

Group Overview

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Cast

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Bicubical directed type theory

- Variant of Riehl-Shulman simplicial type theory
- Simplices $\rightarrow$ "Dedekind" cubes
- Weaver - L. 2020: directed-univalent universe of covariant fibrations (U_{Cov})
Using ideas from
 cubical type theory
 Cavallo - Riehl - Sattler
 Cognard - Ruch - Sattler

Cooltt: proof assistant for cubical
type theory by CMU/UMN sites
Favonia, Sterling, Angioli, Mullanix, Cavallo, Harper,...

directed cooltt:

directed interval (2) + cofibs (Weaver, Mullanix)

Simpler extension types (Weaver's thesis)

plan for UCov, UInner, UCart, ... (Weaver's thesis)

directed circle HIT (Lepeska-True, Weaver)

cofibration entailment with $\begin{matrix} i \sqcap j \\ i \sqcup j \\ r = r' \end{matrix}$ (Rose, Plante)

encode-decode for DHITs (Akubilo, Rose)

DHIT

$I : \text{type}$

$0 : I$

$1 : I$

$\text{Seg} : \text{Hom}_\pm(0, 1)$

$\text{comp} : \text{isSegal}(I)$
(Kan too)

$\left\{ \begin{array}{l} \pi h : \Delta_2 \rightarrow I \\ \pi d : \text{Cof}, \pi t : \Delta \rightarrow (\Delta^2 \rightarrow I) [\Delta_2 \rightarrow h] \\ \pi i, j : \Delta^2. \text{I}[\Delta_2(i, j) \xrightarrow{\alpha} t(i, j)] \end{array} \right.$

$I\text{-rec} : \pi(x : \text{Type}) (s x : \text{isSegal } x)$
 $(z : x) (o : x) (s : \text{Hom}_x(z, o))$
 $\rightarrow (I \rightarrow x)$

$I\text{-elim} : \pi(x : I \rightarrow \text{Type}) (i x : \text{isInner } I \ x)$

init: $\prod_{x:I} \underline{\text{Hom}(0, x)}$
 cou $\rightarrow$ inner

$$\text{init}(0) = \text{id} : \text{Hom}(0, 0)$$

$$\text{init}(1) = \text{seg} : \text{Hom}(0, 1)$$

$$\text{init}(\text{seg}) = \lambda i j. \text{seg}(i \wedge j)$$

$$\underbrace{\text{Hom}_{\text{seg}}^{\text{Hom}(0, x)}(\text{id}, \text{seg})} \equiv \text{id} \downarrow \begin{array}{ccc} & \xrightarrow{\text{id}} & \\ \text{id} \downarrow & & \downarrow \text{seg}(i) \\ & \xrightarrow{\text{seg}(i)} & \end{array}$$

Code: $I \rightarrow U(\omega)$] *segal*

Code 0 = Void

Code 1 = Unit

Code (seg) = fun-to-arr(!) : $\text{Hom}_{U(\omega)}(\text{Void}, \text{Unit})$

enc: $\prod x: I. \text{Hom}(1, x) \longrightarrow \text{Code}(x)$

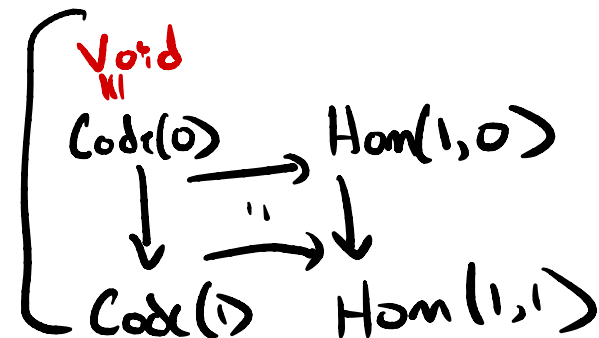
enc $\times$ f = dtransport^{Code} f <>

dec: $\prod x: I. \underline{\text{Code}}(x) \longrightarrow \underline{\text{Hom}}(1, x)$] *inner*

dec 0 = $\lambda ()$

dec 1 = $\lambda \langle \rangle. \frac{\text{id}}{\text{Hom}(1, 1)}$

dec seg =



$$\prod x: I, h: \text{Hom}_{\mathbb{I}}(1, x). \quad h =_{\text{Hom}(1, x)} \text{enc}(\text{dec } h)$$

CON

By dependent yoneda,
 suffices to show

$$\text{id} =_{\text{Hom}(1, 1)} \text{enc}(\text{dec id})$$

Trickier

$$A \xrightarrow{\text{inl}} A+B \xleftarrow{\text{inr}} B$$

$$\text{Hom}_{A+B}(p, q) \stackrel{?}{=} \text{Code}(p, q)$$

[follows from
finiteness of $\mathcal{D}$]

$(A+B)^{\text{op}} \times (A+B) \rightarrow \mathcal{U}(\text{cov})$] should A, B need
to be closed/
crisp?

$$\text{Code}: (A+B) \times (A+B) \rightarrow \text{Type}$$

$$\text{Code}(\text{inl } x, \text{inl } y) = \text{Hom}_A(x, y)$$

$$\text{Code}(\text{inr } x, \text{inr } y) = \text{Hom}_B(x, y)$$

$$\text{Code}(-, -) = \perp$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma^{\text{op}} \vdash A^{\text{op}} \text{ type}}$$

An Interval-Based
Type Theory
For Θ_2 -spaces

Dan Licata

MURI Meeting 2024

	a type*	the whole theory* (types + functions)
Book HoTT Cubical type theory HOTT	$(\infty, 0)$ -category $\equiv$ ∞ -groupoid	$(\infty, 1)$ -category
Simplicial type theory bicubical type theory	$(\infty, 1)$ -category	$(\infty, 2)$ -category
?	$(\infty, 2)$ -category	$(\infty, 3)$ -category

