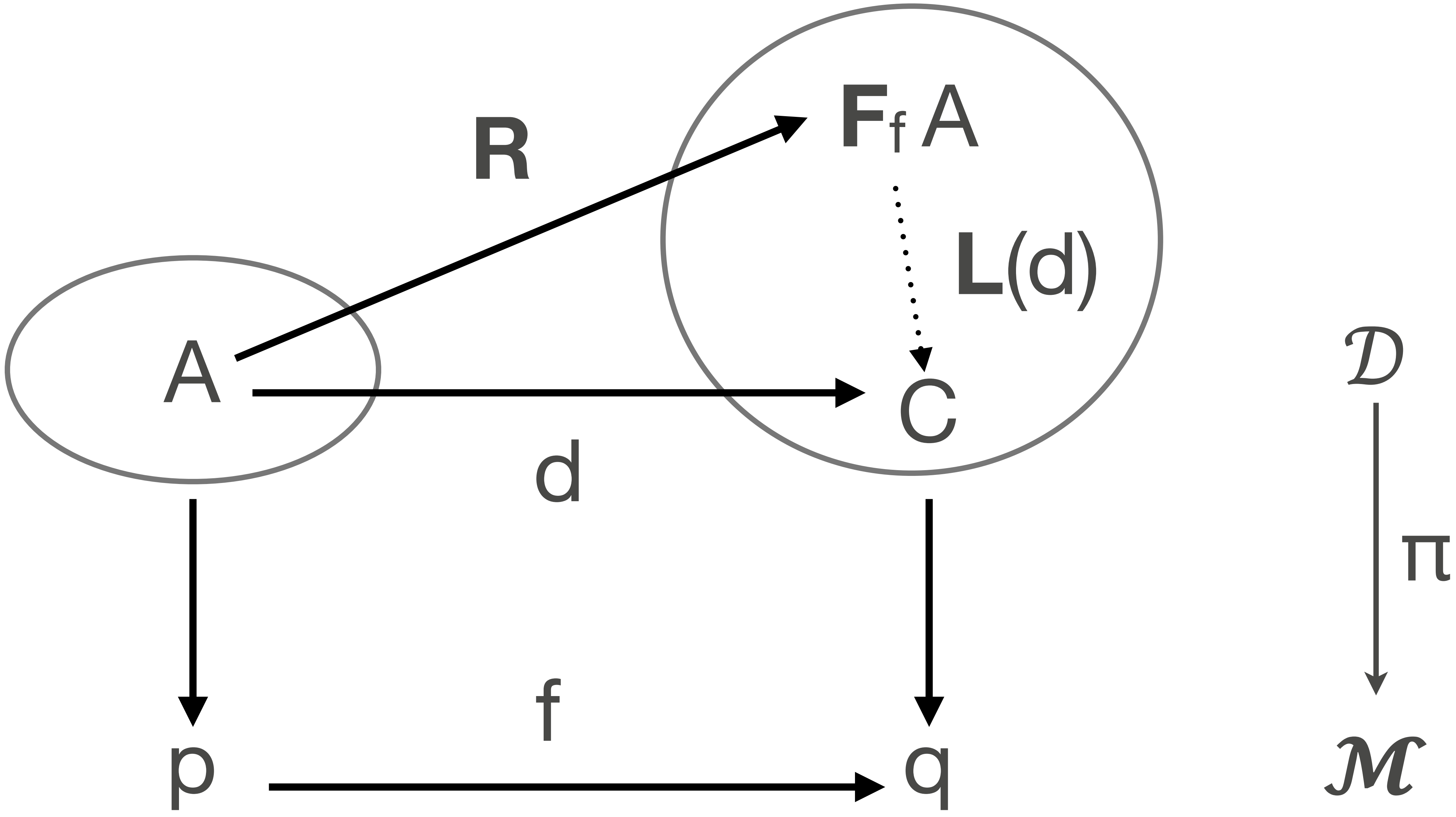
c mode

$\otimes : c, c \rightarrow c$

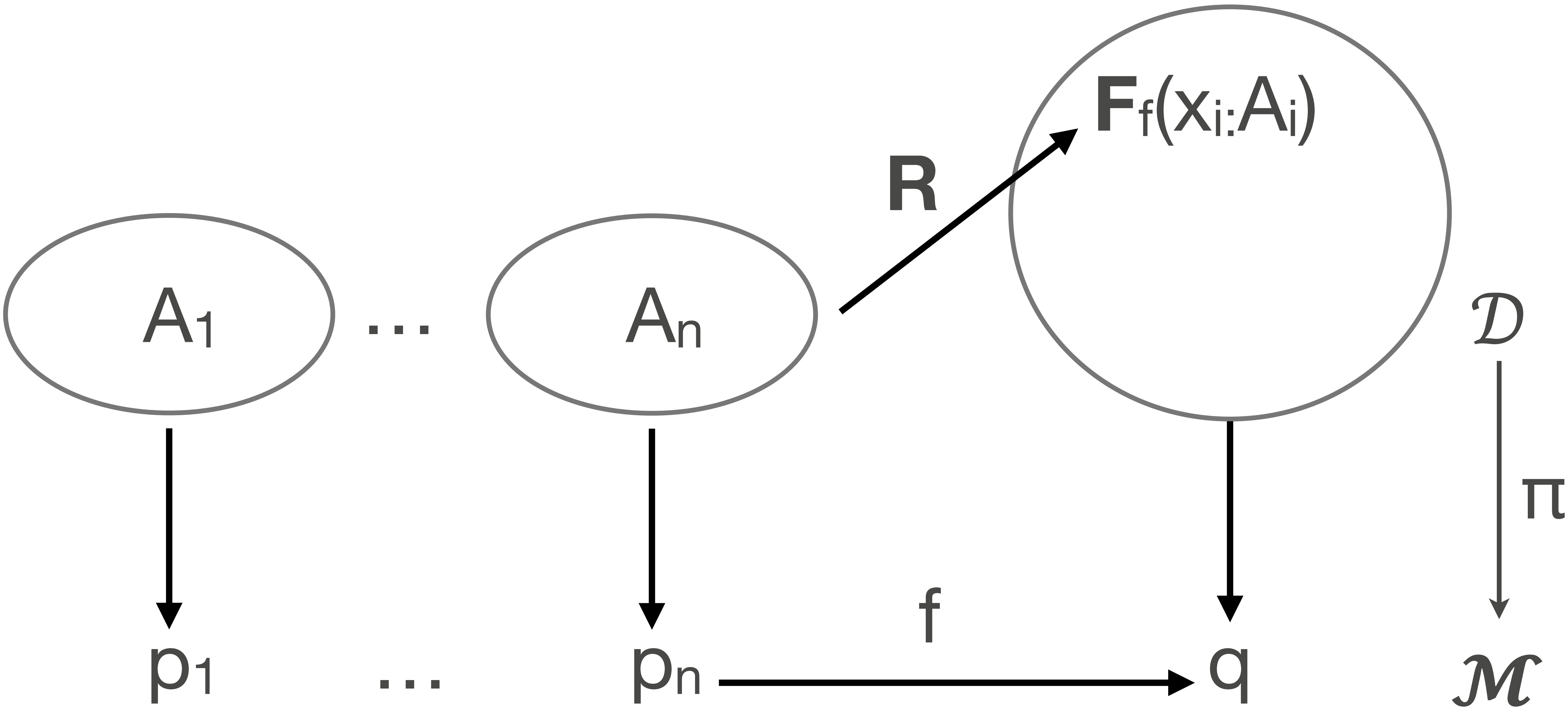
$1 : \cdot \rightarrow c$

… commutative monoid laws …

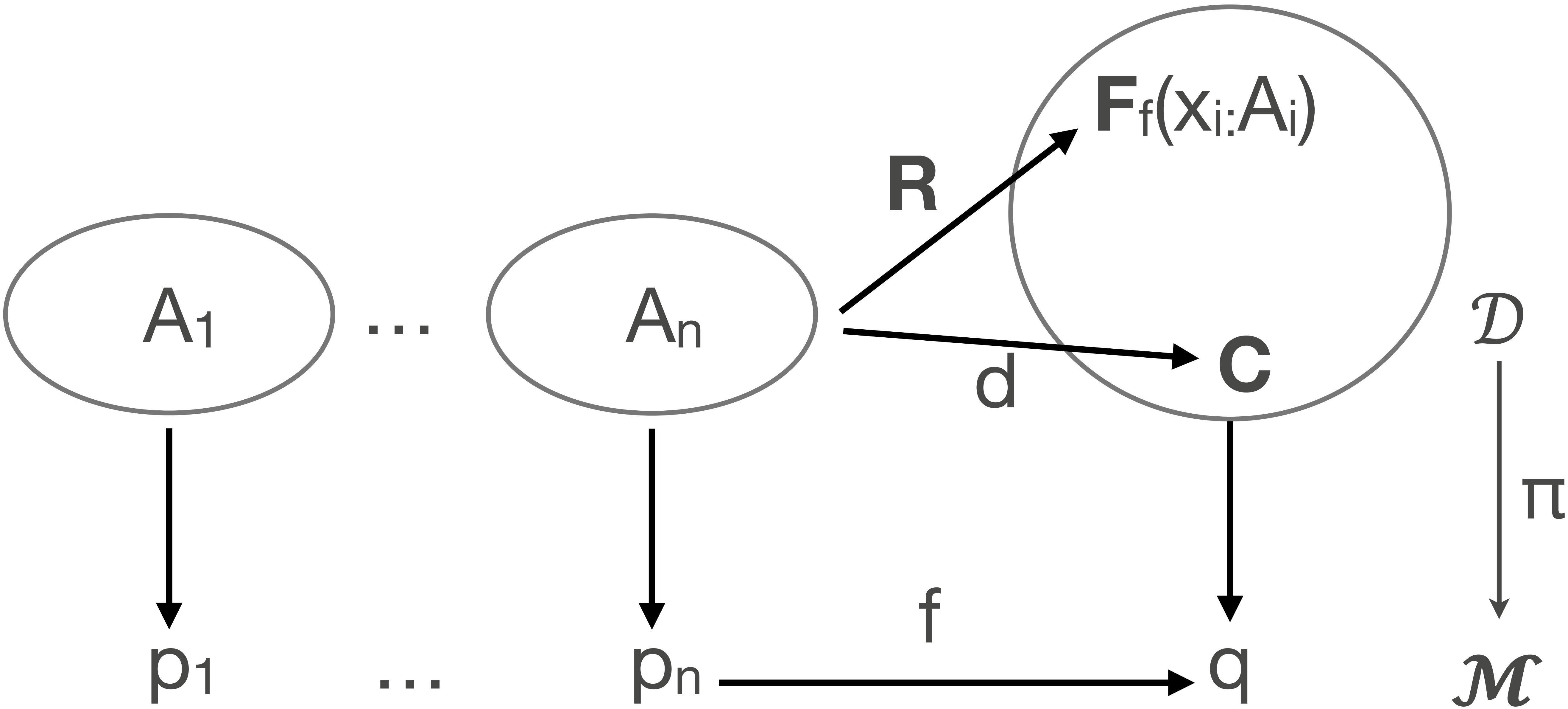
F types: opfibration



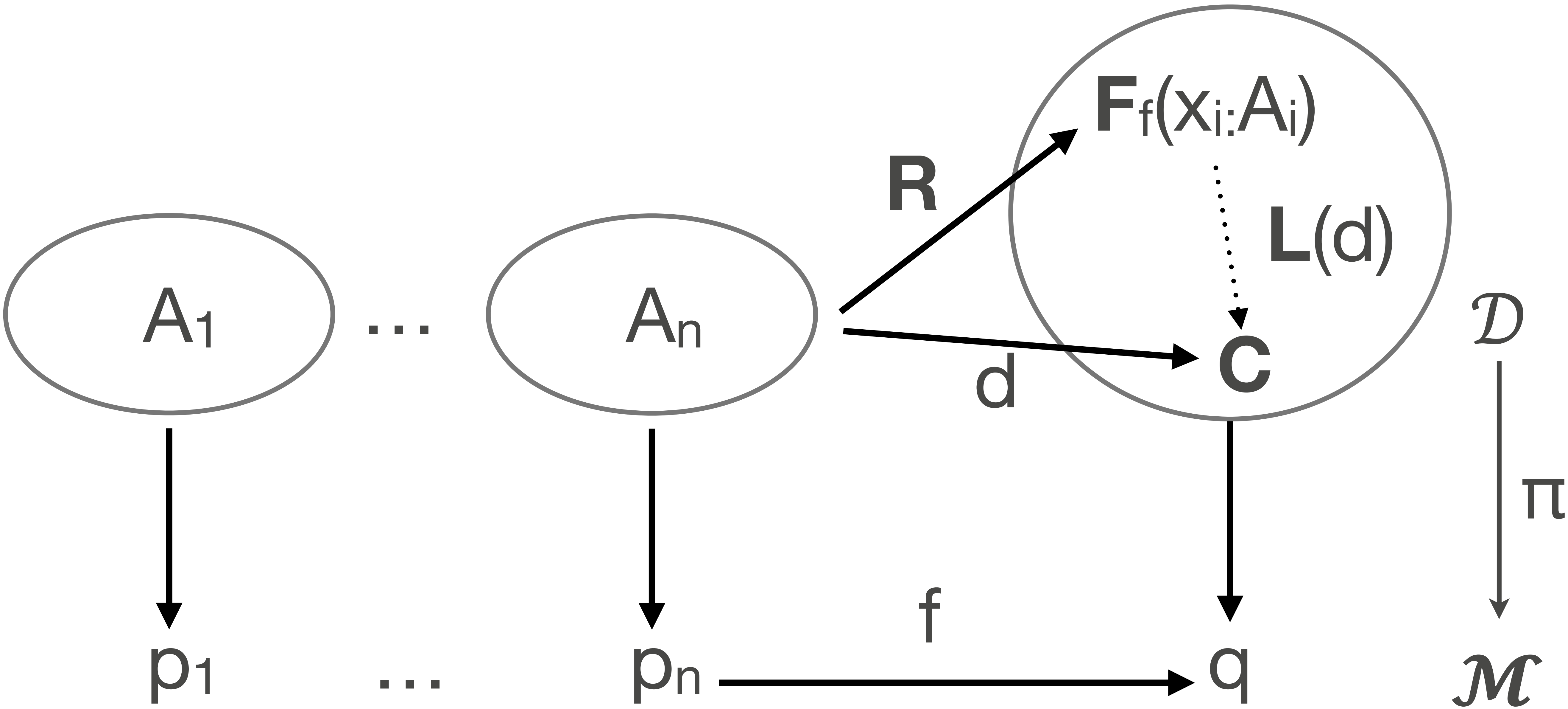
F types



F types



F types



F types

$$\frac{}{A \vdash_{f} \mathbf{F}_{f} A}$$

$$\frac{A \vdash_{g \circ f} C}{\mathbf{F}_{f}(A) \vdash_{g} C}$$

F types

$$\frac{}{x_1{:}A_1,\ldots,x_n{:}A_n \vdash_f \mathbf{F}_f(x_1{:}A_1,\ldots,x_n{:}A_n)}\ \mathbf{FR}$$

$$\frac{A \vdash_{g \circ f} C}{\mathbf{F}_f(A) \vdash_g C}$$

F types

$$\frac{}{x_1{:}A_1,\ldots,x_n{:}A_n \vdash_f \mathbf{F}_f(x_1{:}A_1,\ldots,x_n{:}A_n)}\ \mathbf{FR}$$

$$\frac{\Gamma, x_1{:}A_1,\ldots,x_n{:}A_n \vdash_{g[f/y]} C}{\Gamma, y{:}\mathbf{F}_f(x_1{:}A_1,\ldots,x_n{:}A_n) \vdash_g C}\ \mathbf{FL}$$

⊗ right

$$a : A, b : B \vdash_{a \otimes b} \mathbf{F}_{a \otimes b}(a{:}A, b{:}B)$$

⊗ right

$\Gamma \vdash_{x1 \otimes \ldots \otimes xn} A$

$a : A, b : B \vdash_{a \otimes b} \mathbf{F}_{a \otimes b}(a{:}A, b{:}B)$

$\otimes$ right

$\Gamma \vdash_{x1 \otimes \ldots \otimes xn} A$

$\Gamma \vdash_{y1 \otimes \ldots \otimes ym} B$

$a : A, b : B \vdash_{a \otimes b} \mathbf{F}_{a \otimes b}(a{:}A, b{:}B)$

⊗ right

$$\frac{\Gamma \vdash_{x1\otimes\ldots\otimes xn} A \qquad \Gamma \vdash_{y1\otimes\ldots\otimes ym} B \qquad a : A, b : B \vdash_{a\otimes b} \mathbf{F}_{a\otimes b}(a{:}A, b{:}B)}{\Gamma \vdash_{(x1\otimes\ldots\otimes xn)\otimes(y1\otimes\ldots\otimes ym)} \mathbf{F}_{a\otimes b}(a{:}A, b{:}B)}$$

$\otimes$ right

$$\frac{\Gamma \vdash_{x1\otimes\ldots\otimes xn} A \quad \Gamma \vdash_{y1\otimes\ldots\otimes ym} B \quad a : A, b : B \vdash_{a\otimes b} \mathbf{F}_{a\otimes b}(a{:}A, b{:}B)}{\Gamma \vdash_{(x1\otimes\ldots\otimes xn)\otimes(y1\otimes\ldots\otimes ym)} \mathbf{F}_{a\otimes b}(a{:}A, b{:}B)}$$

$$z_1\otimes\ldots\otimes z_k = (x_1\otimes\ldots\otimes x_n)\otimes(y_1\otimes\ldots\otimes y_m)$$

⊗ right

$$\dfrac{\dfrac{\Gamma \vdash_{x1\otimes\ldots\otimes xn} A \qquad \Gamma \vdash_{y1\otimes\ldots\otimes ym} B \qquad a : A, b : B \vdash_{a\otimes b} \mathbf{F}_{a\otimes b}(a{:}A, b{:}B)}{\Gamma \vdash_{(x1\otimes\ldots\otimes xn)\otimes(y1\otimes\ldots\otimes ym)} \mathbf{F}_{a\otimes b}(a{:}A, b{:}B) \qquad z_1\otimes\ldots\otimes z_k = (x_1\otimes\ldots\otimes x_n)\otimes(y_1\otimes\ldots\otimes y_m)}}{\Gamma \vdash_{z1\otimes\ldots\otimes zk} \mathbf{F}_{\otimes}(a{:}A, b{:}B)}$$

