Computing with Univalence

Dan Licata

Dependent Type Theory

Logical system underlying *proof assistants:* tools for

- Computer-checked math
- Programming

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identity type (=)
identical terms are interchangeable in all contexts

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proofs = programs

Example

```
even-or-odd : Π n:nat. (Σk. Id n 2*k) ∨ (Σk. Id n (2*k+1))
even-or-odd 0         = Even even0
even-or-odd (1 + n) =
  case (even-or-odd n) of
    Even nIsEven → Odd (odd1+ nIsEven)
    Odd nIsOdd  → Even (even1+ nIsOdd)
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```

Computing by calculation:

```
even-or-odd 3 = case (even-or-odd 2) of …
              = <steps elided>
              = case (Even … ) of …
              = Odd …
```

Canonicity Property

“programs don’t get stuck”

If M : nat
then
∃ a numeral k s.t. M $\longmapsto^*$ k

algorithm for =

guides the design of a type theory

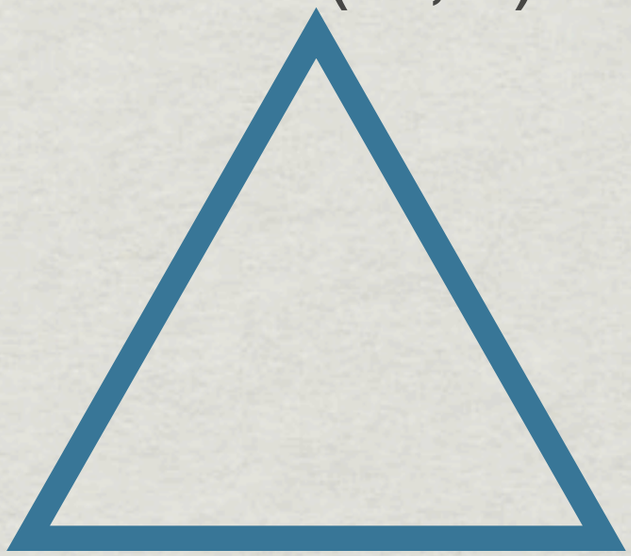
Homotopy Type Theory

dependent type theory

A : type

M : A

$\alpha : Id_A(M,N)$



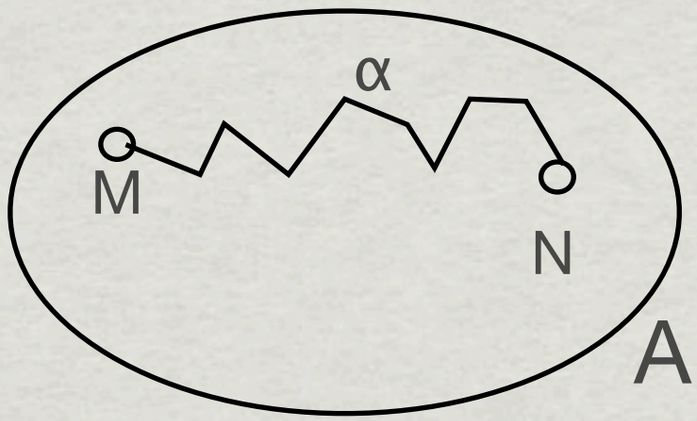
category theory