A type in an $(\infty, 2)$ -category

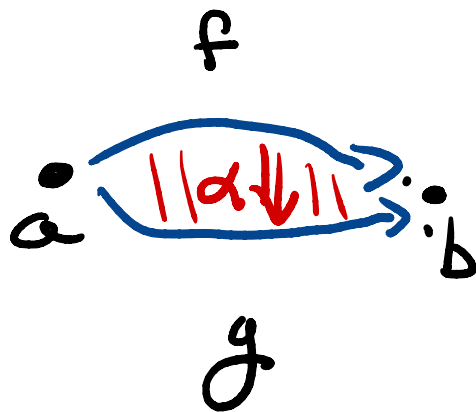
0-cells

a

1-cells



2-cells



3-cells

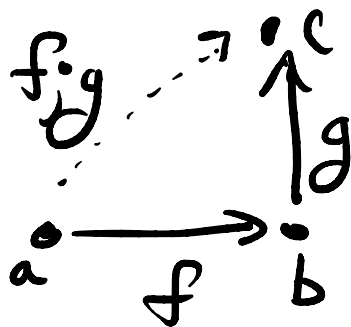
$\alpha \approx \beta$

$\vdots$

not
necessarily
symmetric

symmetric/
paths

1-cells

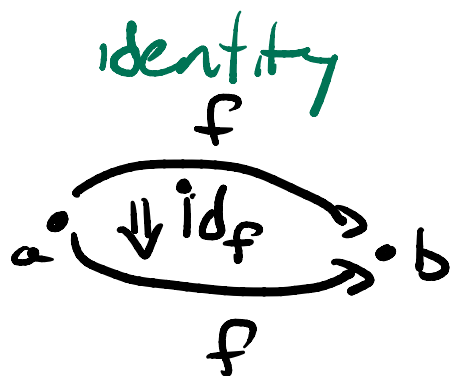


$$Id_a; f = f = f; Id_b$$

$$f; (g; h) = (f; g); h$$

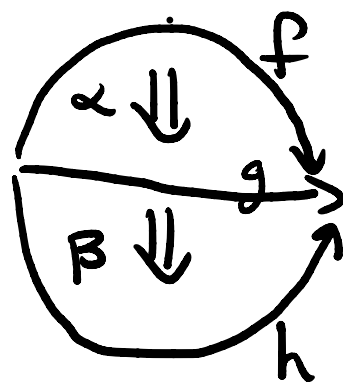
path/iso 2-cell

2-cells



$$Id_f: f \Rightarrow f$$

vertical composites

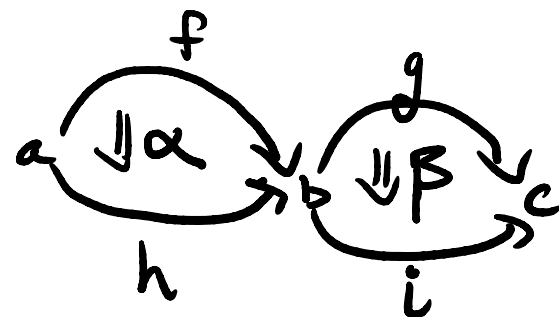


$$\alpha: f \Rightarrow g$$

$$\beta: g \Rightarrow h$$

$$\alpha \cdot \beta: f \Rightarrow h$$

horizontal composites



$$\alpha: f \Rightarrow h$$

$$\beta: g \Rightarrow i$$

$$\alpha; \beta: f; g \Rightarrow h; i$$

(Various laws up to paths)

Background:

$(\infty, 1)$ -categories

Simplex category $\triangle$

- objects $\{0\}, \{0,1\}, \{0,1,2\}, \dots$
- morphisms are order-preserving functions

full subcategory of Cat on free categories
generated by diagrams



[0] .

[1] $\longrightarrow$.

[2] $\bullet \longrightarrow \bullet \longrightarrow \bullet$

Simplicial set

$$X: \Delta^{op} \xrightarrow{\text{functor}} \text{Sets}$$

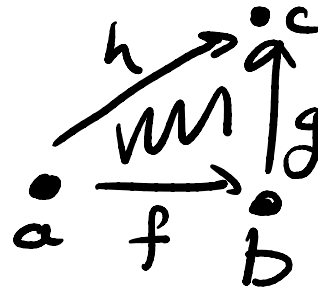
$X(0)$ points

$\bullet a$

$X(1)$ morphisms

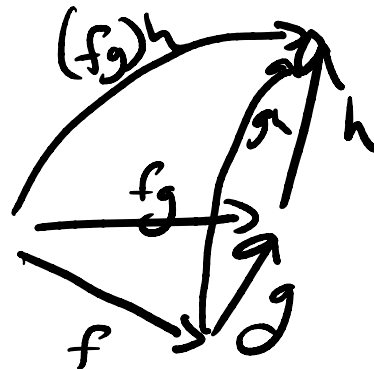
$\bullet a \longrightarrow \bullet b$

$X(2)$ composites



h is $f \circ g$

$X(3)$ associativity

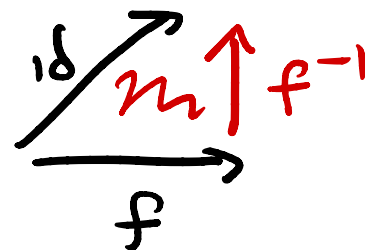
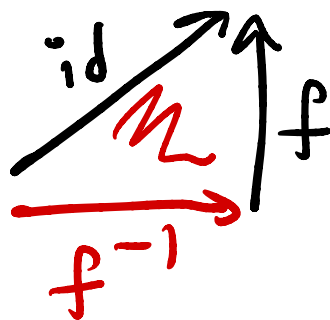


$\vdots$

$(\infty, 0)$ -category $:=$ Kan complex

Identify those simplicial sets that represent an ∞ -groupoid using a filling problem:

Any "horn" can be filled



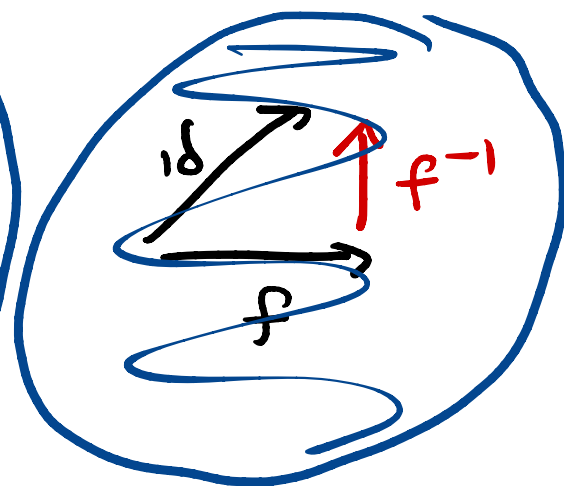
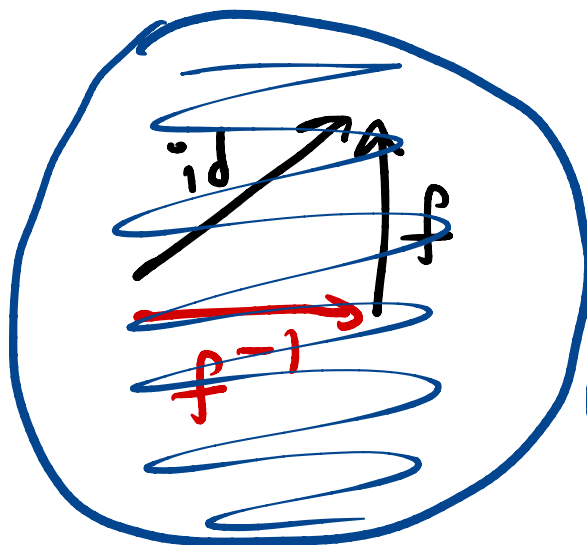
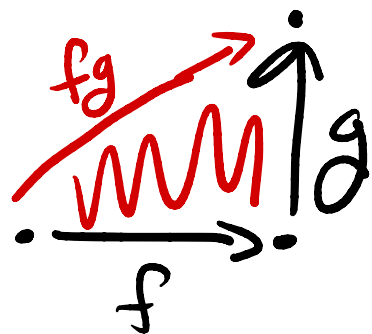
So 1-cells represent paths

$(\infty, 1)$ -category $\coloneqq$ quasicategory

1-cells represent morphisms

Inner horn can be filled

hard
to
make
into a
type
theory



$(\infty, 1)$ -category $\coloneqq$ bisimplicial set

$$\text{Sets}^{\overset{\text{paths}}{\Delta^{op}} \times \overset{\text{morphisms}}{\Delta^{op}}} = \left(\text{Sets}^{\Delta^{op}} \right)^{\Delta^{op}} = \text{Spaces}^{\Delta^{op}}$$