$\otimes$ right

$$\frac{\Gamma = \Delta_1, \Delta_2 \qquad \Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Gamma \vdash A \otimes B}$$

⊗ left

$$\frac{\Gamma, x{:}A, y{:}B \vdash_{z1 \otimes \ldots \otimes (\mathbf{x} \otimes \mathbf{y})} C}{\Gamma, z{:}\mathbf{F}_{x \otimes y}(x{:}A, y{:}B) \vdash_{z1 \otimes \ldots \otimes \mathbf{z}} C}$$

$\otimes$ left

$$\frac{\Gamma, x{:}A, y{:}B \vdash_{z1 \otimes \ldots \otimes \mathbf{(x \otimes y)}} C}{\Gamma, z{:}\mathbf{F}_{x \otimes y}(x{:}A, y{:}B) \vdash_{z1 \otimes \ldots \otimes \mathbf{z}} C}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

⊗ left

$$\frac{\Gamma, x{:}A, y{:}B \vdash_{z1 \ldots zk} C}{\Gamma, z{:}\mathbf{F}_{\otimes}(A,B) \vdash_{z1 \ldots zk} C}$$

⊗ left

$$\frac{\Gamma, x{:}A, y{:}B \vdash_{z1\ldots zk} C}{\Gamma, z{:}\mathbf{F}_{\otimes}(A,B) \vdash_{z1\ldots zk} C}$$

subtlety: **FL** lets you pattern-match z
even if it doesn't occur in the subscript…
we proved a strengthening lemma
that deletes such steps

Relevant ⊗

Let $(p,\otimes,1)$ be a comm. monoid with contraction $x \Rightarrow x \otimes x$

$$\cfrac{\Gamma \vdash_{x1\otimes\ldots\otimes xn} A \qquad \Gamma \vdash_{y1\otimes\ldots\otimes ym} B \qquad a : A, b : B \vdash_{a\otimes b} \mathbf{F}_{a\otimes b}(a{:}A,b{:}B)}{\cfrac{\Gamma \vdash_{(x1\otimes\ldots\otimes xn)\otimes(y1\otimes\ldots\otimes ym)} \mathbf{F}_{a\otimes b}(a{:}A,b{:}B)}{\Gamma \vdash_{z1\otimes\ldots\otimes zk} \mathbf{F}_{\otimes}(a{:}A,b{:}B)}}$$

Relevant $\otimes$

Let $(p,\otimes,1)$ be a comm. monoid with contraction $x \Rightarrow x \otimes x$

$$\Gamma \vdash_{x1\otimes\ldots\otimes xn} A$$

$$\Gamma \vdash_{y1\otimes\ldots\otimes ym} B$$

$$a : A, b : B \vdash_{a\otimes b} \mathbf{F}_{a\otimes b}(a:A,b:B)$$

$$\overline{\Gamma \vdash_{(x1\otimes\ldots\otimes xn)\otimes(y1\otimes\ldots\otimes ym)} \mathbf{F}_{a\otimes b}(a:A,b:B)}$$

$$z_1\otimes\ldots\otimes z_k \Longrightarrow (x_1\otimes\ldots\otimes x_n)\otimes(y_1\otimes\ldots\otimes y_m)$$

$$\overline{\Gamma \vdash_{z1\otimes\ldots\otimes zk} \mathbf{F}_{\otimes}(a:A,b:B)}$$

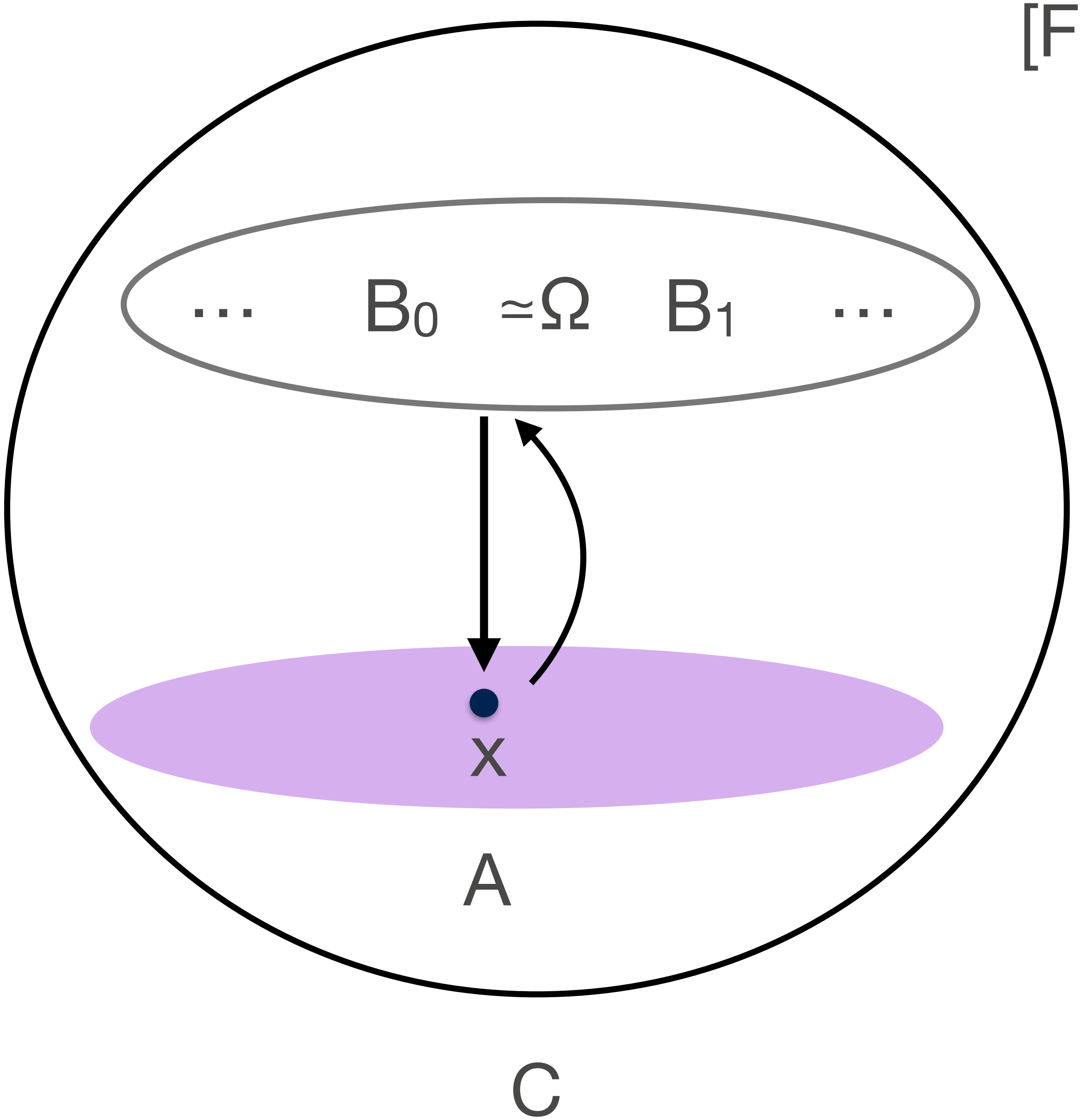
Relevant ⊗

Let (p,⊗,1) be a comm. monoid with contraction $x \Longrightarrow x \otimes x$

$$\frac{\Gamma \vdash_{x \otimes y} A \qquad \Gamma \vdash_{y \otimes z} B \qquad x \otimes y \otimes z \Longrightarrow (x \otimes y) \otimes (y \otimes z)}{\Gamma \vdash_{x \otimes y \otimes z} \mathbf{F}_{\otimes}(a{:}A, b{:}B)}$$