A is a groupoid

M is an object

$\alpha : M \to N$ in A

homotopy theory

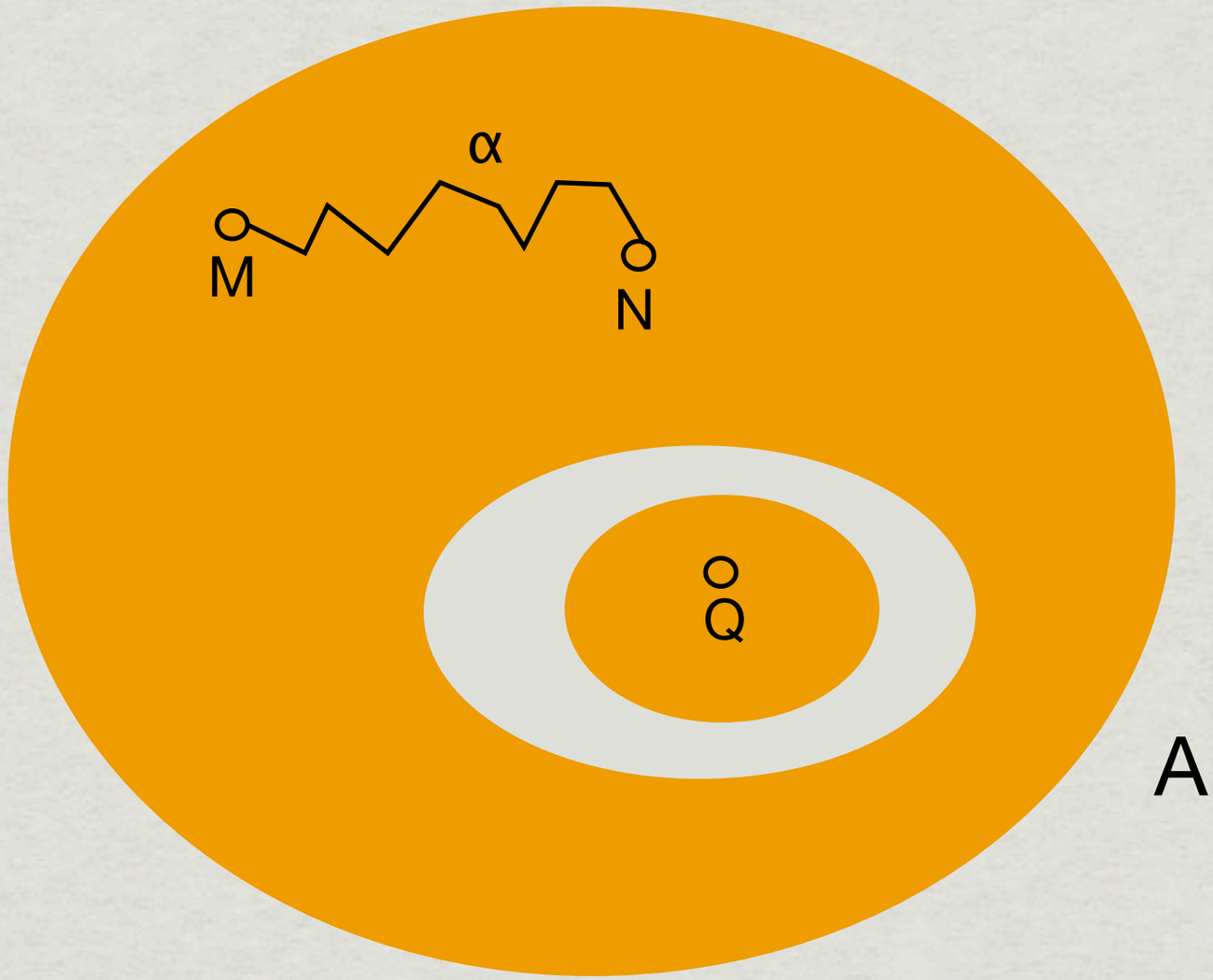


Homotopy Type Theory

Theorem: Martin-Löf's intensional type theory has semantics in homotopy theory (spaces as types) and category theory (groupoids as types)

[Hofmann&Streicher,Awodey, Warren,Lumsdaine,Garner, Voevodsky, 1990's and 2000's]

Types as Spaces