- path interval $I := \mathcal{Y}([1], [0])$
- morphism interval $R := \mathcal{Y}([0], [1])$
- Kan w.r.t. paths $\longrightarrow$ "contractable"
"equivalence"
- Segal condition for morphism composition
- Rezk condition relates paths to iso-morphisms

Simplicial type theory

$\Delta^{op} \times \Delta^{op}$
Sets

[Riehl, Shulman]

- "Book HoTT"
- morphism interval $\mathbb{R}$
- carve out the categorics (Segal/Rezk) by formulas

Interval theory for simplices [Riehl, Shulman]

(pre)type $\mathbb{Q}$

(Δ_1)

terms $0, 1 : \mathbb{Q}$

$(\Delta_0 \xrightarrow[1]{0} \Delta_1)$

tope / cofibration formula

$x : \mathbb{Q}, y : \mathbb{Q} \vdash x \leq y$ cofib

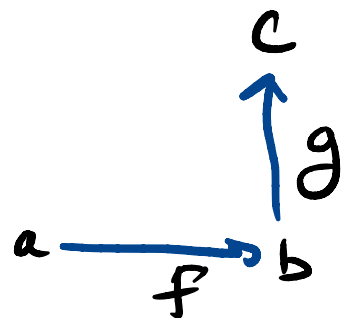
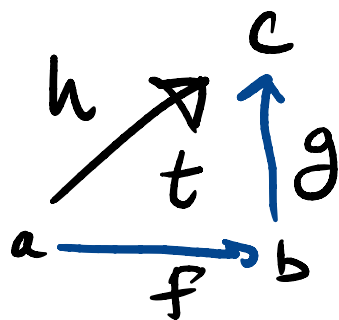
$(\begin{array}{c} \Delta_2 \\ \downarrow \\ \mathbb{Q} \times \mathbb{Q} \end{array})$

$\leq$ is a total order bounded by $0, 1$

Segal Idea: A :Type is segal

iff an n -simplex in A is
determined by its spine

(up to paths)



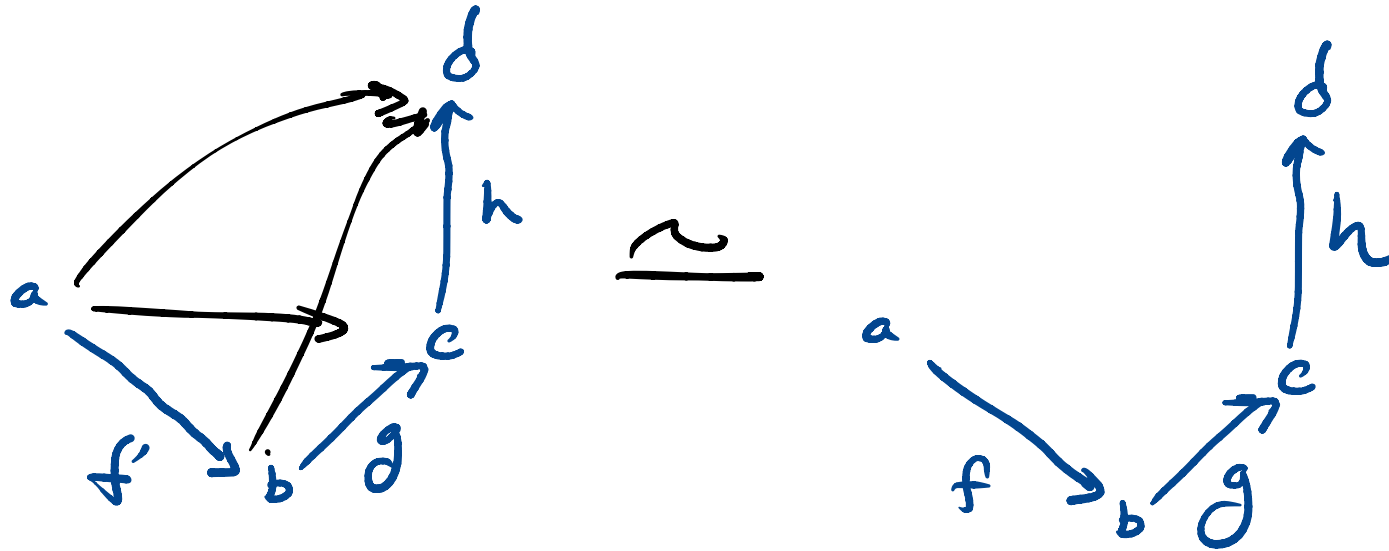
$(\Delta^2 \rightarrow A)$

$\simeq$

$\sum a, b, c: A.$
 $\text{Hom}(a, b)$
 $\times \text{Hom}(b, c)$

Definition
in [RS]

next level:



Interval Theory For

$(\infty, 2)$ - categories

2-fold Segal space

[Barwick,
Lurie]

paths 1-cells 2-cells
 $\Delta^{0p} \times \Delta^{0p} \times \Delta^{0p}$
 Sets

II

$\text{Path}_A(a, b)$

$\mathcal{Q}^h$

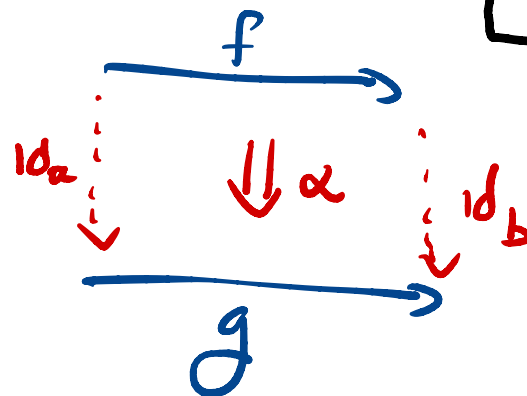
$1\text{Cell}_A(a, b)$

$\pi_x: \mathcal{Q}^h \cdot A \left[\begin{array}{l} x=0 \rightarrow a \\ x=1 \rightarrow b \end{array} \right]$