Parametrized spectra

[Finster,L.,Morehouse,Riley]



C ∧ C' := product in base, smash product of spectra in the fiber

Parametrized spectra

[Finster,L.,Morehouse,Riley]

Let $(p,\otimes,1)$ be comm. monoid

Parametrized spectra

[Finster,L.,Morehouse,Riley]

Let $(p,\otimes,1)$ be comm. monoid

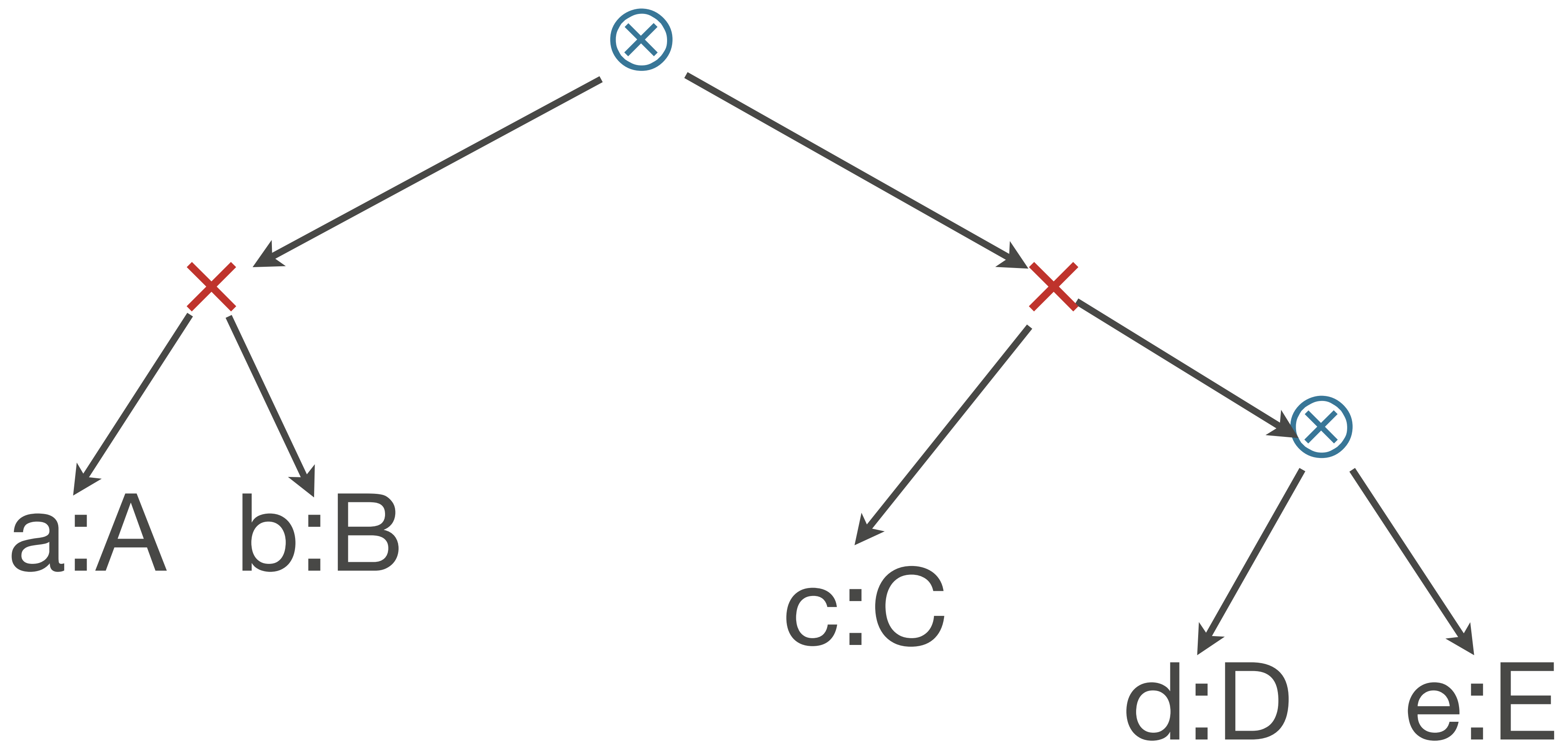
Let $(p,\times,\top)$ be a cartesian monoid

Parametrized spectra

[Finster,L.,Morehouse,Riley]

Let (p,⊗,1) be comm. monoid

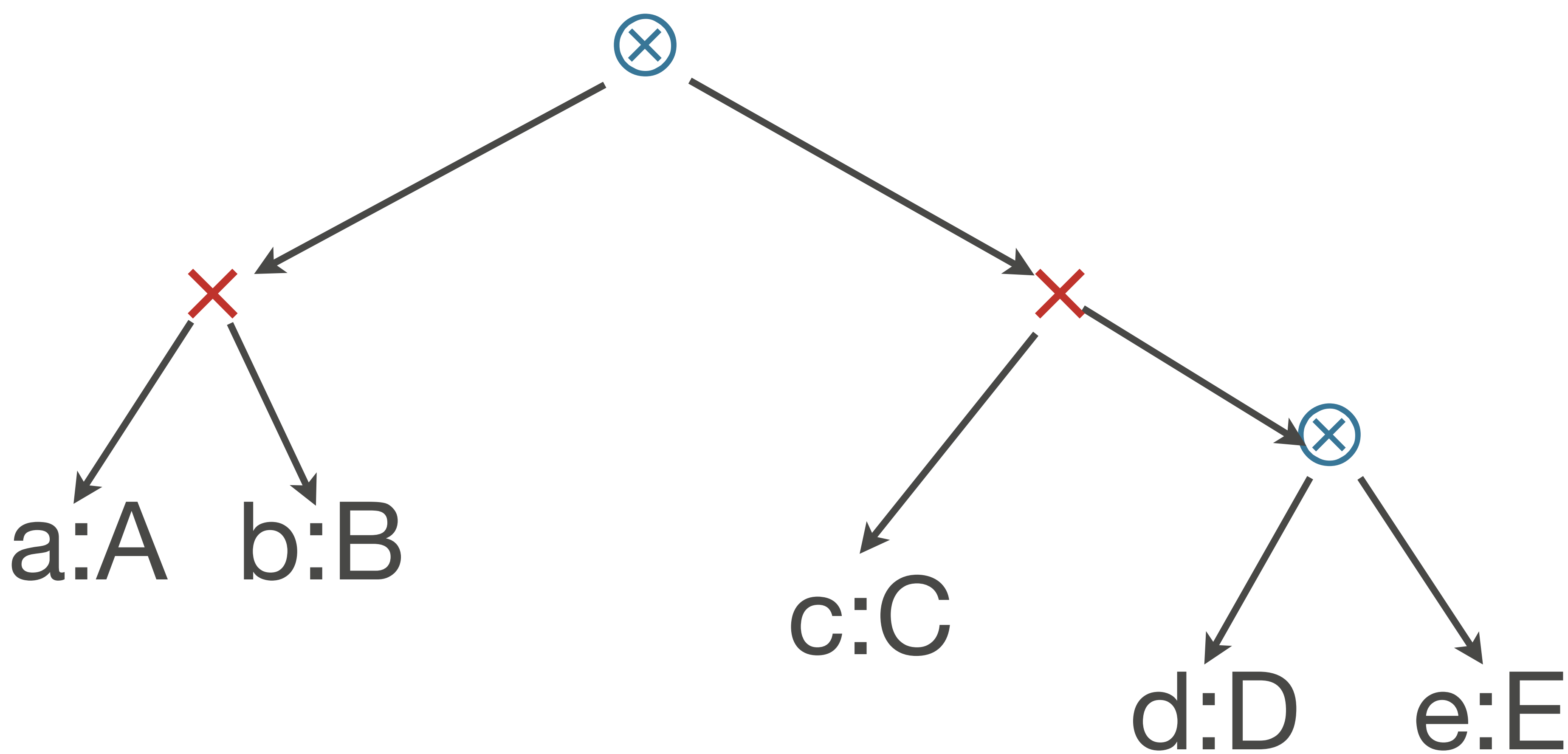
Let (p,×,⊤) be a cartesian monoid



Parametrized spectra

[Finster,L.,Morehouse,Riley]

$$a{:}A, b{:}B, c{:}C, d{:}D, e{:}E \vdash_{(a \times b) \otimes (c \times (d \otimes e))} F$$



Benton's LNL

m mode

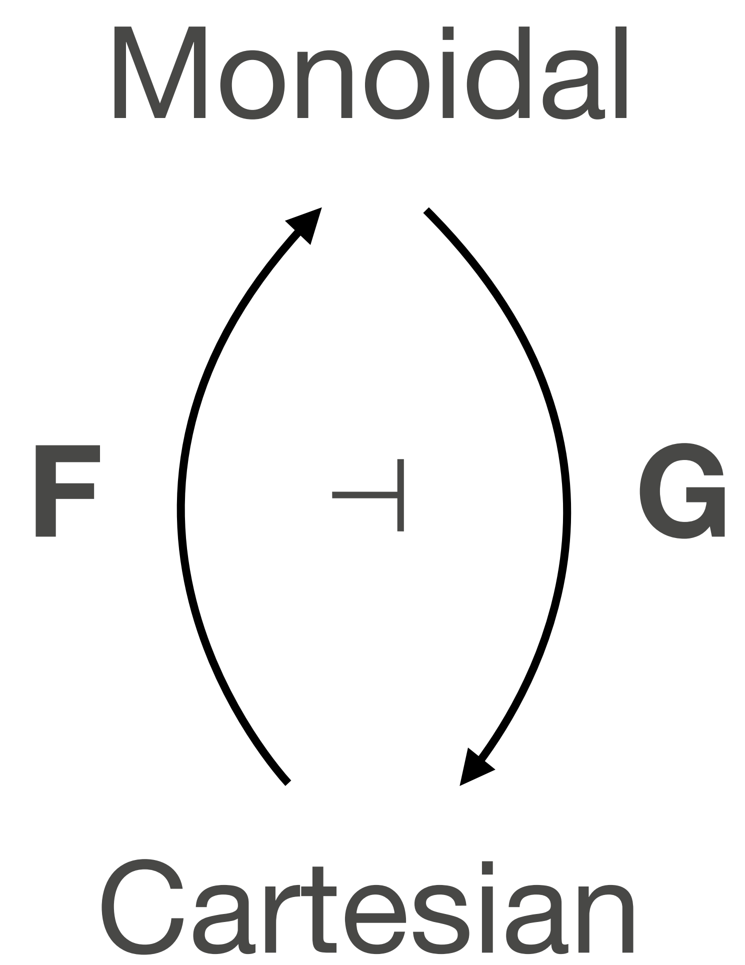
$(m, \otimes, 1)$ comm. monoid

c mode

$(m, \times, \top)$ cart. monoid

$f : c \to m$

$f(x \times y) = f(x) \otimes f(y)$



$!A := \mathbf{FG}\, A$

$$x{:}\ F_f\, A \otimes F_f\, B \vdash_x F_f\,(A \times B)$$

$$A{\otimes}B := F_{y\otimes z}(y{:}A, z{:}B) \qquad A{\times}B := F_{y\times z}(y{:}A, z{:}B)$$

$$\frac{y{:}F_f\ A,\ z{:}F_f\ B \vdash_{y \otimes z} F_f\ (A \times B)}{x{:}\ F_f\ A \otimes F_f\ B \vdash_x F_f\ (A \times B)}$$

$A \otimes B := F_{y \otimes z}(y{:}A, z{:}B)$ $\quad$ $A \times B := F_{y \times z}(y{:}A, z{:}B)$

$$\dfrac{\dfrac{y{:}A,\ z{:}F_f\ B \vdash_{f(y)\otimes z} F_f\ (A \times B)}{y{:}F_f\ A,\ z{:}F_f\ B \vdash_{y\otimes z} F_f\ (A \times B)}}{x{:}\ F_f\ A \otimes F_f\ B \vdash_{x} F_f\ (A \times B)}$$

$A\otimes B := F_{y\otimes z}(y{:}A, z{:}B) \quad A\times B := F_{y\times z}(y{:}A, z{:}B)$

$$\cfrac{\cfrac{\cfrac{y{:}A,\ z{:}B \vdash_{f(y)\otimes f(z)} F_f\,(A \times B)}{y{:}A,\ z{:}F_f\,B \vdash_{f(y)\otimes z} F_f\,(A \times B)}}{y{:}F_f\,A,\ z{:}F_f\,B \vdash_{y\otimes z} F_f\,(A \times B)}}{x{:}\ F_f\,A \otimes F_f\,B \vdash_{x} F_f\,(A \times B)}$$

$A\otimes B := F_{y\otimes z}(y{:}A,z{:}B)$ $\quad$ $A\times B := F_{y\times z}(y{:}A,z{:}B)$

$f(y) \otimes f(z) = f(?)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y{:}A,\ z{:}B \vdash_{f(y)\otimes f(z)} F_f\,(A \times B)}}{y{:}A,\ z{:}F_f\,B \vdash_{f(y)\otimes z} F_f\,(A \times B)}}{y{:}F_f\,A,\ z{:}F_f\,B \vdash_{y \otimes z} F_f\,(A \times B)}}{x{:}\ F_f\,A \otimes F_f\,B \vdash_{x} F_f\,(A \times B)}$$