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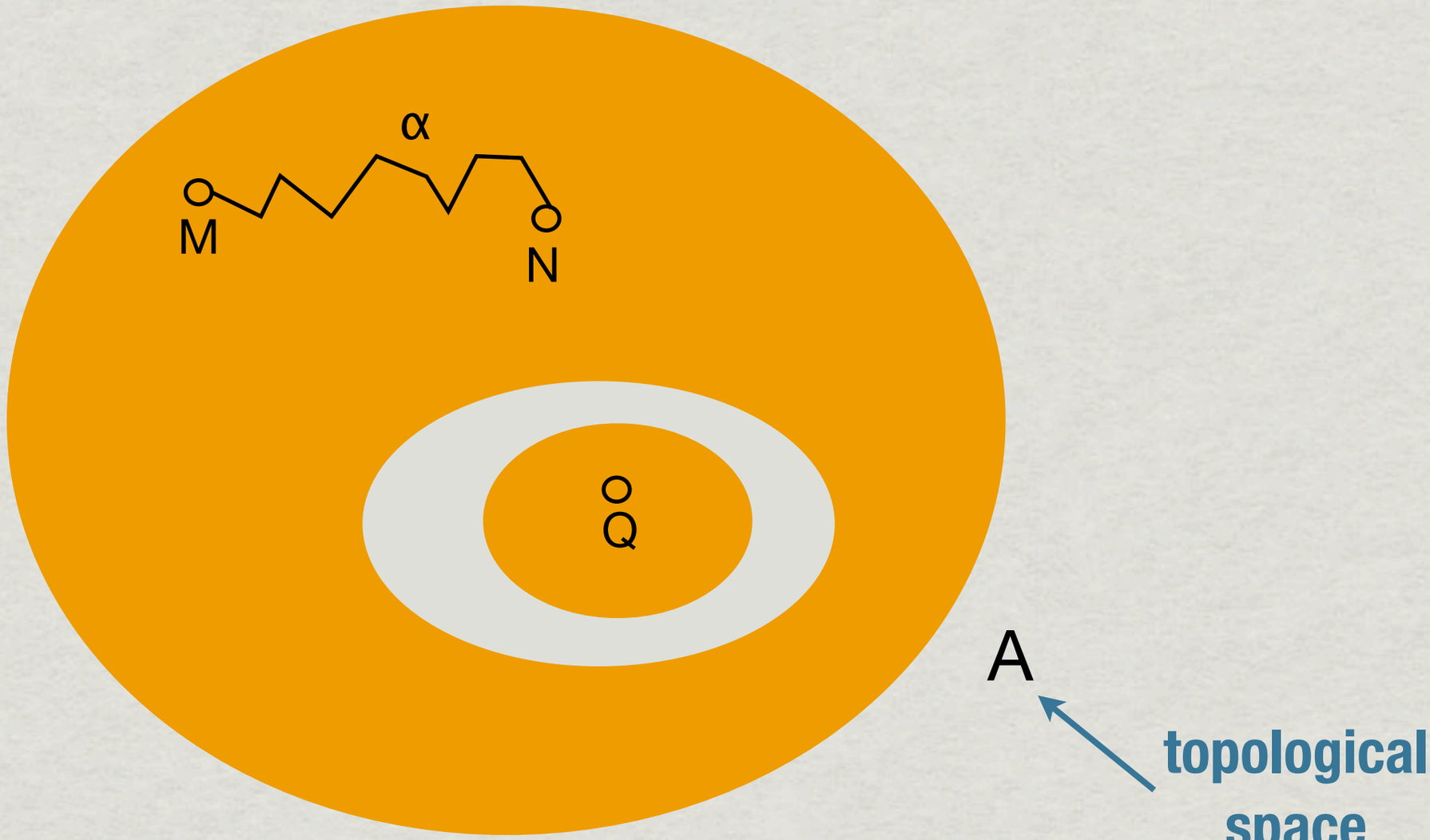

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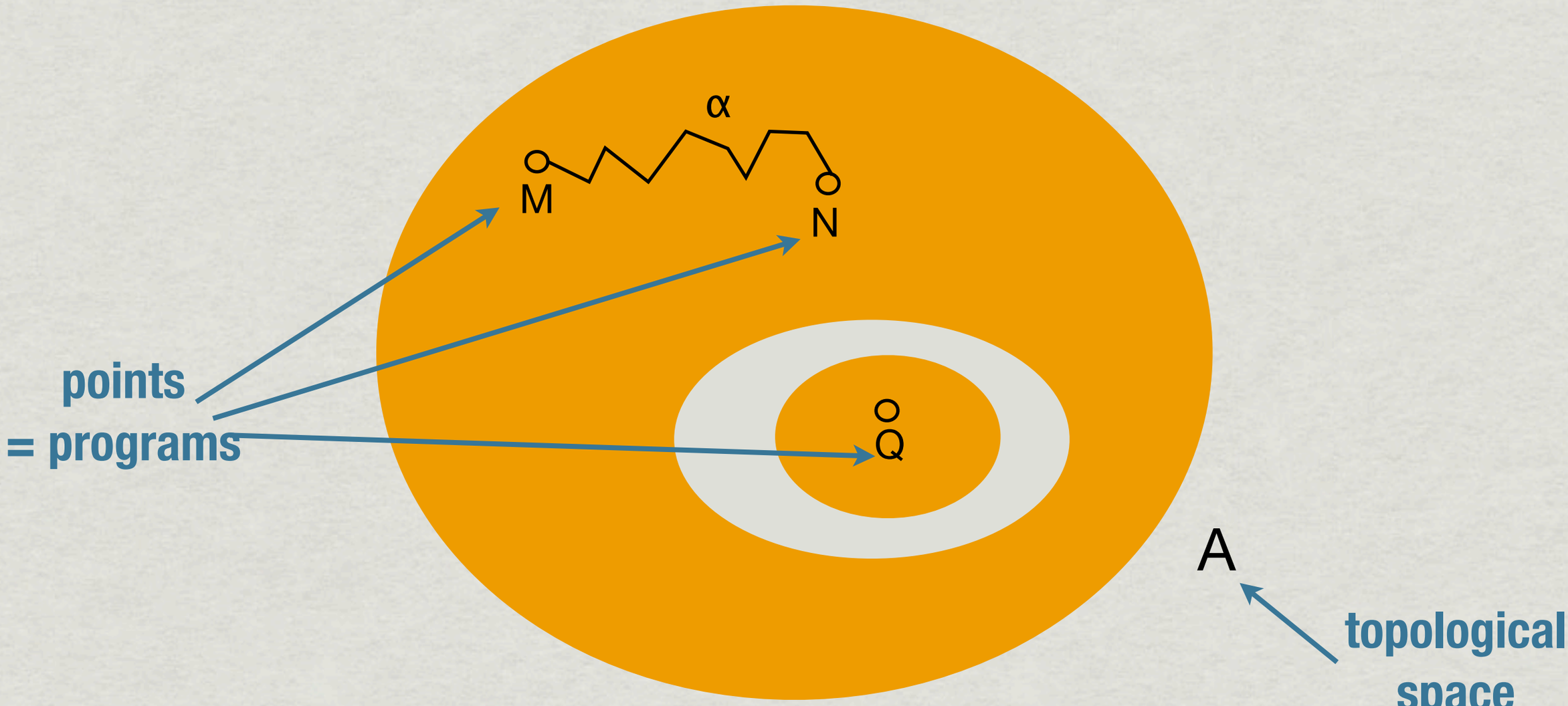

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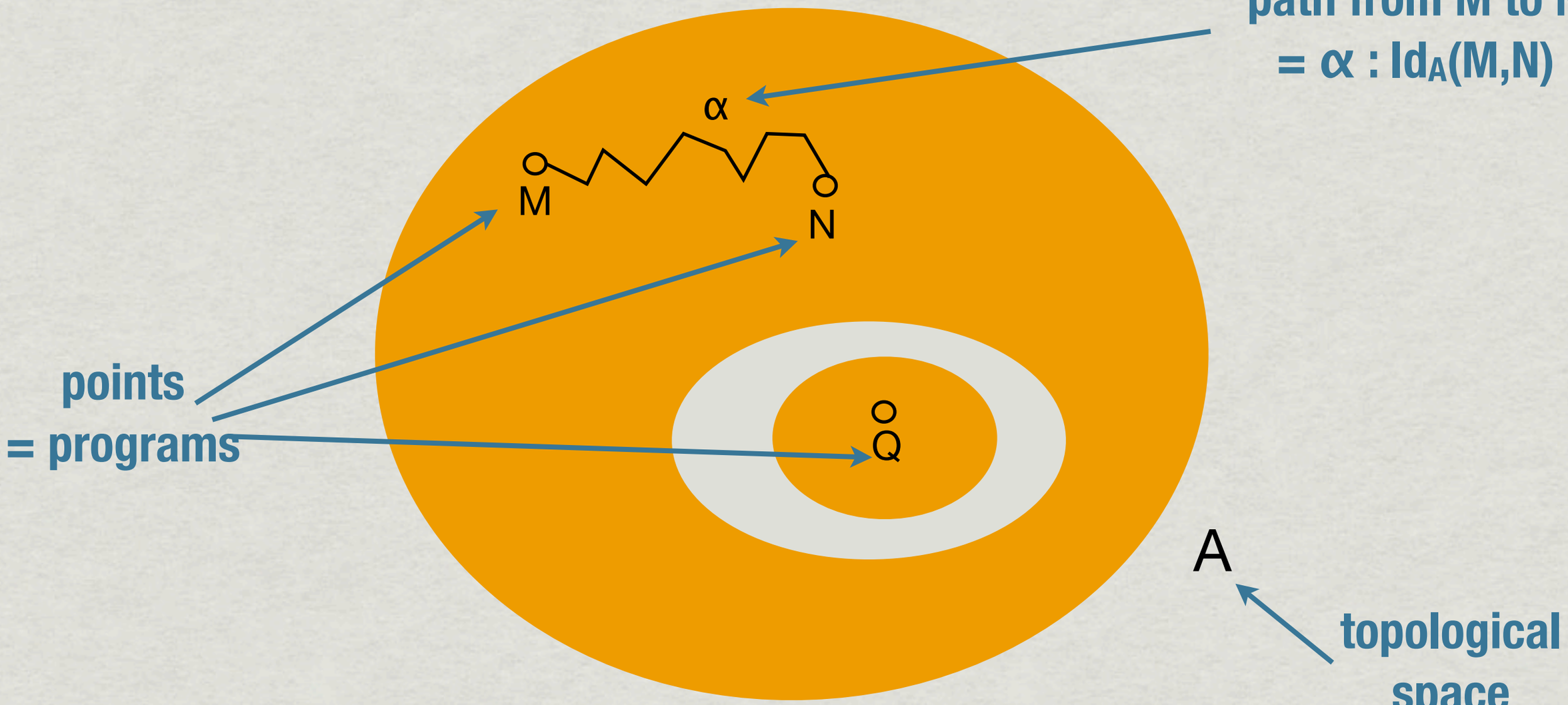
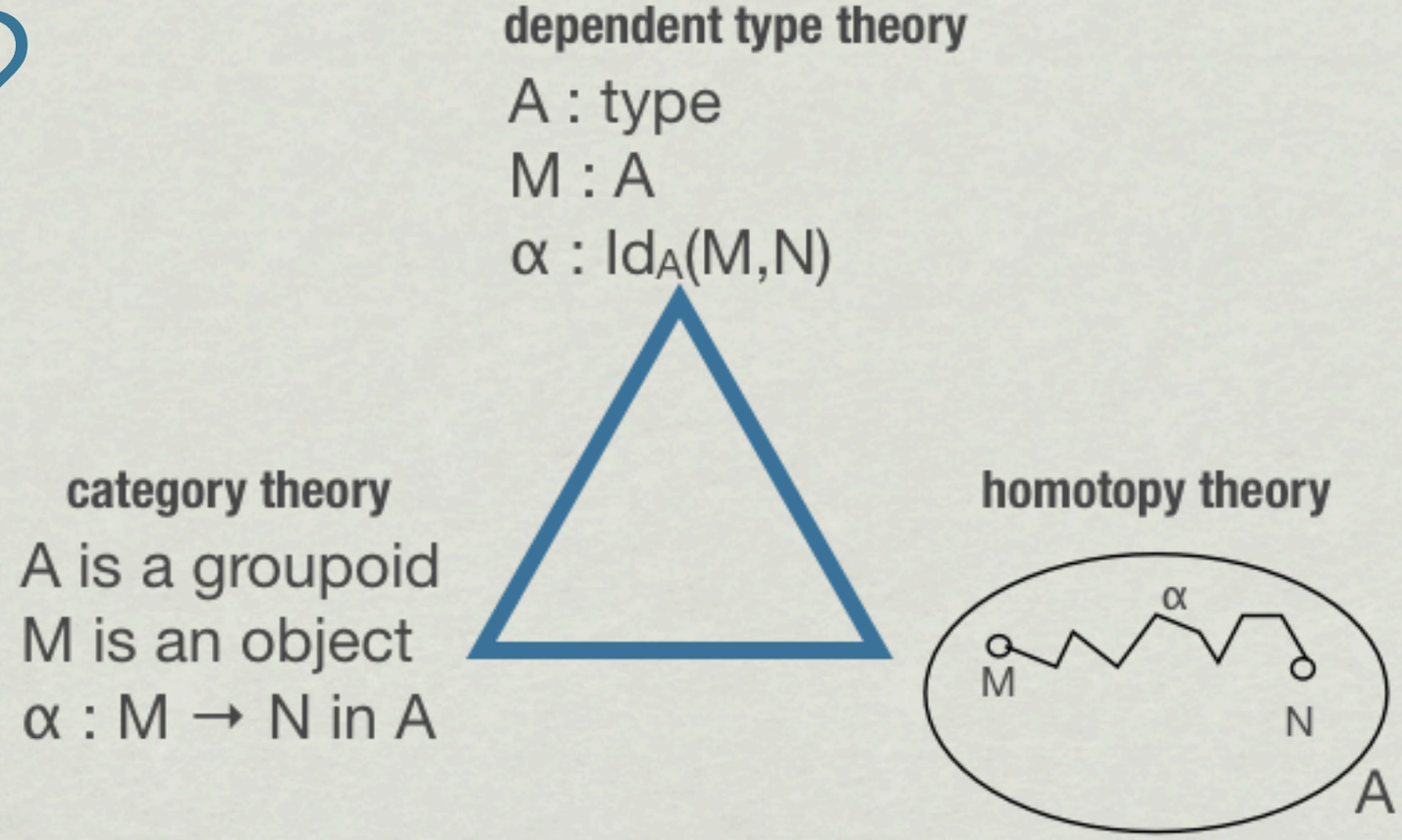


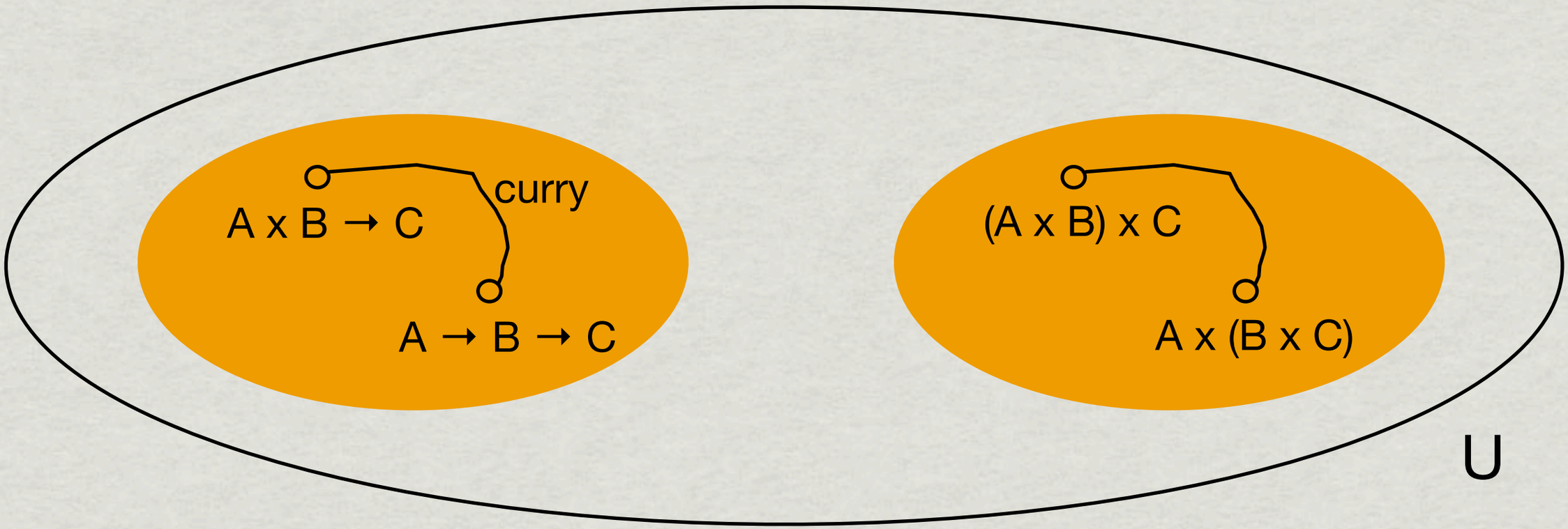
So what?