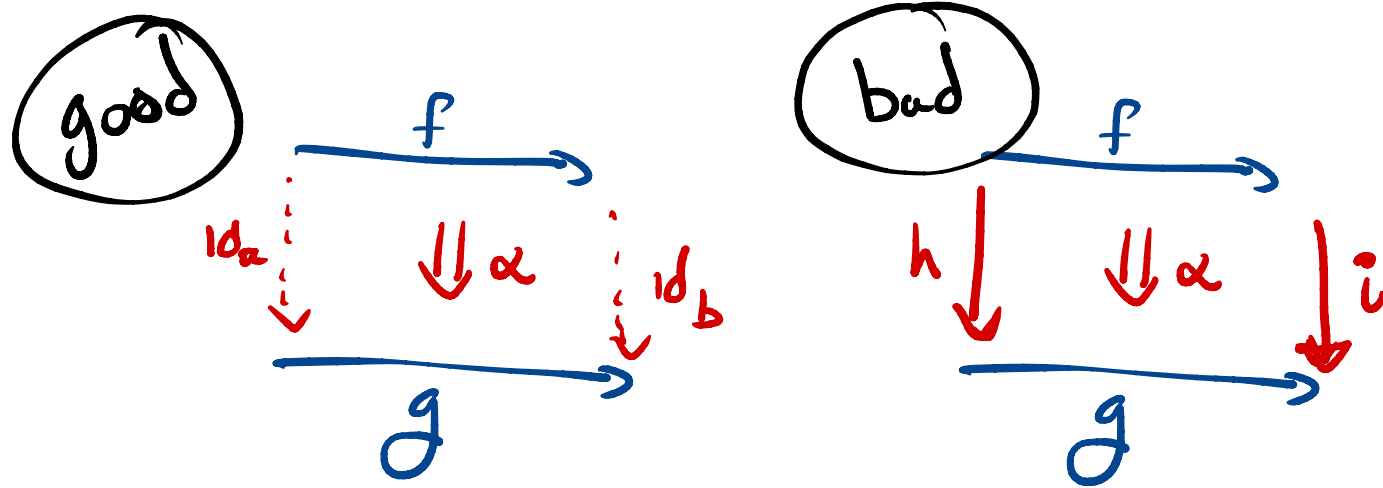
$\mathcal{Q}^v$

$2\text{Cell}_A(a, b, f, g)$

$\pi_x: \mathcal{Q}^h \cdot \pi_y: \mathcal{Q}^v \cdot A \left[\begin{array}{l} x=0 \rightarrow id_a \\ x=1 \rightarrow id_b \\ y=0 \rightarrow f \\ y=1 \rightarrow g \end{array} \right]$



Problem: $D^{\checkmark} \rightarrow A$ is extra data
that shouldn't be there



Solution 1:

Can impose a discreteness condition
to avoid them

but not preserved by $A \rightarrow B$

Solution 2: Θ_2 -spaces [Joyal, Rezk] $\mathbf{Sets}^{\Delta^{op} \times \Theta_2^{op}}$
1 cells
2 cells

change the shapes of the index
category to avoid unwanted data

Θ_2 full subcategory of strict 2-Cat
on free 2-cats generated by
pasting diagrams



$m [n_1, n_2, \dots, n_m]$

0[]



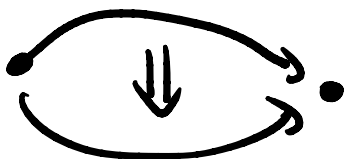
1[0]



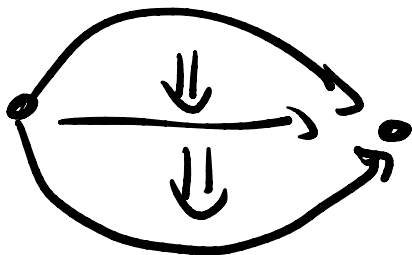
2[0,0]



1[1]



1[2]



2[1,1]



0[]



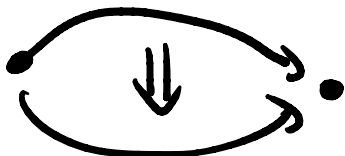
1[0]



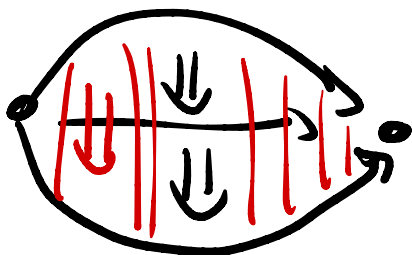
2[0,0]



1[1]



1[2]



2[1,1]



Interval Theory for Θ_2

horizontal interval as in simplicial type theory

$\mathcal{Q}^h$ pretype

$0, 1 : \mathcal{Q}^h$

$x, y : \mathcal{Q}^h \vdash x \leq y \text{ cofib} \Big] \text{ bounded total order}$

Vertical interval as a dependent type

$x: \mathbb{Q}^h \vdash \mathbb{Q}^v(x) \text{ pretype}$

$x: \mathbb{Q}^h \vdash 0_v : \mathbb{Q}^v(x)$

$x: \mathbb{Q}^h \vdash 1_v : \mathbb{Q}^v(x)$

$x: \mathbb{Q}^h, a: \mathbb{Q}^v(x), b: \mathbb{Q}^v(x) \vdash a \leq_v b \text{ cof. } b$
bounded total order

$a: \mathbb{Q}^v(0), b: \mathbb{Q}^v(0) \vdash a \equiv b \text{ true}$

$a: \mathbb{Q}^v(1), b: \mathbb{Q}^v(1) \vdash a \equiv b \text{ true}$

Idea:

$x: \mathcal{D}^n$

column of the pasting
diagram

$x \leq y$

order of the columns

$i: \mathcal{D}^v(x)$

2-cell in column x

$i \leq_v j$

order of 2-cells in column

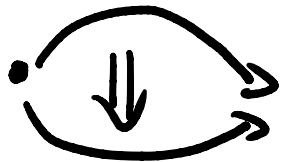


$0[]$



$1[0]$

$x: \mathcal{D}^h$



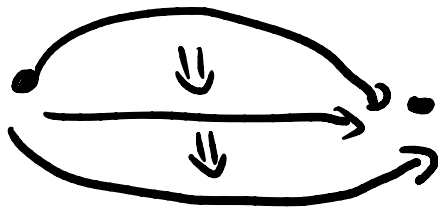
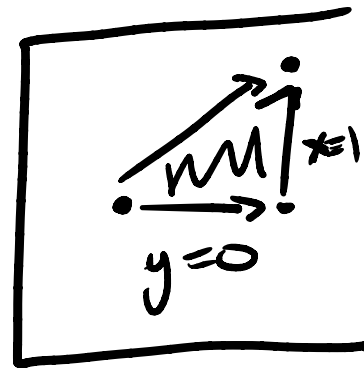
$1[1]$