$A \otimes B := F_{y \otimes z}(y{:}A, z{:}B) \quad A \times B := F_{y \times z}(y{:}A, z{:}B)$

$$\frac{\dfrac{\dfrac{\overline{y{:}A,\ z{:}B \vdash_{f(y)\otimes f(z)} F_f\,(A \times B)}}{y{:}A,\ z{:}F_f\,B \vdash_{f(y)\otimes z} F_f\,(A \times B)}}{y{:}F_f\,A,\ z{:}F_f\,B \vdash_{y\otimes z} F_f\,(A \times B)}}{x{:}\ F_f\,A \otimes F_f\,B \vdash_{x} F_f\,(A \times B)}$$

$$A\otimes B := F_{y\otimes z}(y{:}A,z{:}B) \qquad A\times B := F_{y\times z}(y{:}A,z{:}B)$$

$$f(y)\otimes f(z)=f(y\times z)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y{:}A,\ z{:}B \vdash_{f(y)\otimes f(z)} F_f\,(A\times B)}}{y{:}A,\ z{:}F_f\,B \vdash_{f(y)\otimes z} F_f\,(A\times B)}}{y{:}F_f\,A,\ z{:}F_f\,B \vdash_{y\otimes z} F_f\,(A\times B)}}{x{:}\ F_f\,A\otimes F_f\,B \vdash_{x} F_f\,(A\times B)}$$

$$A\otimes B := F_{y\otimes z}(y{:}A,z{:}B) \qquad A\times B := F_{y\times z}(y{:}A,z{:}B)$$

$$\dfrac{f(y)\otimes f(z)=f(y\times z) \qquad \dfrac{}{y{:}A,z{:}B \vdash_{y\times z} A \times B}}{\dfrac{y{:}A,\ z{:}B \vdash_{f(y)\otimes f(z)} F_f\,(A \times B)}{\dfrac{y{:}A,\ z{:}F_f\,B \vdash_{f(y)\otimes z} F_f\,(A \times B)}{\dfrac{y{:}F_f\,A,\ z{:}F_f\,B \vdash_{y\otimes z} F_f\,(A \times B)}{x{:}\ F_f\,A \otimes F_f\,B \vdash_{x} F_f\,(A \times B)}}}}$$

$$A\otimes B := F_{y\otimes z}(y{:}A,z{:}B) \qquad A\times B := F_{y\times z}(y{:}A,z{:}B)$$

mode theory axiomatizes whether F preserves ⊗ (strictly,iso,laxly, not at all)

$$
\dfrac{f(y)\otimes f(z)=f(y\times z) \qquad \dfrac{}{y{:}A, z{:}B \vdash_{y\times z} A \times B}}{\dfrac{y{:}A,\ z{:}B \vdash_{f(y)\otimes f(z)} F_f\,(A \times B)}{\dfrac{y{:}A,\ z{:}F_f\,B \vdash_{f(y)\otimes z} F_f\,(A \times B)}{\dfrac{y{:}F_f\,A,\ z{:}F_f\,B \vdash_{y\otimes z} F_f\,(A \times B)}{x{:}\ F_f\,A \otimes F_f\,B \vdash_{x} F_f\,(A \times B)}}}}
$$

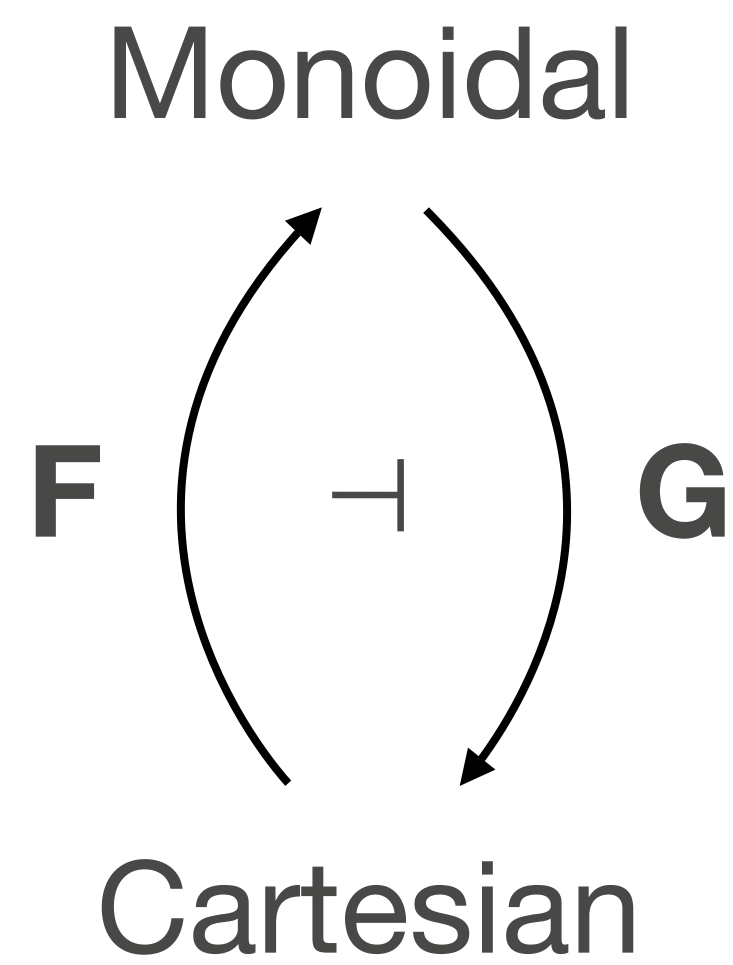
$$
A\otimes B := F_{y\otimes z}(y{:}A,z{:}B) \qquad A\times B := F_{y\times z}(y{:}A,z{:}B)
$$

Benton's LNL

m mode
$(m, \otimes, 1)$ comm. monoid

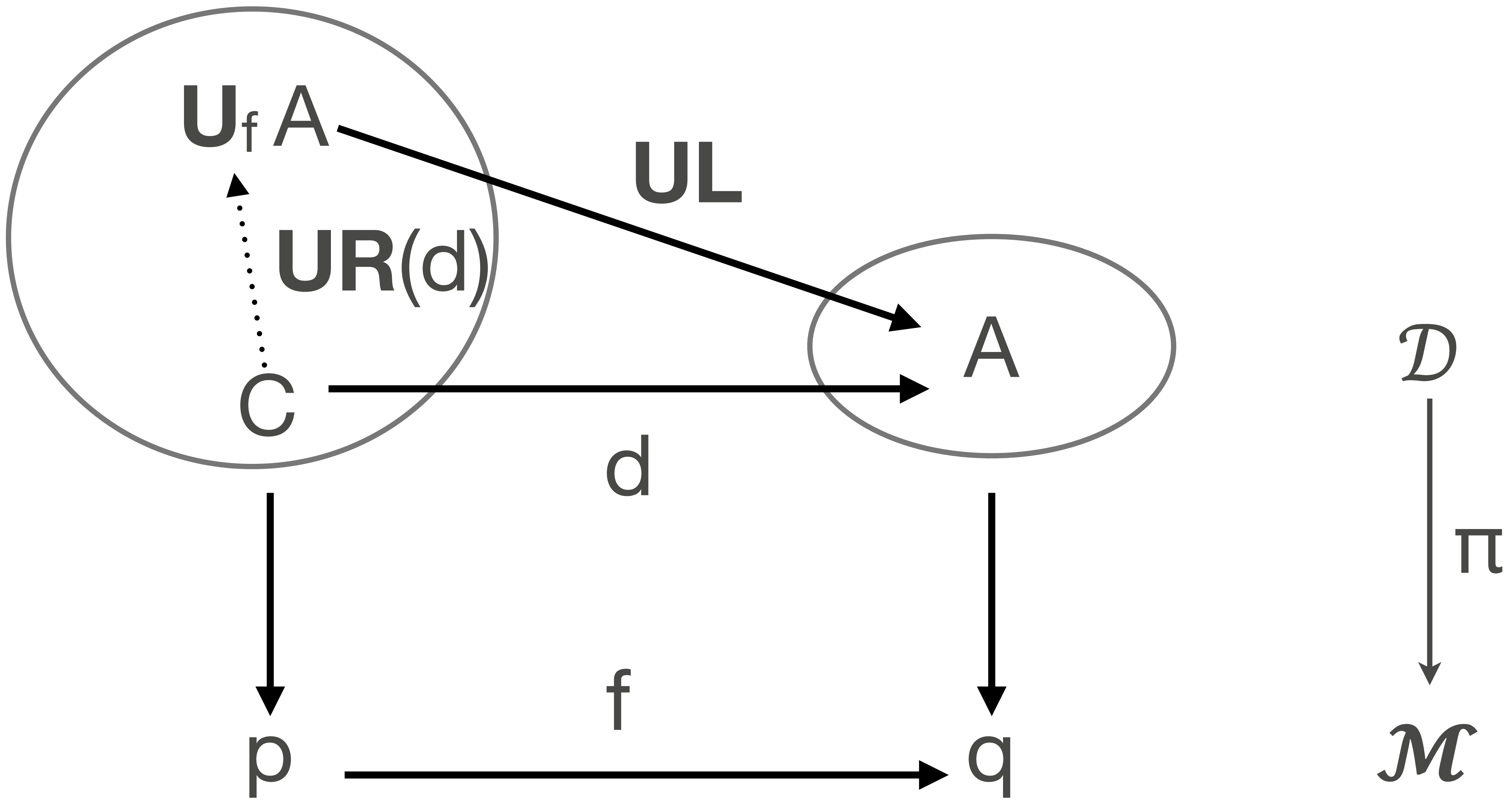
c mode
$(m, \times, \top)$ cart. monoid

$f : c \to m$
$f(x \times y) = f(x) \otimes f(y)$



$!A := FG\,A$

U types: fibration



U types

$$\frac{}{\Gamma \vdash_{y1 \times \ldots \times yn} \mathbf{U_f}(A)}$$

U types

$$\frac{\Gamma \vdash_{f(y1 \times \ldots \times yn)} A}{\Gamma \vdash_{y1 \times \ldots \times yn} \mathbf{U_f}(A)}$$

U types

$$\dfrac{\dfrac{\Gamma \vdash_{f(y1) \otimes \ldots \otimes f(yn)} A}{\Gamma \vdash_{f(y1 \times \ldots x\ yn)} A}}{\Gamma \vdash_{y1 \times \ldots x\ yn} \mathbf{U_f}(A)}$$

U types

non-monoidal: stop here, see Bas's talk next! →

$$\frac{\dfrac{\Gamma \vdash_{f(y1) \otimes \ldots \otimes f(yn)} A}{\Gamma \vdash_{f(y1 \times \ldots x\ yn)} A}}{\Gamma \vdash_{y1 \times \ldots x\ yn} \mathbf{U_f}(A)}$$

intro

$$\frac{\Delta, \Gamma \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^{\sharp} : \sharp A}$$

a map into the #A type can use **each** variable flatly

$$\flat \dashv \sharp$$

$\flat$ idem comonad

$\sharp$ idem monad

Mode theory

c mode

$\flat : c \to c$

counit : $\flat \Longrightarrow 1_c$

$\flat\flat = \flat$ [+ triangle]

$$\flat \dashv \sharp$$

$\flat$ idem comonad
$\sharp$ idem monad

Mode theory

c mode

$\flat : c \to c$

$\text{counit} : \flat \Rightarrow 1_c$

$\flat\flat = \flat$ **[+ triangle]**

$(c, \times, \top)$ cart. monoid

$\flat(y \times z) = \flat(y) \times \flat(z)$

intro

$$\frac{}{\Gamma \vdash_{y \times \flat(z)} \mathbf{U}_\flat\, C}$$

$$\frac{\Delta, \Gamma \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^\sharp : \sharp A}$$