- Type theory can be used as a formal logical calculus for proving results in higher-category theory and homotopy theory
- Correspondence suggests new logical principles to add to type theory

Univalence

[Hofmann&Streicher, Voevodsky]



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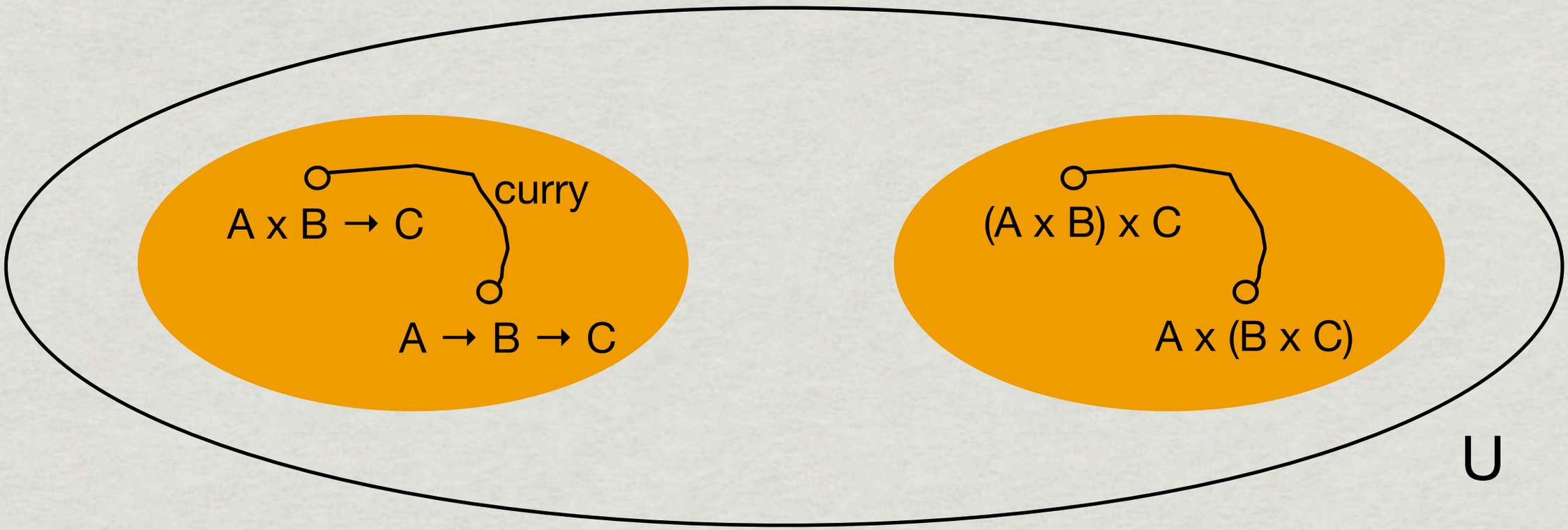
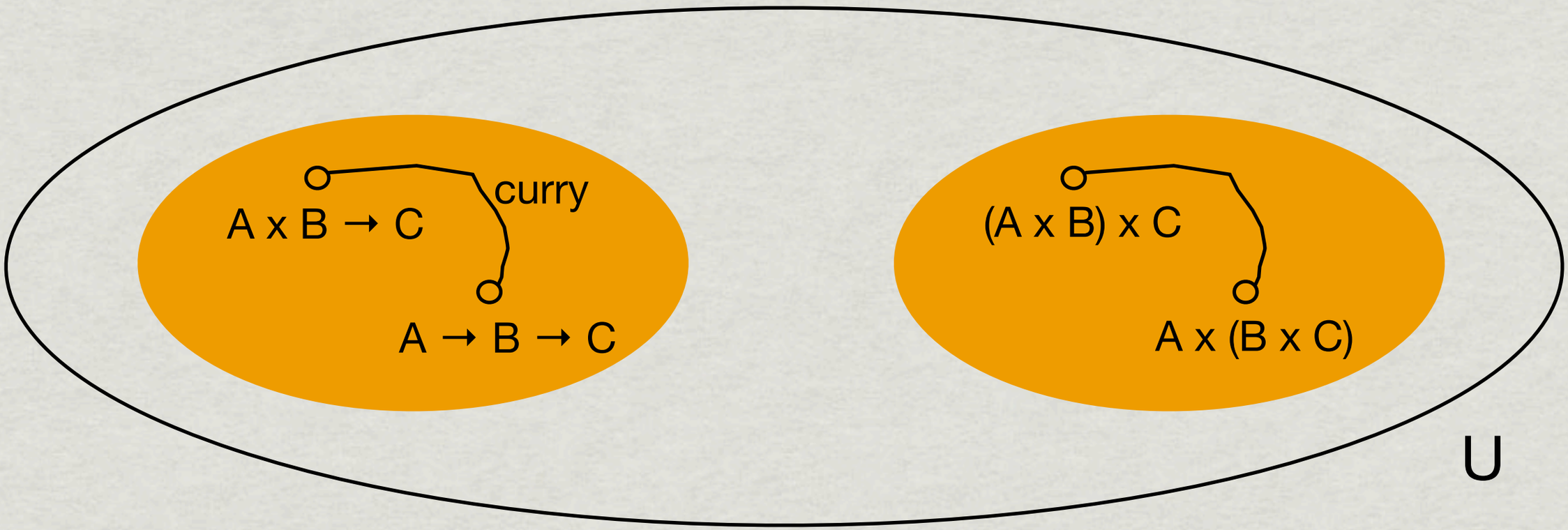


U = universe (type of types)
where *paths are isomorphisms*

Univalence

[Hofmann&Streicher, Voevodsky]



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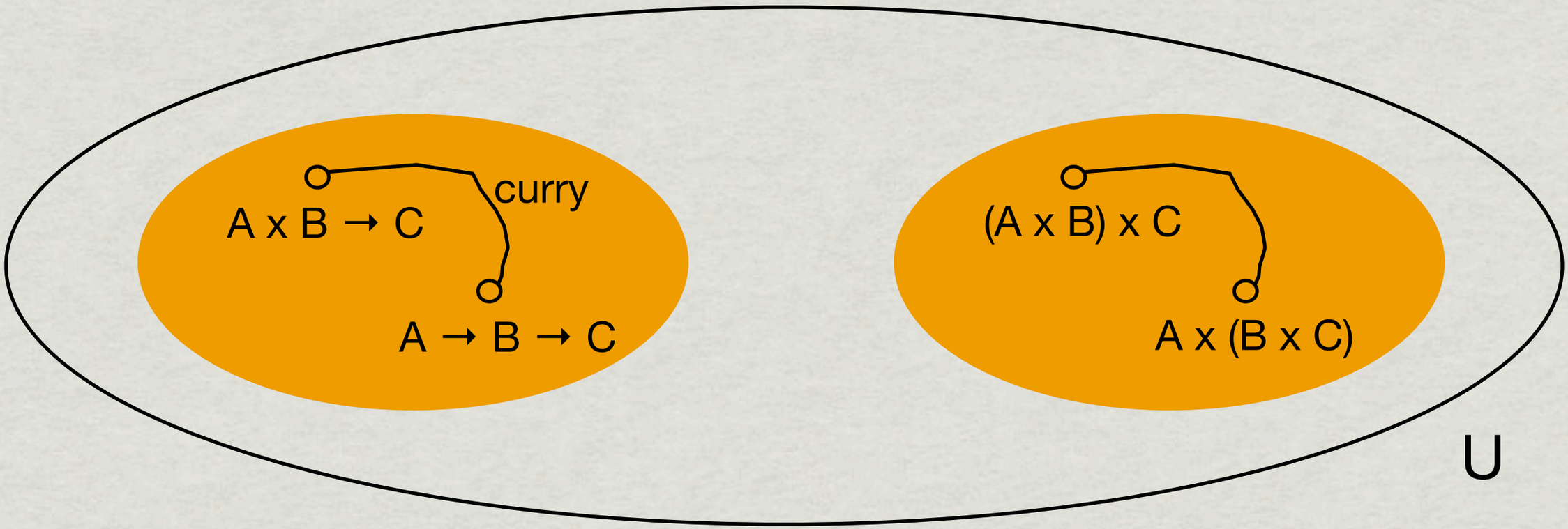
Isomorphic types are interchangeable in all contexts; or, all constructions respect isomorphism

Univalence

[Hofmann&Streicher, Voevodsky]



U = universe (type of types)
where *paths are isomorphisms*
(really weak equivalences)

Isomorphic types are interchangeable in all contexts; or, all constructions respect isomorphism

Work “up-to-iso”

Monoid = Σ X:U. MonoidStr[X]
MonoidStr[X:U] : U =
Σ ⊙ : X → X → X.
Σ u : X.
(Πx,y,z. Id (x ⊙ (y ⊙ z)) ((x ⊙ y) ⊙ z))) ×
(Πx.Id (u ⊙ x) x) ×
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If A ≃ B then MonoidStr[A] ≃ MonoidStr[B] ?