$x: \mathcal{D}^h, i: \mathcal{D}^v(x)$



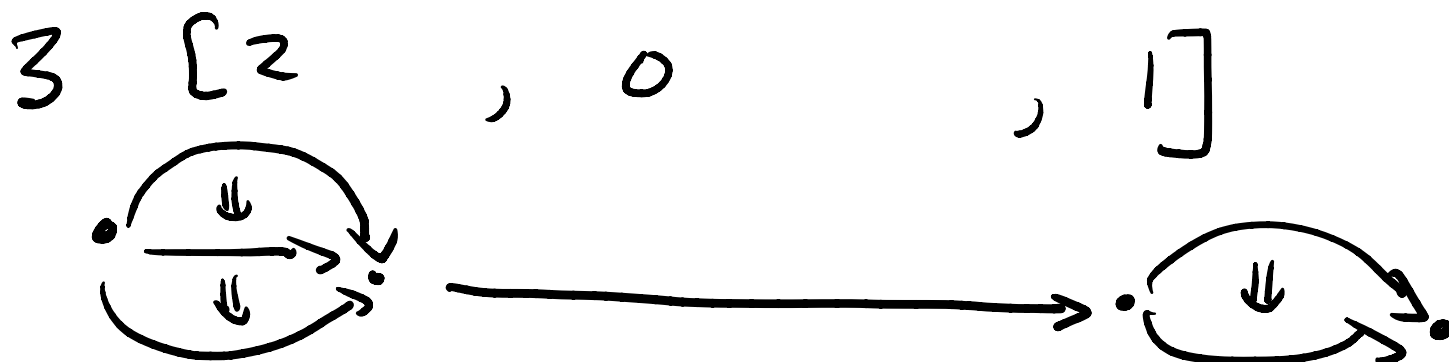
$2[0,0]$

$x: \mathcal{D}^h, y: \mathcal{D}^h, y \leq x$



$1[2]$

$x: \mathcal{D}^h, i: \mathcal{D}^v(x), j: \mathcal{D}^v(x), j \leq_v i$



$x: \mathcal{Q}^h,$
 $i: \mathcal{Q}^u(x)$
 $j: \mathcal{Q}^u(x)$
 $j \leq i$

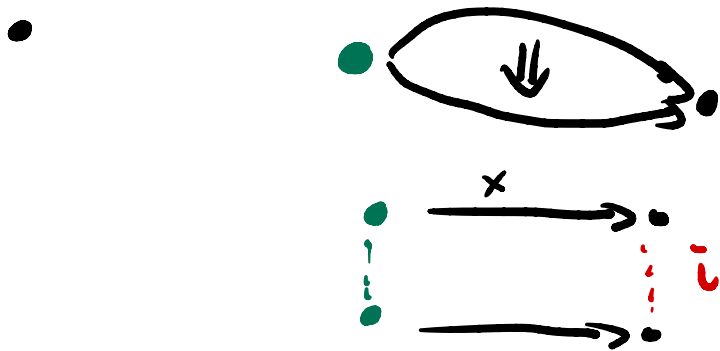
$y: \mathcal{Q}^h, y \leq x,$ $z: \mathcal{Q}^h, z \leq y,$
 $k: \mathcal{Q}^v(z)$

Faces

$$0 \equiv 1 : \mathcal{D}^v(0)$$

$$0 \equiv 1 : \mathcal{D}^v(1)$$

$$0 \longrightarrow 1[\]$$



$$\bullet \vdash (0, \underset{///}{0}) : x : \mathcal{D}^h, i : \mathcal{D}^v(x)$$

$$\bullet \vdash (0, \underset{///}{1}) : x : \mathcal{D}^h, i : \mathcal{D}^v(x)$$

$$0 \longrightarrow 1[\]$$

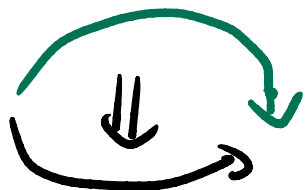


$$\bullet \vdash (1, \underset{///}{0}) : x : \mathcal{D}^h, i : \mathcal{D}^v(x)$$

$$\bullet \vdash (1, \underset{///}{1}) : x : \mathcal{D}^h, i : \mathcal{D}^v(x)$$

$l[0] \longrightarrow l[1]$

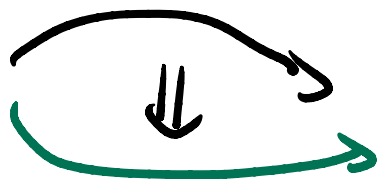
$\bullet \longrightarrow \bullet$



$$x: \mathcal{D}^h \vdash (x, \textcolor{red}{0}) : x: \mathcal{D}^h, \textcolor{red}{i}: \mathcal{D}^v(x)$$

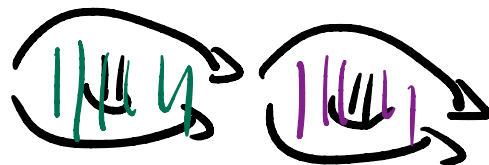
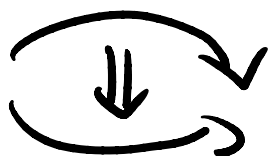
$l[0] \longrightarrow l[1]$

$\bullet \longrightarrow \bullet$



$$x: \mathcal{D}^h \vdash (x, \textcolor{green}{1}) : x: \mathcal{D}^h, \textcolor{red}{i}: \mathcal{D}^v(x)$$

$1[1] \longrightarrow 2[1,1]$



$z: \mathcal{D}^n, k: \mathcal{D}^v(z) \vdash (z, k, z, k)$

$(z, k, 0, !)$

$(1, !, z, k)$

$\left[\begin{array}{l} x: \mathcal{D}^n, \\ i: \mathcal{D}^v(x), \\ y: \mathcal{D}^n, \\ j: \mathcal{D}^v(y), \\ y \leq x \end{array} \right]$

n-ary Segal Conditions

[Rezk,
Bergner]

$(\infty, 1)$ -cats: A is segal iff

$$\frac{(\Delta^n \rightarrow A)}{n\text{-simplex}} \simeq \underbrace{(\Delta^1 \rightarrow A) \times (\Delta^1 \rightarrow A) \times \dots}_{n \text{ composable } 1\text{-cells}}$$

$(\infty, 2)$ -cats: A is segal iff

$$(m[n_1 \dots n_m] \rightarrow A)$$

(includes all composites)

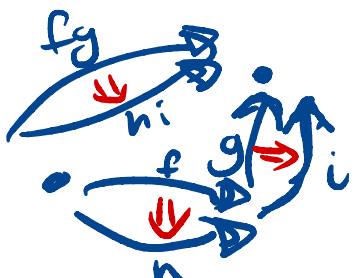
$\simeq$ cells of the pasting diagram
in A

Can be decomposed into horizontal and vertical

Horizontal: $(n[n_1 \dots n_n] \rightarrow A)$ break up columns

$$\simeq (1[n_1] \rightarrow A) \times (1[n_2] \rightarrow A) \times \dots$$

e.g. $\left(\begin{array}{c} \text{graph with two nodes and two edges} \end{array} \rightarrow A \right)$



$$\simeq \left(\begin{array}{c} \text{graph with one node and one edge} \end{array} \rightarrow A \right) \times \left(\begin{array}{c} \text{graph with one node and one edge} \end{array} \rightarrow A \right)$$

α β

$\alpha(1!) \equiv \beta(0!)$

Vertical:

break up one column

$$I[n] \rightarrow A \\ \cong (I[1] \rightarrow A) \times (I[1] \rightarrow A) \times \dots$$

e.g. $\left(\begin{array}{c} \text{---} \downarrow \text{---} \\ \text{---} \downarrow \text{---} \end{array} \rightarrow A \right)$

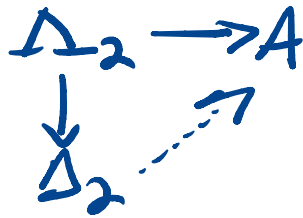
$$\cong \underbrace{\left(\begin{array}{c} \text{---} \downarrow \text{---} \\ \text{---} \downarrow \text{---} \end{array} \rightarrow A \right)}_{\alpha} \times \underbrace{\left(\begin{array}{c} \text{---} \downarrow \text{---} \\ \text{---} \downarrow \text{---} \end{array} \rightarrow A \right)}_{\beta}$$

$$\underline{\alpha(x, 1) \equiv \beta(x, 0)}$$

How simple can the Segal condition be?