a map into the #A type can use **each** variable flatly

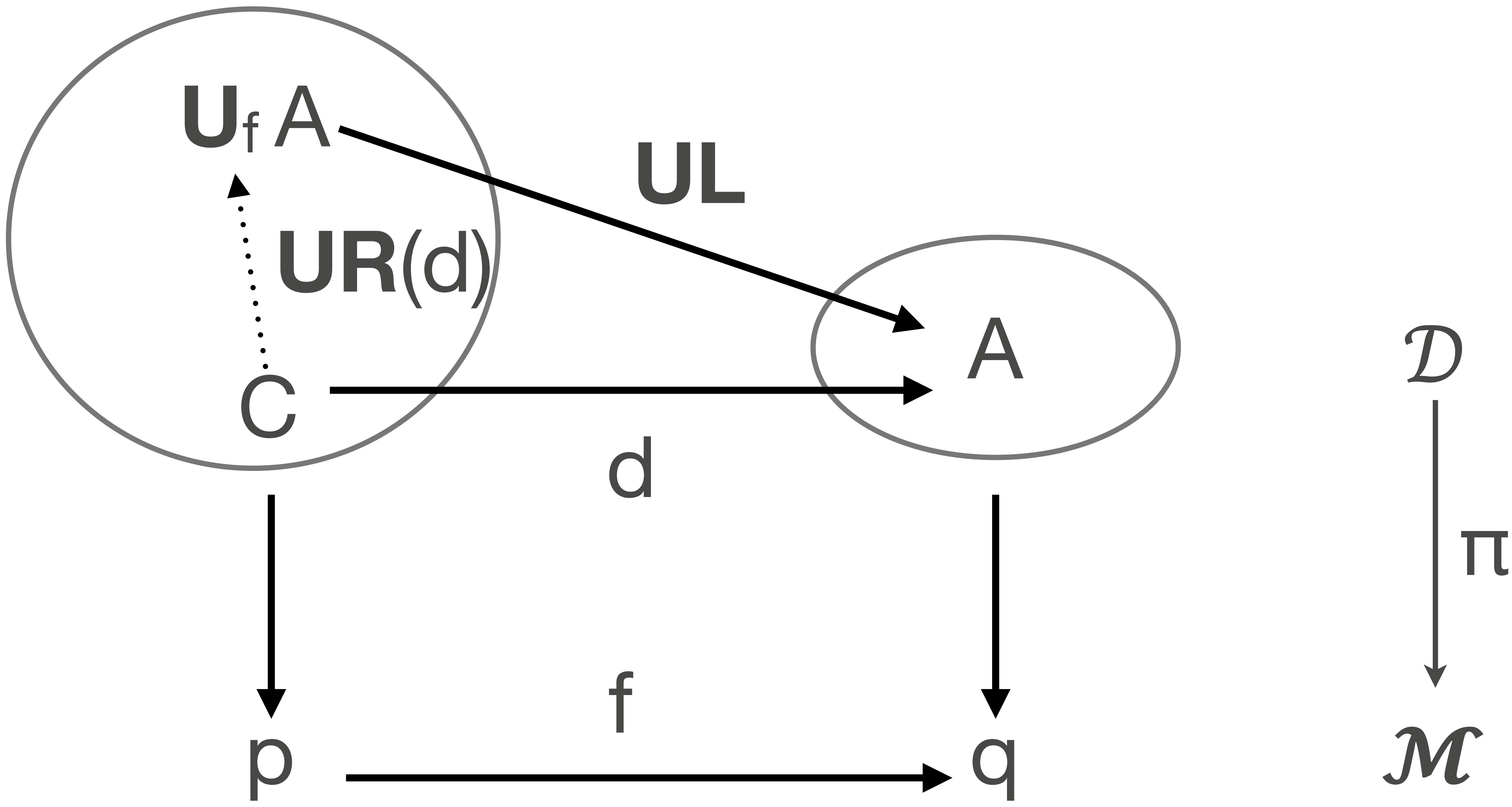
intro

$$\frac{\Gamma \vdash_{\flat(y\times\flat(z)) = \flat(y)\times\flat\flat(z) = \flat(y)\times\flat(z)} C}{\Gamma \vdash_{y \times \flat(z)} \mathbf{U}_{\flat} C}$$

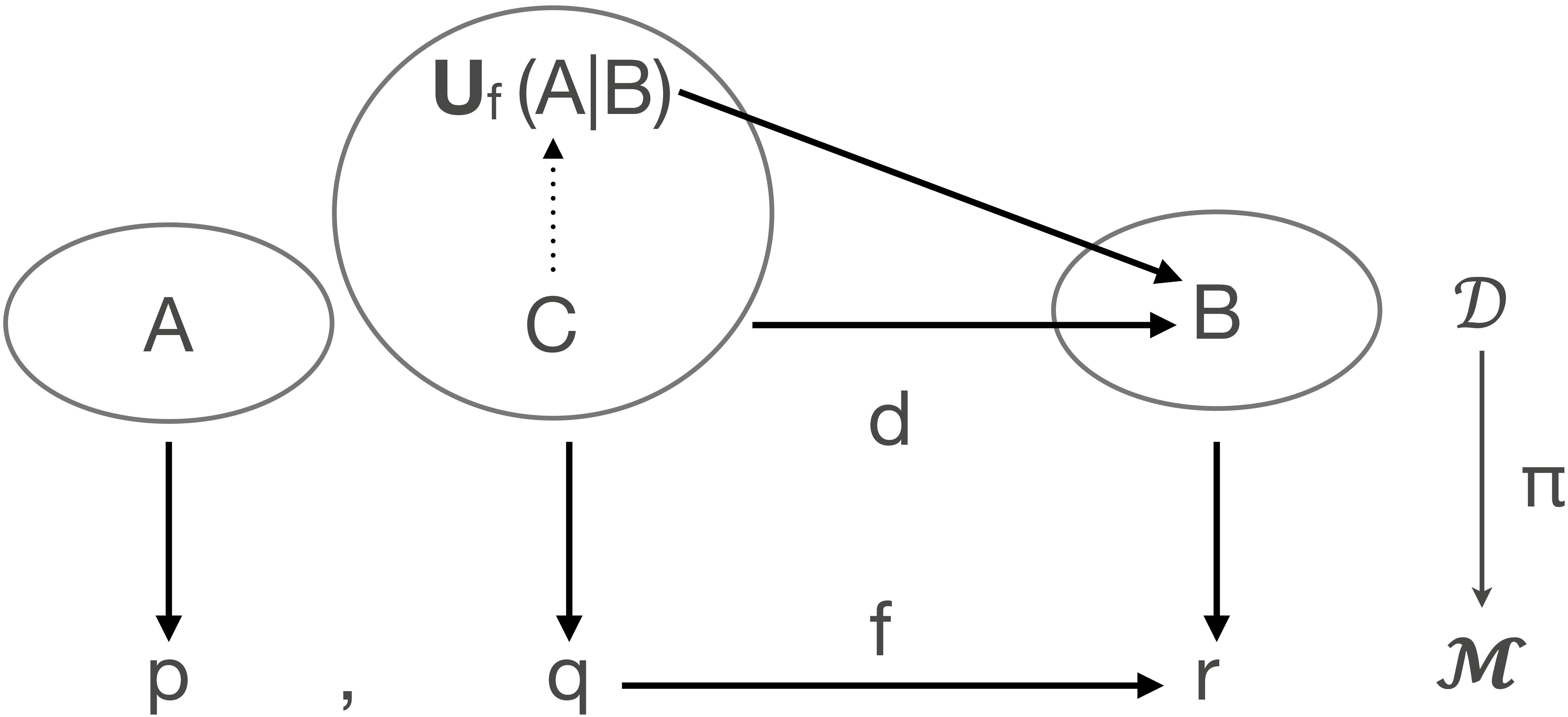
$$\frac{\Delta, \Gamma \mid \cdot \vdash M : A}{\Delta \mid \Gamma \vdash M^{\sharp} : \sharp A}$$

a map into the #A type can use **each** variable flatly

U types: fibration



U types [Atkey'04]



U types

$$\frac{}{x{:}A,\ y{:}\mathbf{U}_{y.f}\,(x{:}A \mid B) \vdash_f B}$$

U types

$$\frac{}{x{:}A,\; y{:}\mathbf{U}_{y.f}(x{:}A \mid B) \vdash_f B}$$

e.g. $A \multimap B := \mathbf{U}_{y.y \otimes x}(x{:}A \mid B)$

U types

$$\frac{}{x{:}A,\ y{:}\mathbf{U}_{y.f}\,(x{:}A \mid B) \vdash_f B}$$

e.g. $A \multimap B := \mathbf{U}_{y.y \otimes x}(x{:}A \mid B)$

$A \xrightarrow{\flat} B := \mathbf{U}_{y.y \times \flat(x)}(x{:}A \mid B)$

U types

$$\frac{}{x{:}A,\ y{:}\mathbf{U}_{y.f}(x{:}A \mid B) \vdash_f B}$$

$$\text{e.g. } A \multimap B := \mathbf{U}_{y.y \otimes x}(x{:}A \mid B)$$

$$A \xrightarrow{\flat} B := \mathbf{U}_{y.y \times \flat(x)}(x{:}A \mid B)$$

“crisp-argument functions” implemented by Vezzosi in agda-flat

Multiplicatives and exponentials are the same connective

- $\mathbf{F}_f(x_1 : A_1, \ldots, x_n : A_n)$ unifies $\flat$ and $\otimes$
- $\mathbf{U}_f(x_1 : A_1, \ldots, x_n : A_n \mid B)$ unifies $\sharp$ and $\multimap$
- Cut-free sequent calculus with subformula property
- Sound and complete for local discrete bifibrations of cartesian 2-multicategories
- Soundness of usual rules for lots of examples, completeness for some

[L., Shulman, Riley, ’17]

A Framework for Modal Dependent Type Theories

[L., Riley, Shulman]

Dependency

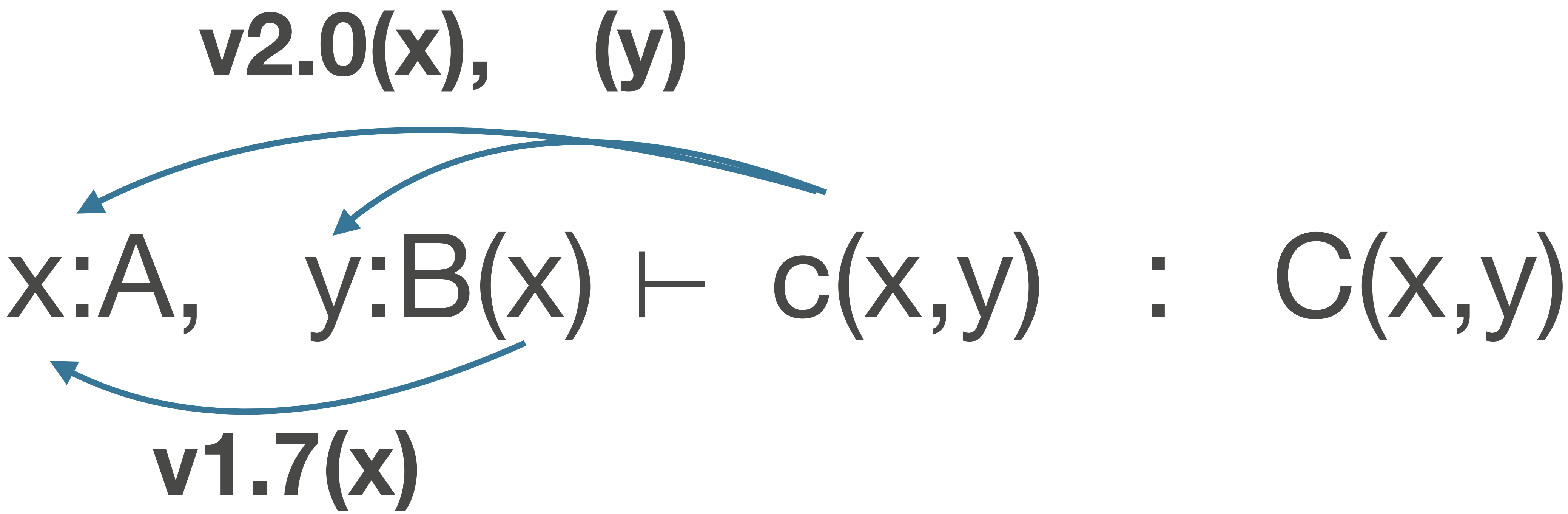
$$x{:}A, \quad y{:}B(x) \vdash c(x,y) \quad : \quad C(x,y)$$

Dependency

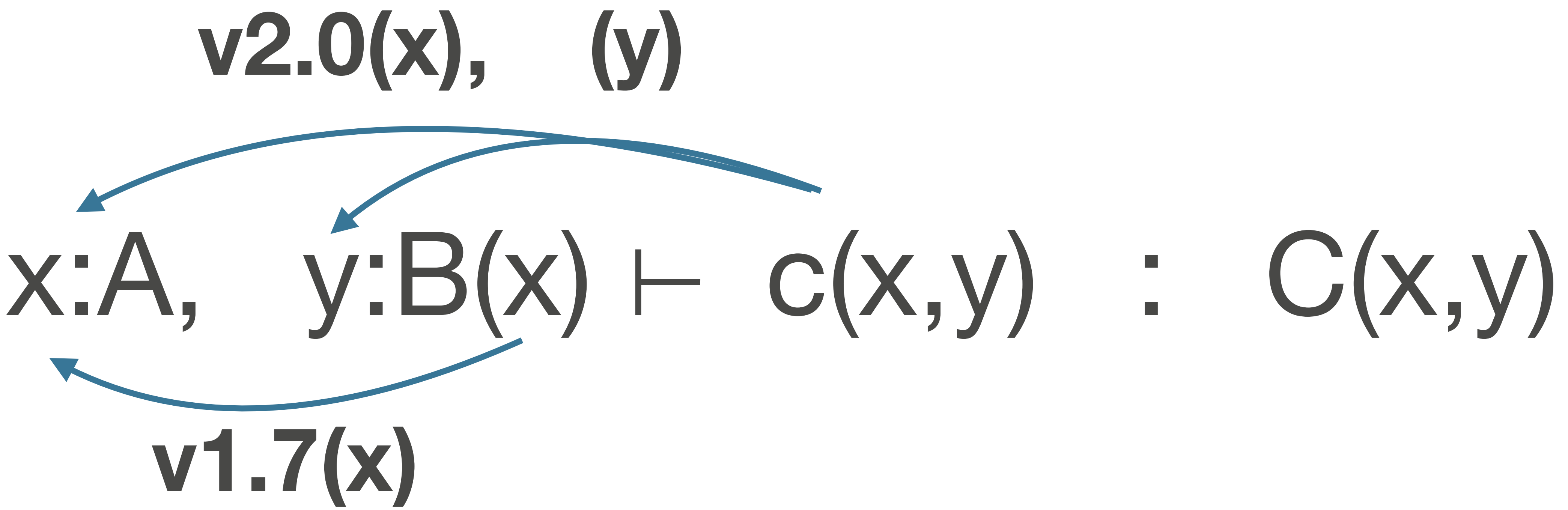
$$x{:}A, \quad y{:}B(x) \vdash c(x,y) \; : \; C(x,y)$$

v1.7(x)