“the hard way”

MonoidStr : type → type
MonoidStr X = Σ ⊙:X→X→X. Σ u:X. …

Given (f,f^{-1},α,β) : Iso(A, B)
and (⊙,u,…) : MonoidStr[A]
make (⊙’,u’,…) : MonoidStr[B]

f : A → B
f^{-1} : B → A
α : Id (f o f^{-1}) id
β : Id (f^{-1} o f) id

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$y_1 \odot' y_2 = f ((f^{-1} y_1) \odot (f^{-1} y_2))$
$u' = f\ u$

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$y_1 \odot' y_2 = f ((f^{-1} y_1) \odot (f^{-1} y_2))$
$u' = f u$

$(y \odot' u') = f ((f^{-1} y) \odot (f^{-1} (f u)))$
$= f (f^{-1} y \odot u)$ by α
$= f (f^{-1} y)$ by unit law for ⊙ and u
$= y$ by β

“The easy way”

1. Each individual construction in type theory without univalence already respects isomorphism

2. Univalence gives generic name to this principle, and ensures that hypotheses/extensions still do.
   To make MonoidStr[A] → MonoidStr[B] just write

$$\mathrm{transport}_{\mathrm{MonoidStr}}(\mathrm{univalence}\ (f, f^{-1}, \alpha, \beta))$$

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iso determines Id

- Formalized math: don’t need to prove this by hand for each type
- Programming: automatically compute monoid structure on B from monoid structure on A using a *generic program*

Computational Interpretation

- We can **prove** that
  $\mathrm{transport}_{\mathrm{MonoidStr}}(\mathrm{univalence}(f, f^{-1}, \alpha, \beta))$
  is Id-entical to what we wrote out by hand

- But for type theory + univalence, as currently formulated, it doesn't **compute**: programs get stuck

- Formally: canonicity fails

Canonicity Property

“programs don’t get stuck”

algorithm for =

If M : nat

then

∃ a numeral k s.t. M $\longmapsto^{*}$ ***k***

guides the design of a type theory

Status

- **Theorem (Licata&Harper,2012):**
  Canonicity for special case of a 2-dimensional type theory with univalent universe of h-sets
  … up to equational deduction
- In progress: extends to an algorithm for 2TT
- One the goals for this special year:
  canonicity for full higher-dimensional univalence

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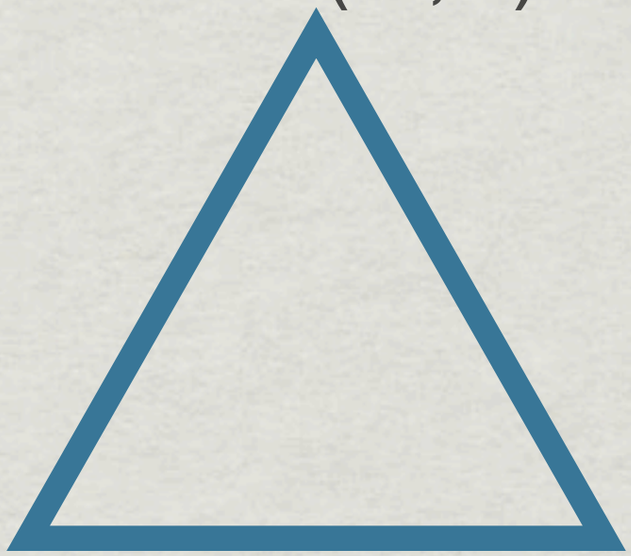

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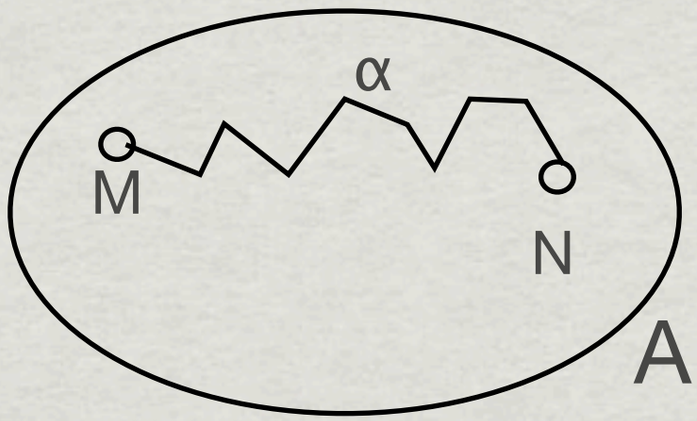

Dimension Hierarchy

A type is an

- h-proposition: all terms are identical
  (in the sense of Id-type)
- h-set: unique identities
  e.g. nat
- h-groupoid: unique identities between identities
  e.g. universe of h-sets
- h-2-groupoid: and so on
  e.g. universe of h-groupoids

Status

types are at most h-groupoids

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- In progress: extends to an algorithm for 2TT

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