For $(\infty, 1)$ -categories:

A is binary Segal [RS]

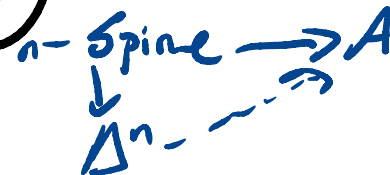


doesn't work for Θ_2 ?

A has post-whiskering



A is n-ary Segal

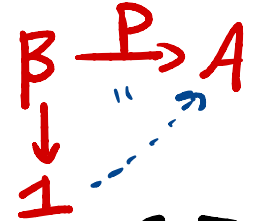


works for Θ_2

Working in internal language of 1-topos Sets $\mathbb{C}^{op} \times \Theta_2^{op}$ à la Orton-Pitts

Contractible: $Type \rightarrow Type$

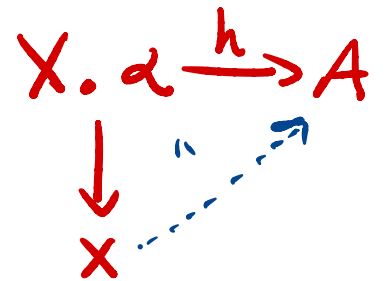
$Contractible(A) := \prod \beta : Cof. \prod p : (\beta \rightarrow A) \cdot A[\beta \rightarrow p]$



$Fill : \prod (x : Type) (\alpha : x \rightarrow Cof) (A : Type) \rightarrow Type$

$Fill\ X\ \alpha\ A :=$

$\prod (h : \prod x : X. \alpha(x) \rightarrow A).$



$Contractible(\prod x : X. \underline{A[\alpha x \rightarrow h x]})$

$\Delta : \text{Nat} \rightarrow \text{pre type}$

$$\Delta(n) = x_1 : \mathcal{D}, x_2 : \mathcal{D}, x_2 \leq x_1, \dots, x_n : \mathcal{D}, x_n \leq x_{n-1}$$

Mutual

$$\Delta 0 = \text{Unit}$$

$$\Delta(n+1) = \sum s : \Delta(n). \exists x_{n+1} : \mathcal{D}. x_{n+1} \leq \text{least}(s)$$

$$\text{least} : \prod (n : \text{Nat}). \Delta(n) \rightarrow \mathcal{D}$$

$$\text{least } 0 * = 1$$

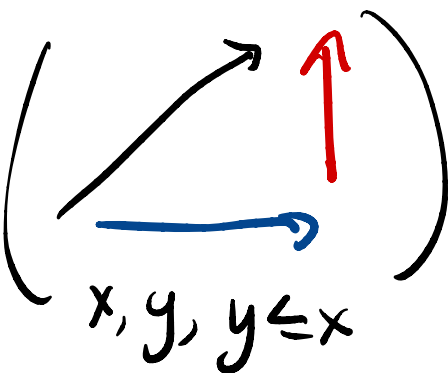
$$\text{least}(n+1) (-, x_{n+1}) = x_{n+1}$$

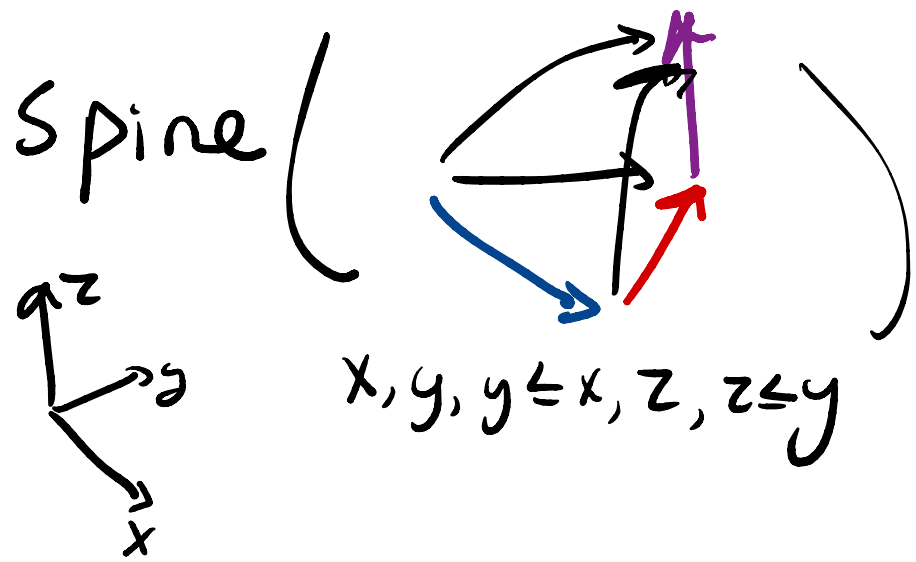
e.g.

$$\Delta(1) = \sum_{* : \underbrace{\Delta(0)}_{\equiv \text{Unit}}} \cdot \sum_{x_1 : \mathcal{D}} \cdot x_1 \leq \underbrace{\text{least}(*)}_{\equiv 1}$$
$$\equiv \mathcal{D}$$

$$\Delta(2) \equiv \sum_{x_1 : \mathcal{D}} \cdot \sum_{x_2 : \mathcal{D}} \cdot x_2 \leq \underbrace{\text{least}(*, x_1)}_{\equiv x_1}$$

$$Spine: \Pi(n: Nat). \Delta(n) \rightarrow CoF$$

$$Spine \left(\begin{array}{c} \nearrow \\ \xrightarrow{\text{blue}} \\ x, y, y \leq x \end{array} \right) = \underline{\underline{y=0 \vee x=1}}$$


$$Spine \left(\begin{array}{c} \nearrow \\ \xrightarrow{\text{blue}} \\ \nwarrow \\ \xrightarrow{\text{red}} \\ x, y, y \leq x, z, z \leq y \end{array} \right) = \begin{array}{l} \underline{\underline{(y=0 \wedge z=0)}} \\ \vee \underline{\underline{(x=1 \wedge z=0)}} \\ \vee \underline{\underline{(x=1 \wedge y=1)}} \end{array}$$


$$= \underline{\underline{(y=0 \vee x=1)}} \wedge (z=0) \vee (y=1)$$

$$\text{Spine } 0 * = T$$

$$\text{Spine } (n+1) (\delta, x_{n+1}) =$$

$$(\text{Spine } (n) \wedge (x_{n+1} = 0)) \vee (\text{least}(\delta) = 1)$$

E.g.

$$\text{Spine } 2 (*, x, y) = ((\text{Spine } 1 x) \wedge (y = 0)) \vee x = 1$$

$$\begin{aligned} \text{Spine } 1 (*, x) &= ((\text{Spine } 0 *) \wedge (x = 0)) \vee 1 = 1 \\ &= T \end{aligned}$$