Dependency

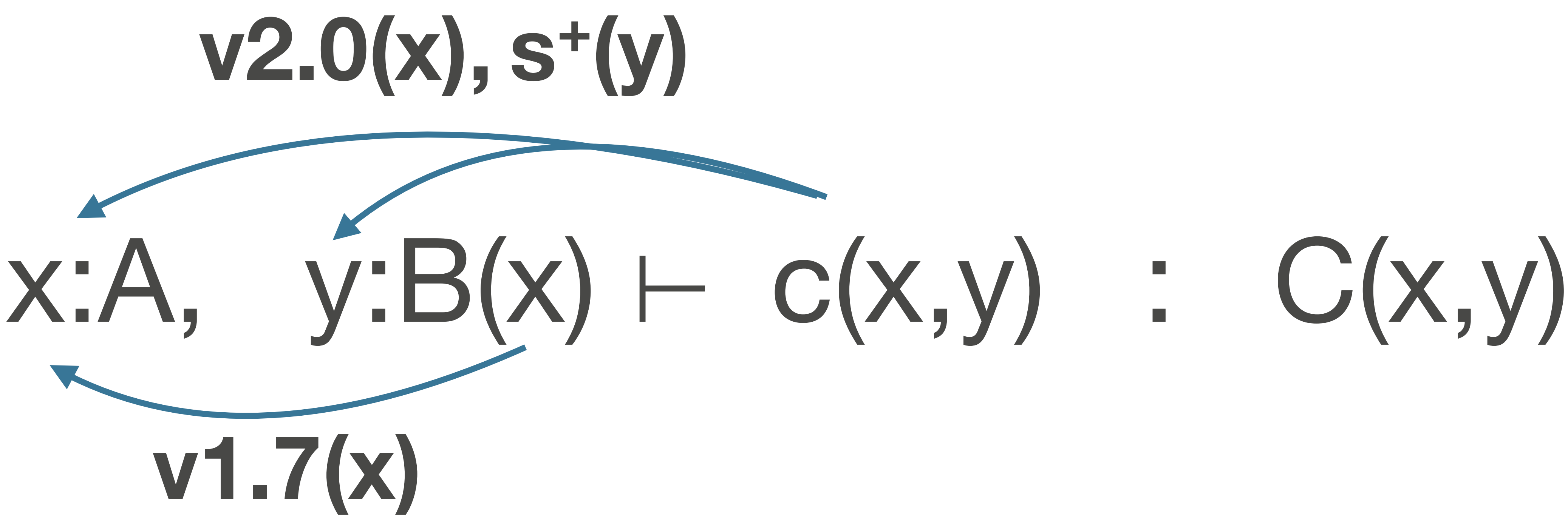


Dependency



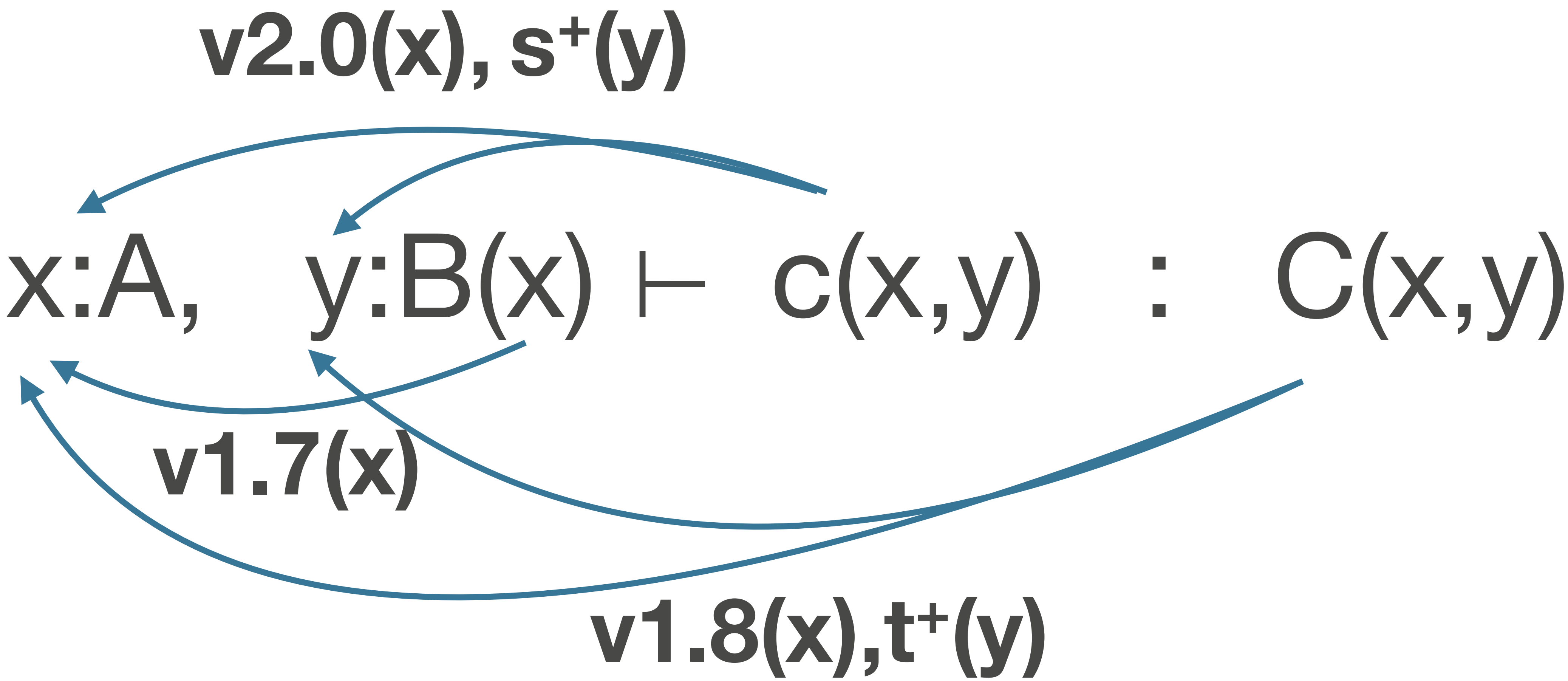
$$\mathbf{s : v2.0(x) \Longrightarrow v1.7(x)}$$

Dependency



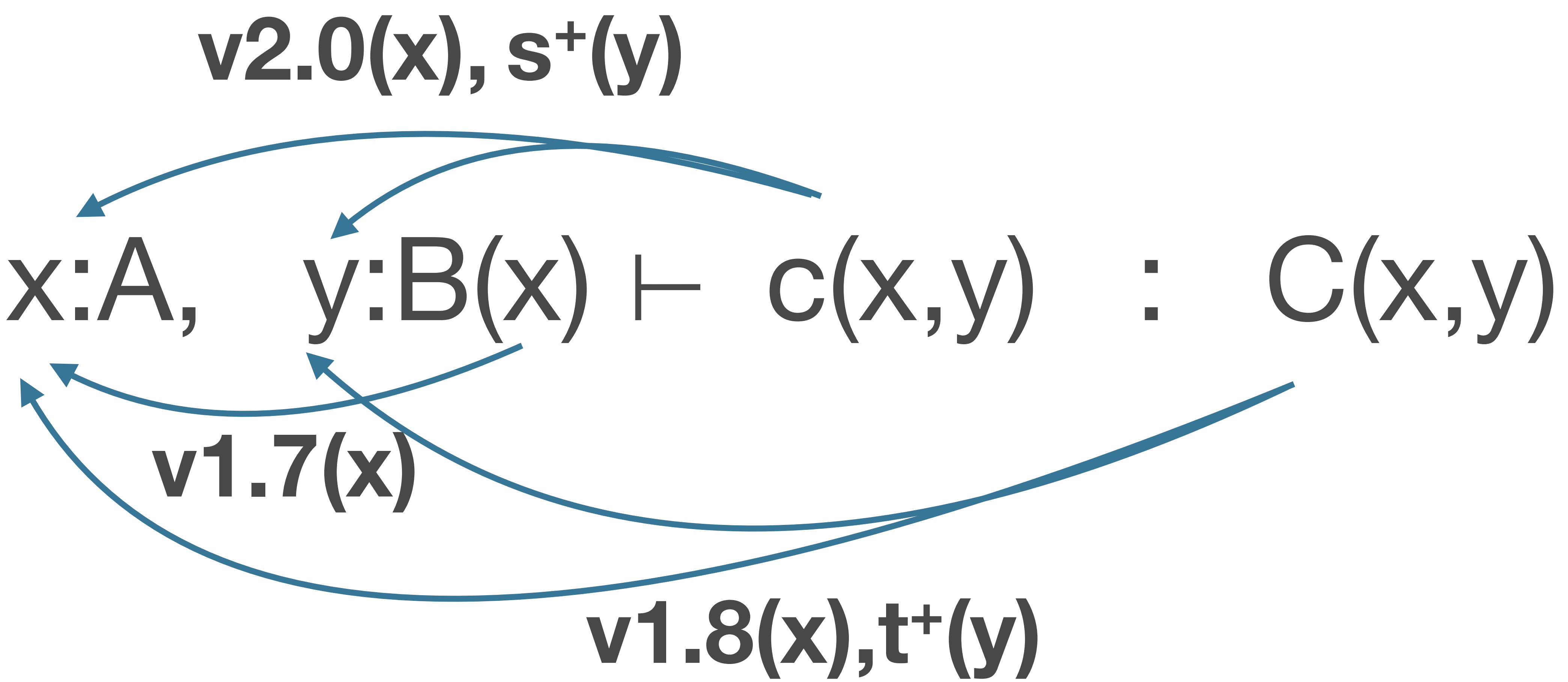
$$\mathbf{s : v2.0(x) \Longrightarrow v1.7(x)}$$