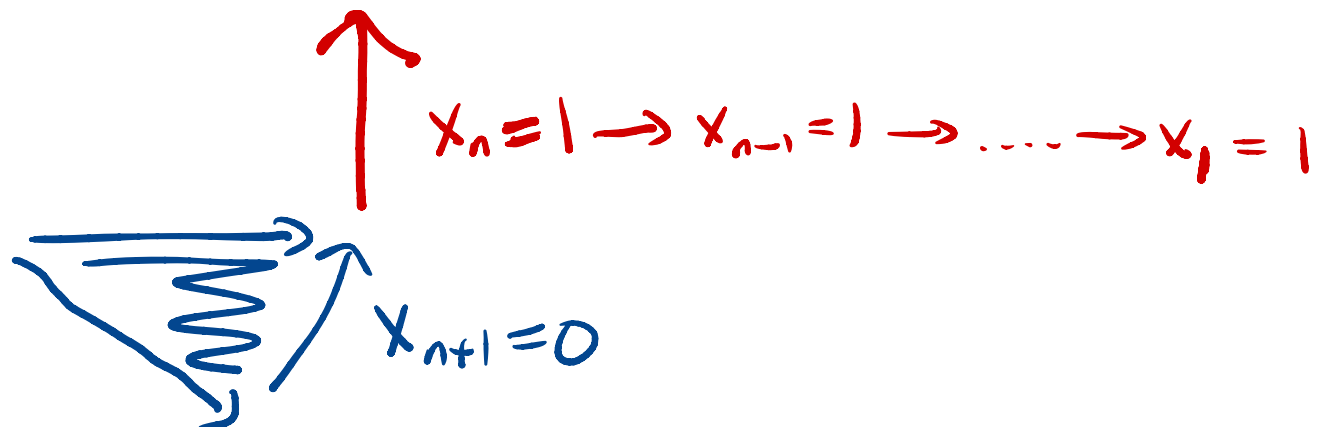
$$\text{Spine } 3 (x, x, y, z) = (\text{Spine } 2(x, y) \wedge z = 0) \vee y = 1$$

A is binary Segal $\coloneqq$ Fill $\Delta_2 \rhd_2 A$ 

A is n -ary Segal $\coloneqq$ $\prod_{n:\text{Nat.}}$
 Fill $\Delta(n) \text{ spine}(n) A$

A has post-whiskering $\coloneqq$

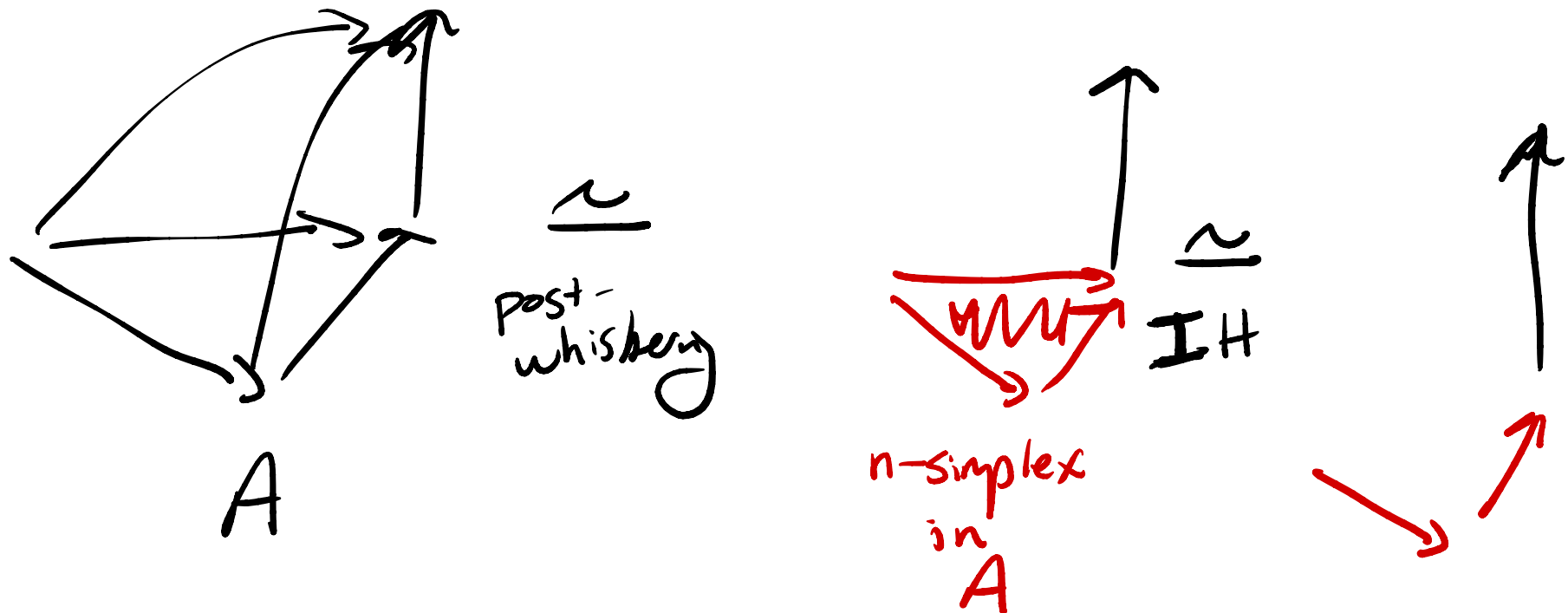
$\prod_{n:\text{Nat.}}$ Fill $\Delta(n+1) (\delta, x_{n+1} \cdot \underbrace{(x_{n+1}=0)}_{x_n} \vee \underbrace{(\text{least}(\delta)=1)}_{x_n}) A$



If A has post-whiskering then
 A is n -ary Segal

Induction on n . 0-spine is trivial $\begin{array}{ccc} 1 \cdot T & \rightarrow & A \\ \downarrow & \nearrow & \\ 1 & & \end{array}$

For $n+1$ (intuition):



Now, for θ_2 :

$m[n_1, \dots, n_m] : \prod m : \text{Nat}. \text{Vec Nat } m \rightarrow \text{pre type}$

$$m[n_1, \dots, n_m] = x_1 : \mathbb{Q}^h, \vec{i}_1 : \Delta_{n_1}^V(x_1), x_2 : \mathbb{Q}^h, \vec{i}_1 : \Delta_{n_2}^V, x_2 \leq x_1, \dots$$

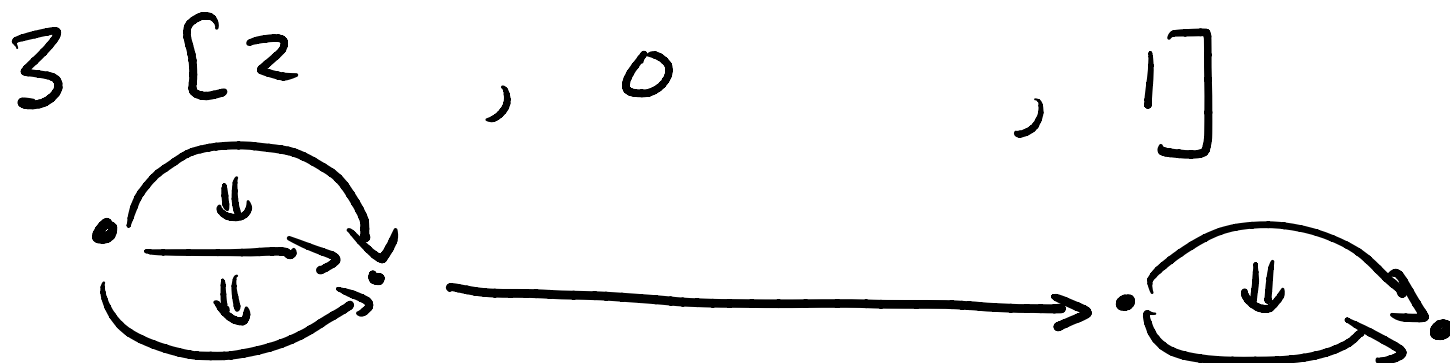
$$m+1[\vec{\pi}, n_{m+1}] = \sum (\delta : m[\vec{n}], x_{m+1} : \mathbb{Q}^h, \Delta_{n_{m+1}}^V(x_{m+1}) \times (x_{m+1} \leq x_n)$$

$\Delta^V : \mathbb{Q}^h \rightarrow \text{Nat} \rightarrow \text{pre type}$

$$\Delta_n^V(x) = i_1 : \mathbb{Q}^V(x), i_2 : \mathbb{Q}^V(x), i_2 \leq i_1, \dots, i_n : \mathbb{Q}^V(x), i_n \leq i_{n-1}$$

$$\Delta_{n+1}^V(x) = \sum \delta : \Delta_n^V(x), \sum i_{n+1} : \mathbb{Q}^V(x), i_{n+1} \leq \text{least}(\delta)$$

$$\Delta_0^V = \text{Unit}$$



$x: \mathcal{Q}^h,$
 $i: \mathcal{Q}^u(x),$
 $j: \mathcal{Q}^u(x),$
 $j \leq i,$

$y: \mathcal{Q}^h, y \leq x,$ $z: \mathcal{Q}^h, z \leq y,$
 $k: \mathcal{Q}^v(z)$

Define the horizontal spine as before:

$$\text{HSpine} : \prod m : \text{Nat}, \vec{n} : \text{Vec nat } m \rightarrow m[\vec{n}] \rightarrow \text{Cof}$$
$$\text{Hspine } 0 \times \times = \top$$
$$\text{HSpine } (m+1) (\vec{n}, n_{m+1}) (\delta, x_{m+1}, \vec{i}) = \\ (\text{HSpine } m \vec{n} \delta \wedge (x_{m+1} = 0)) \vee x_{m+1} = 1$$

Morally: $m[n_1, \dots, n_m]$'s spine is

$$1[n_1] + 1[n_2] + 1[n_3] + \dots + 1[n_m]$$

Lemma



A has horizontal post-whiskering $\Leftrightarrow$

$$\prod m: \text{Nat}. \prod \vec{n}: \text{Vec Nat } (m+1)$$

$$\text{Fill } (m+1[\vec{n}]) \left(\delta, x_{m+1}, \vec{i}_{m+1} \cdot \left(x_{m+1} = 0 \right) \bigvee \left(x_m = 1 \right) \right) A$$

implies

A has horizontal spine fillers $\Leftrightarrow$

$$\prod m: \text{Nat}. \prod \vec{n}: \text{Vec Nat } (m+1).$$

$$\text{Fill } (m[\vec{n}]) (\text{HSpine } m \vec{n}) A$$

Proof same as before.

So, take this to be definition of Segal

A has horizontal post-whiskering $\coloneqq$

$\prod m: \text{Nat}. \prod \vec{n}: \text{Vec Nat}(m+1)$

$\text{Fill}(m+1[\vec{n}]) \left(\delta, x_{m+1}, \vec{l}_{m+1} \cdot \left(x_{m+1} = 0 \right) \vee \left(x_m = 1 \right) \right) A$



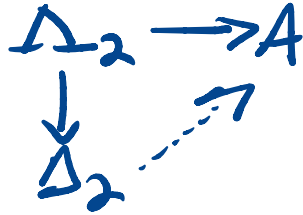
Pro: only composing 2 things

Con: base is m -dimensional, so type theory
needs shapes dependent on a nat/vec
(cofibrant exo-nat)

How simple can the Segal condition be?

For $(\infty, 1)$ -categories:

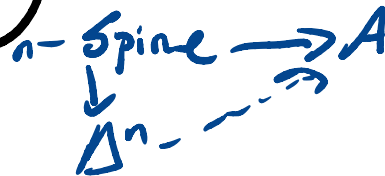
A is binary Segal [RS]



doesn't work for Θ_2 ?

A has post-whiskering

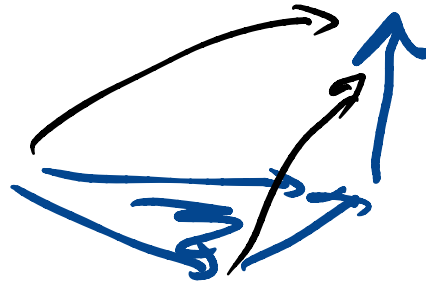
A is n -ary Segal



works for Θ_2

For $(\infty, 1)$ -cats, binary Segal suffices

$F, \parallel \Delta_2 \Delta_2 A \rightarrow A$ has post-whiskering



Retraction:
e.g. $n=3$

Δ_2
 $\downarrow$
 $\Delta_2 \times \Delta_2$



$z=0 \vee y=1$
 $\downarrow$

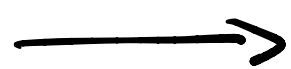
Δ_3

$(x, y), (y, z)$



$x, y, y \leq x, z, z \leq y$

$(a, b), (c, d)$
 $b \leq a \quad d \leq c$



$(a \vee c, b \vee c, d)$
 $a \vee c$ $b \vee c$

Doesn't seem to be "connection"

$$I[m] \times I[n] \longrightarrow I[?(m,n)]$$

?

Vertical Segal condition

Define $VSpine: \Pi n: \text{Nat. } 1[n] \rightarrow \text{Cof}$

$$Vspine_{n+1}(x: \mathbb{Q}^h, \delta, i: \mathbb{Q}^v(x)) \\ = (VSpine_n(x, \delta) \wedge i = 0) \vee (\text{leastv}(\delta) = 1)$$

A has vertical post-whiskering $:=$

$\Pi_n. \text{Fill} (1[n+1]) (x, (\delta, i). i=0 \vee \text{leastv}(\delta)=1)$

Implies

A has vertical spine files $:=$

$\Pi_n. \text{Fill} (1[n]) (\text{vs}_{\text{spine } n}) A$

So define A is 2-segal. $\coloneqq$

A has horizontal

and vertical post-whiskering



Θ_2 -space with composites

Directions

- $\Theta_3 \quad x: \mathcal{Q}^1, \quad i: \mathcal{Q}^2(x), \quad \beta: \mathcal{Q}^3(x, i)$

- Rezk conditions

$$\text{Path}_{\text{1Cell}(a,b)}(f,g) \xrightarrow{\sim} \text{Iso2Cell}(f,g)$$

$$\text{Path}_A(a,b) \xrightarrow{\sim} \text{Equiv1Cell}(a,b)$$

- If A is an $(\infty, 2)$ -cat then

$$\text{Hom}_A(x,y) \text{ is an } (\infty, 1)\text{-cat} \left(\begin{array}{l} \text{Path}(f,g) \\ \xrightarrow{\sim} 2\text{Cell}(f,g) \end{array} \right)$$

[Not $\mathcal{Q}^n \rightarrow A$?