Dependency



$$s : v2.0(x) \Longrightarrow v1.7(x)$$

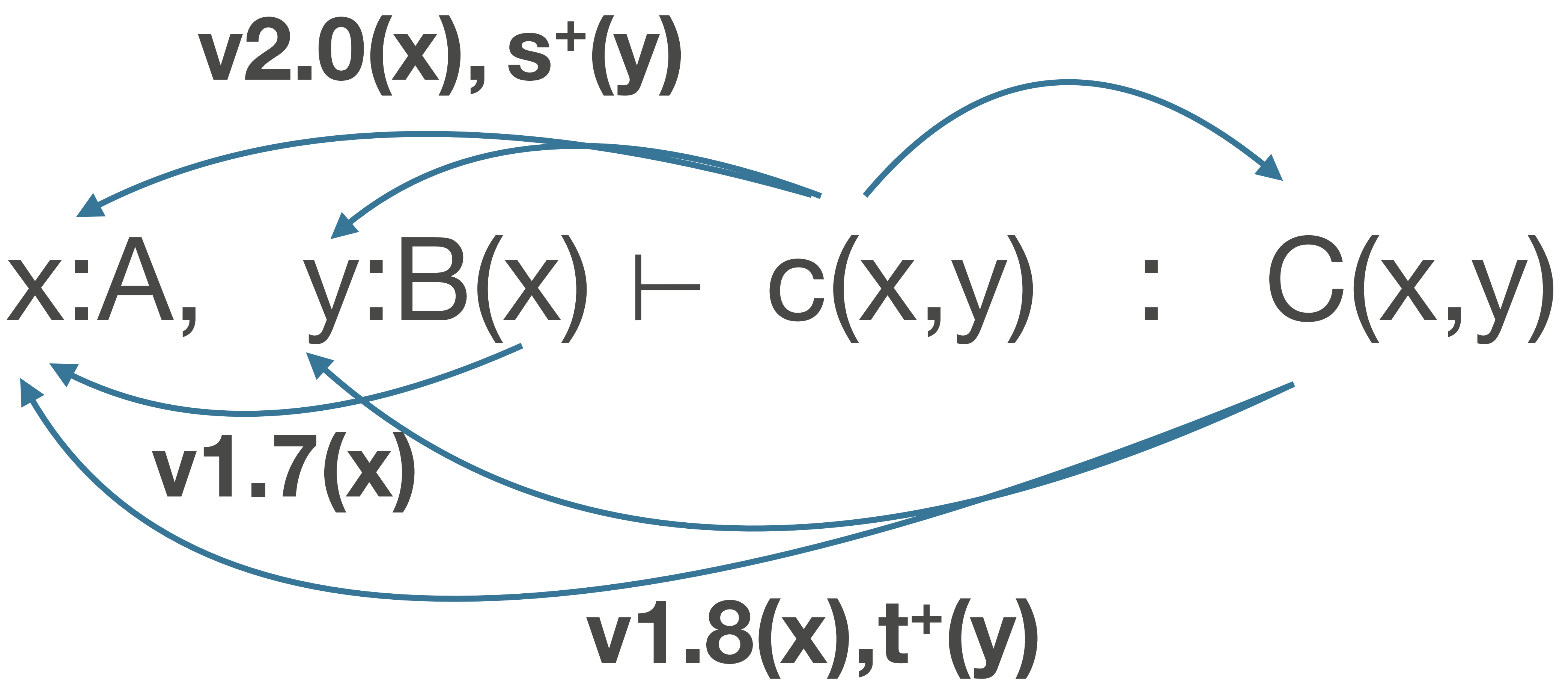
Dependency



$$\mathbf{s : v2.0(x) \Longrightarrow v1.7(x)}$$

$$\mathbf{t : v1.8(x) \Longrightarrow v1.7(x)}$$

Dependency



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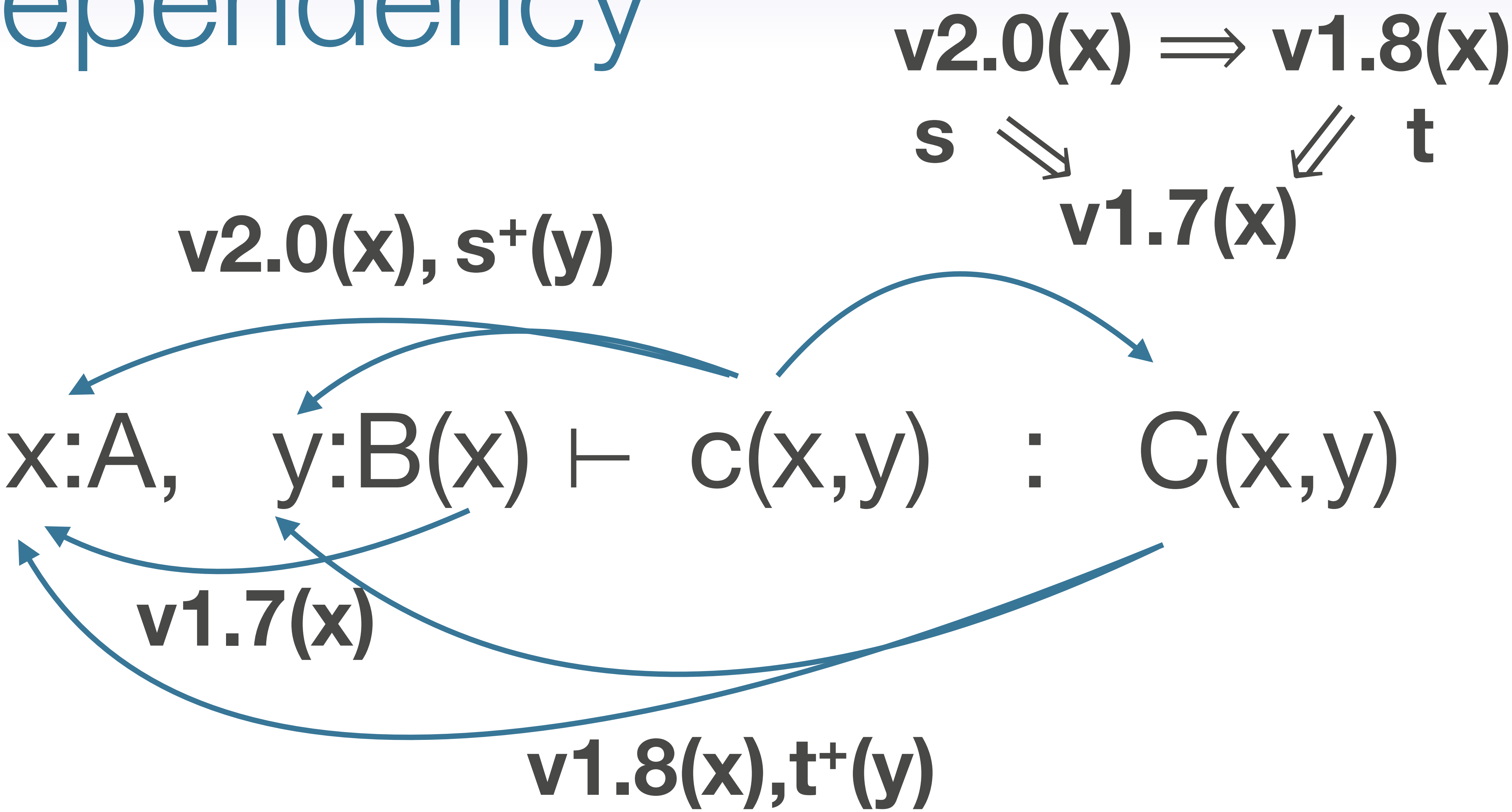

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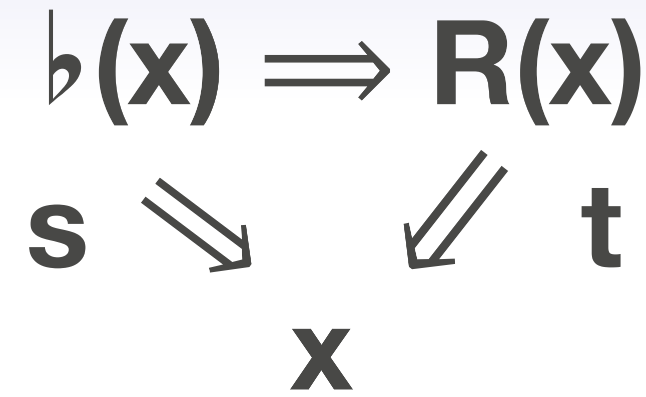
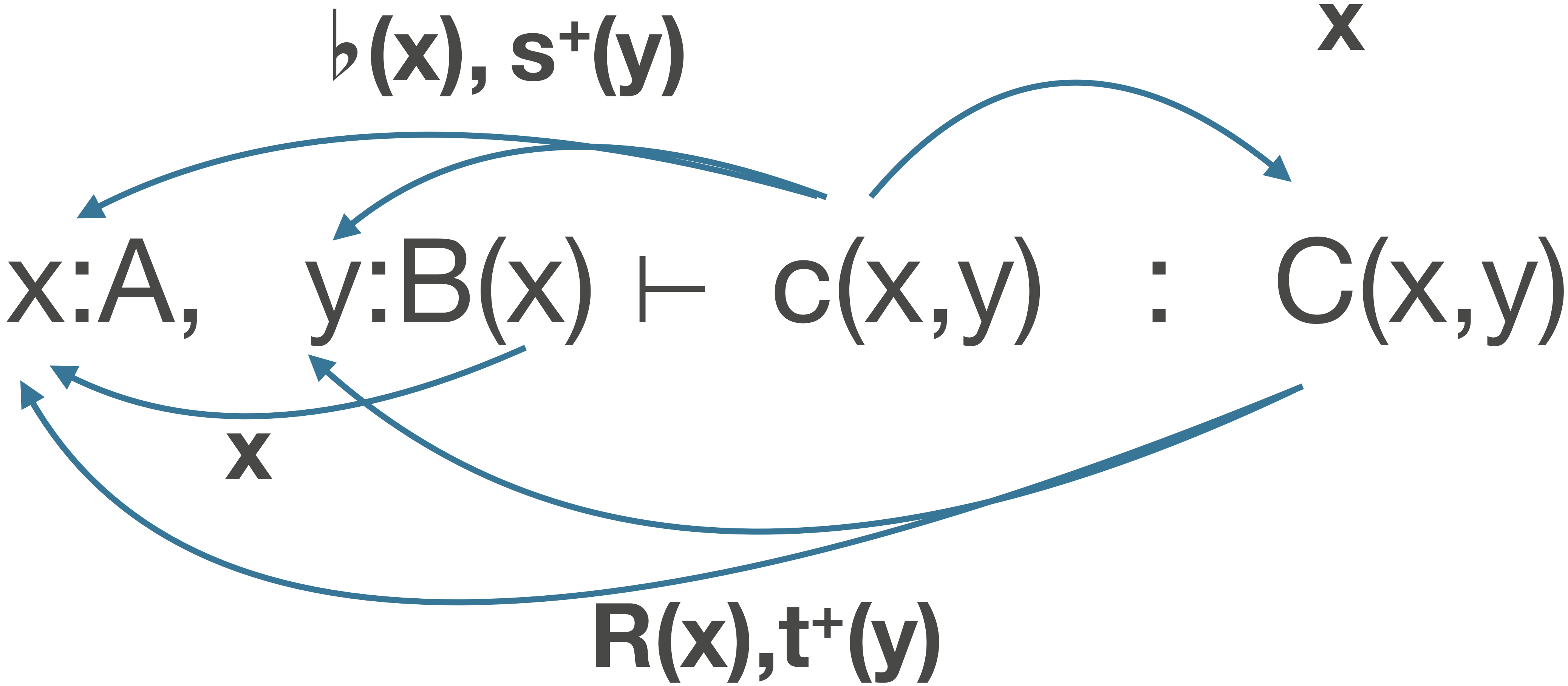
Dependency





$$s : \flat(x) \Longrightarrow x$$

$$t : R(x) \Longrightarrow x$$

Unary type theory

local discrete bifibration of 2-categories

$$\begin{array}{c} \mathcal{D} \\ \Big\downarrow \pi \\ \mathcal{M} \end{array}$$

base is 2-categorical (natural transformaitions)

Simple type theory

top includes ordinary simple type theory

$$\mathcal{D} \xrightarrow{\;\pi\;} \mathcal{M}$$

local discrete bifibration of cartesian 2-multicategories

base is 2-categorical (natural transformaitions)

Dependent type theory

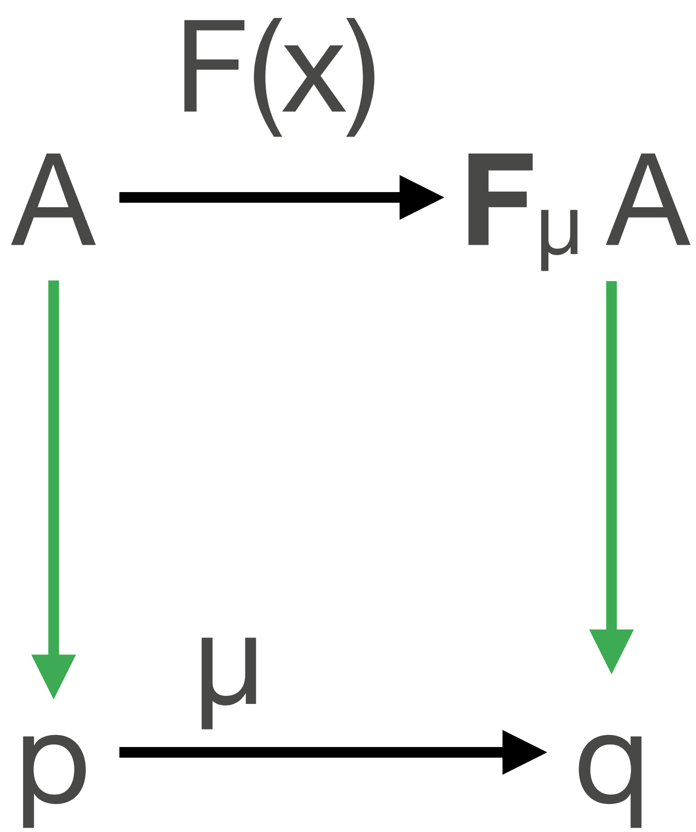
local discrete bifibration of comprehension bicategories

$$\begin{array}{c} \mathcal{D} \\ \downarrow \pi \\ \mathcal{M} \end{array}$$

top is a dependent type theory

base is 2-categorical

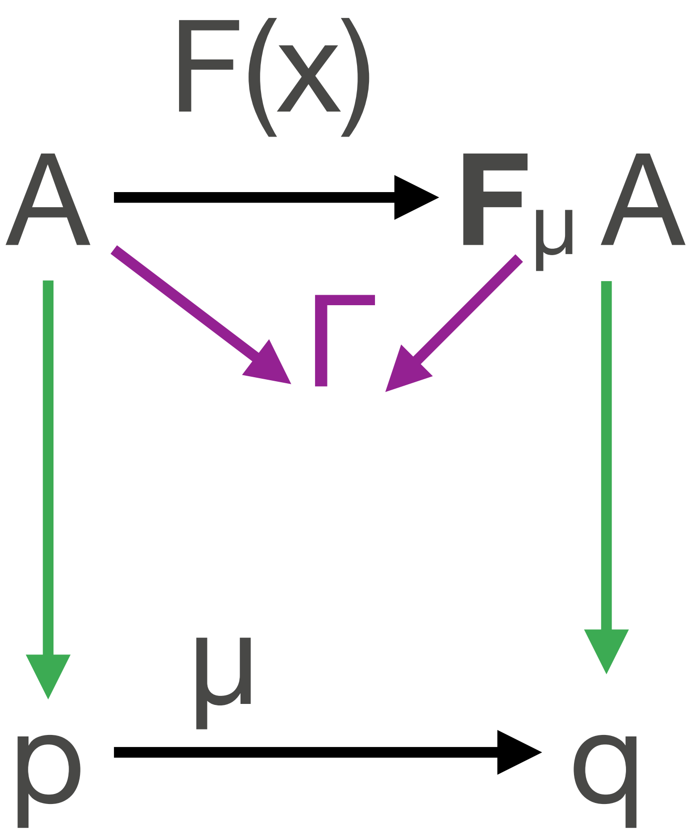
F Types (Opfibrations)



$$\frac{\Gamma \vdash_p A\ \text{Type}}{\Gamma, x{:}A \vdash_\mu F(x) : \mathbf{F}_\mu\, A}$$

$$\frac{\gamma \vdash p,\ q\ \text{type}}{\gamma, x{:}p \vdash \mu : q}$$

F Types (Opfibrations)



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$$\frac{\gamma \vdash p, q \text{ type}}{\gamma, x{:}p \vdash \mu : q}$$

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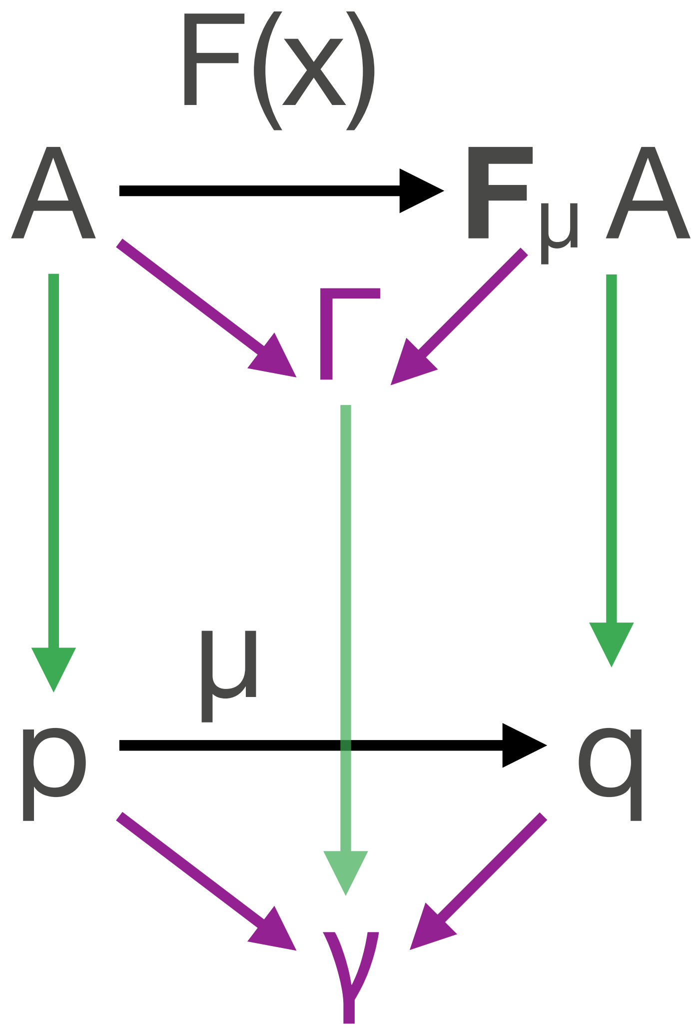
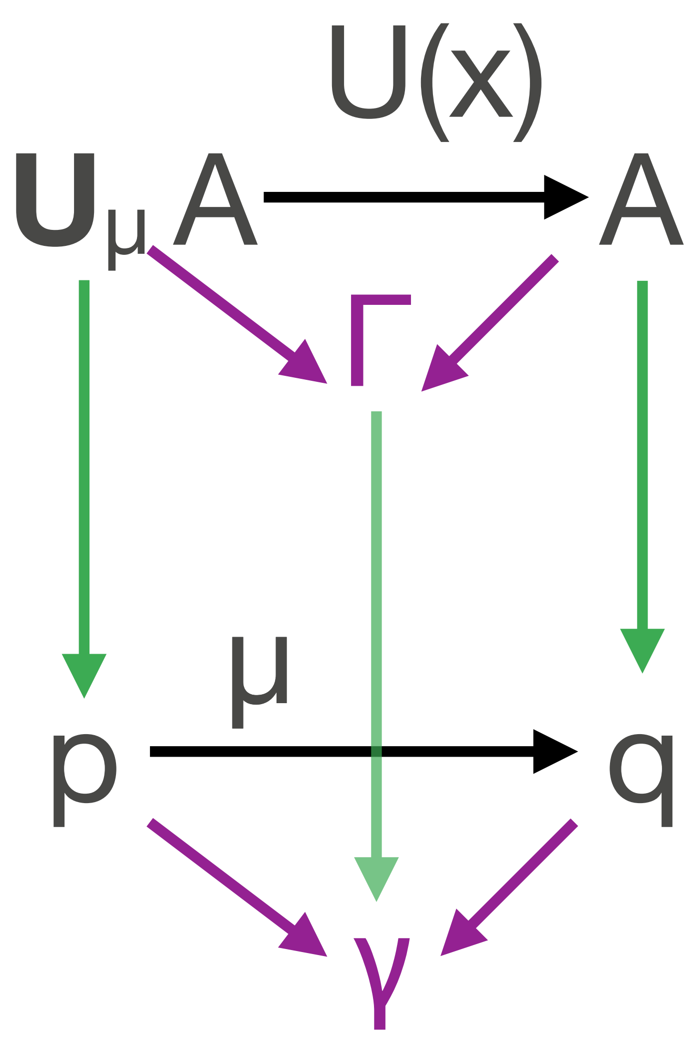


$$\frac{\Gamma \vdash_p A \text{ Type}}{\Gamma, x{:}A \vdash_\mu F(x) : \mathbf{F}_\mu A}$$

$$\frac{\gamma \vdash p, q \text{ type}}{\gamma, x{:}p \vdash \mu : q}$$

U Types (Fibrations)



$$\frac{\Gamma \vdash_q A \text{ Type}}{\Gamma, x{:}\, \mathbf{U}_\mu A \vdash_\mu U(x) : A}$$

$$\frac{\gamma \vdash p, q \text{ type}}{\gamma, x{:}p \vdash \mu : q}$$

Subtle Parts

- Action of 2-cells on upstairs types
- Action of 2-cells inside the mode theory itself (basic *directed* dependent type theory)
- What mode theories should be

Mode Theory for Simple Types

describes the "algebra" of contexts

- Linear: ($\mathbf{p}$,⊗,1) a symmetric monoid object in $\mathcal{M}$
- Cartesian: ($\mathbf{q}$,×,⊤) a monoid object in $\mathcal{M}$
- Comonads: $\flat:\mathbf{p}\rightarrow\mathbf{p}$ a idempotent comonad in $\mathcal{M}$

Mode Theory for Dependent Types

Let $(\mathbf{p}, \mathbf{T}, \varnothing, .)$ be a comprehension object with $\Sigma, =$ in $\mathcal{M}$:

Mode Theory for Dependent Types

Let (**p**,**T**,∅,.) be a comprehension object with Σ,= in $\mathcal{M}$:

- a type **p** of contexts
- a dependent type **T**(α : **p**) of dependent types
- for α:**p** and x:**T**(α), a comprehension α.x : **p** with a projection 2-cell α.x ⟹ α
- "substitution" along projection **T**(α) → **T**(α.x)
- with 1, Σ types left adjoint to projection $\Sigma_\alpha(x)$: **T**(α.x) → **T**(α)

Mode Theory for Dependent Types

$$x{:}A, y{:}B, z{:}C \vdash_{x \otimes (y \otimes z)} D$$ **uses all three**

Mode Theory for Dependent Types

$$x{:}A, y{:}B(x), z{:}C(y) \vdash D(x,y,z) \text{ type}$$

Mode Theory for Dependent Types

$$x{:}A, y{:}B(x), z{:}C(y) \vdash D(x,y,z) \text{ type}$$

$$x{:}A,\ y{:}B(x),\quad z{:}C(x,y) \quad \vdash_{\mathbf{T}(\varnothing.x.y.z)} D(x,y,z) \text{ type}$$

Mode Theory for Dependent Types

$$x{:}A, y{:}B(x), z{:}C(y) \vdash D(x,y,z)\ \text{type}$$

$$x{:}A,\ y{:}B(x),\quad z{:}C(x,y) \quad \vdash_{\mathbf{T}(\varnothing.x.y.z)} D(x,y,z)\ \text{type}$$

$$x{:}\mathbf{T}(\varnothing), y{:}\mathbf{T}(\varnothing.x), z{:}\mathbf{T}(\varnothing.x.y) \vdash \mathbf{T}(\varnothing.x.y.z)\ \text{mode}$$

Mode Theory for Dependent Types

$$x{:}A, y{:}B(x), z{:}C(x,y) \vdash d(x,y,z) : D(x,y,z)$$

Mode Theory for Dependent Types

$$x{:}A, y{:}B(x), z{:}C(x,y) \vdash d(x,y,z) : D(x,y,z)$$

$$x{:}A, y{:}B(x), z{:}C(x,y) \vdash_1 d(x,y,z) : D(x,y,z)$$

Mode Theory for Dependent Types

$$x{:}A, y{:}B(x), z{:}C(x,y) \vdash d(x,y,z) : D(x,y,z)$$

$$x{:}A, y{:}B(x), z{:}C(x,y) \vdash_1 d(x,y,z) : D(x,y,z)$$

$$x{:}\mathbf{T}(\varnothing), y{:}\mathbf{T}(\varnothing.x), z{:}\mathbf{T}(\varnothing.x.y) \vdash 1 : \mathbf{T}(\varnothing.x.y.z)$$

Mode Theory for Dependent Types

$$x{:}A, y{:}B(x), z{:}C(x,y) \vdash\ d(x,y,z) : D(x,y,z)$$

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$$x{:}\mathbf{T}(\varnothing), y{:}\mathbf{T}(\varnothing.x), z{:}\mathbf{T}(\varnothing.x.y) \vdash 1 : \mathbf{T}(\varnothing.x.y.z)$$

strict monoid $\approx$ $\alpha.1 = \alpha$

Modalities

Let $\flat$: $\mathbf{p} \to \mathbf{p}$ idempotent comonad

$\flat'$: (α : $\mathbf{p}$) $\to$ $\mathbf{T}(\alpha) \to \mathbf{T}(\flat\, \alpha)$

be a morphism of comprehension objects

Modalities

Let ♭ : **p** → **p** idempotent comonad

♭' : (α : **p**) → **T**(α) → **T**(♭ α)

be a morphism of comprehension objects

$$\frac{\Delta \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \flat A : \mathsf{Type}} \qquad \frac{\Delta, \Gamma \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \sharp A : \mathsf{Type}}$$

Modalities

Let $\flat : \mathbf{p} \to \mathbf{p}$ idempotent comonad

$\flat' : (\alpha : \mathbf{p}) \to \mathbf{T}(\alpha) \to \mathbf{T}(\flat\, \alpha)$

be a morphism of comprehension objects

$$\frac{\Gamma \vdash_{\mathbf{T}(\flat\, \alpha)} A \text{ Type}}{\Gamma \vdash_{\mathbf{T}(\alpha)} \mathbf{U}_{\flat'(\alpha,-)} A \text{ Type}}$$

$$\frac{\Delta \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \flat A : \mathsf{Type}} \qquad \frac{\Delta, \Gamma \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \sharp A : \mathsf{Type}}$$

Modalities

Let $\flat : \mathbf{p} \to \mathbf{p}$ idempotent comonad

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$$\frac{\Gamma \vdash_{\mathbf{T}(\flat\, \alpha)} A \text{ Type}}{\Gamma \vdash_{\mathbf{T}(\flat\, \alpha)} \mathbf{F}_{\flat'(\alpha,-)} A \text{ Type}} \qquad \frac{\Gamma \vdash_{\mathbf{T}(\flat\, \alpha)} A \text{ Type}}{\Gamma \vdash_{\mathbf{T}(\alpha)} \mathbf{U}_{\flat'(\alpha,-)} A \text{ Type}}$$

$$\frac{\Delta \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \flat A : \mathsf{Type}} \qquad \frac{\Delta, \Gamma \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \sharp A : \mathsf{Type}}$$

Modal dependent type theories

local discrete bifibration of comprehension bicategories

$$\mathcal{D} \xrightarrow{\ \pi\ } \mathcal{M}$$

top is a dependent type theory

base is 2-categorical dependent type theory