A Fibrational Framework for Substructural and Modal Logics

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Substructural Logic

Weakening

$$\frac{\Gamma \vdash B}{\Gamma, x{:}A \vdash B}$$

Exchange

$$\frac{\Gamma, y{:}B, x{:}A \vdash B}{\Gamma, x{:}A, y{:}B \vdash C}$$

Contraction

$$\frac{\Gamma, x{:}A, y{:}A \vdash B}{\Gamma, x{:}A \vdash B}$$

Modal Logic

$$\frac{\emptyset \vdash A}{\Gamma \vdash \Box A} \qquad \frac{A \vdash \Diamond C}{\Diamond A \vdash \Diamond C}$$

Modal Logic

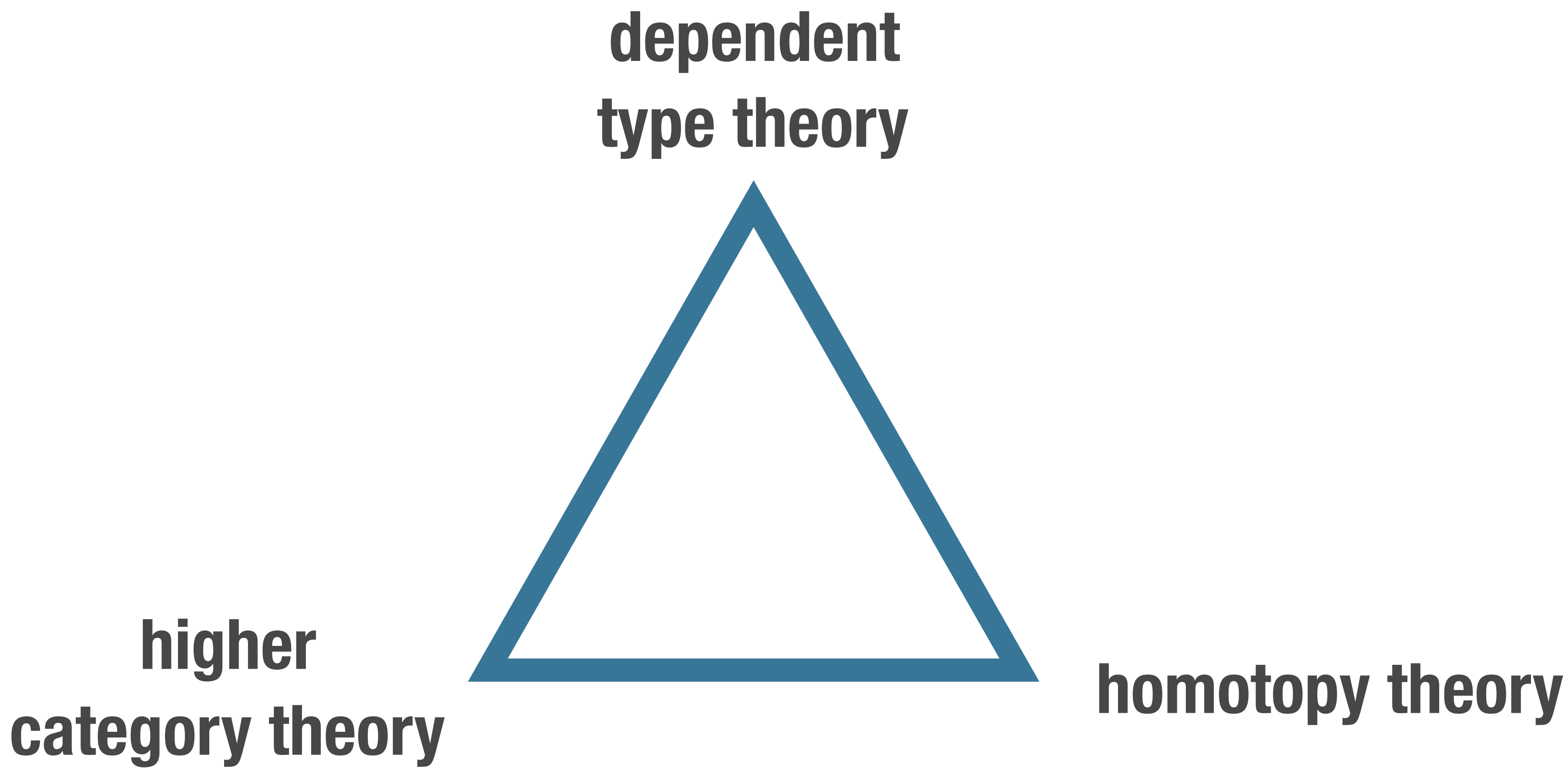
$$\frac{\varnothing \vdash A}{\Gamma \vdash \Box A} \qquad \frac{A \vdash \Diamond C}{\Diamond A \vdash \Diamond C}$$

$(\Diamond A) \times B$ vs. $\Diamond(A \times B)$

Intuitionistic substructural and modal logics/type systems

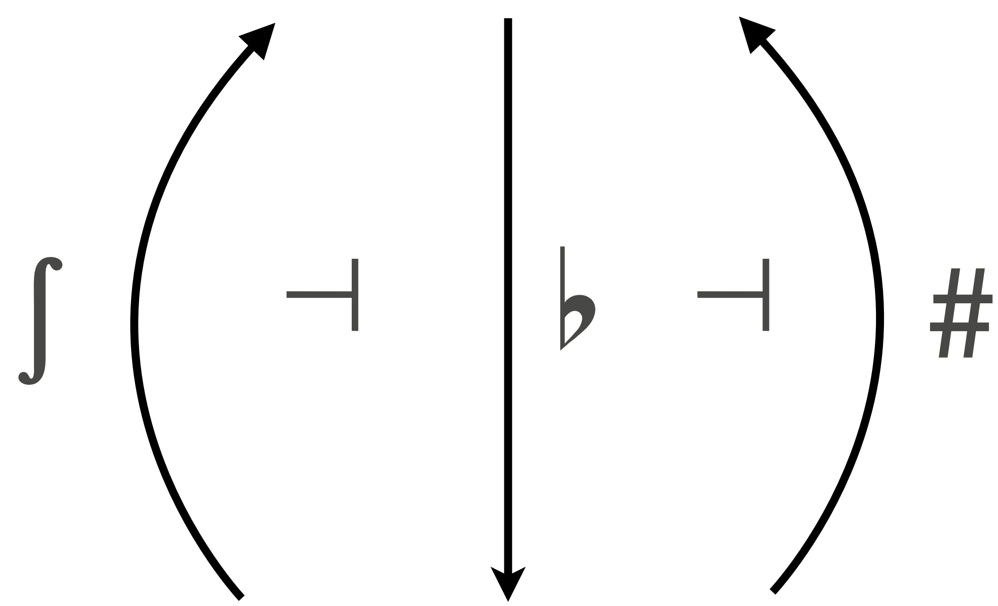
- **Linear/affine:** use once (state, sessions)
- **Relevant:** strictness annotations
- **Ordered:** linguistics
- **Bunched**: separation logic
- **Comonads:** staging, metavariables, coeffects
- **Monads:** effects
- Interactions between products and modalities

Homotopy type theory



Cohesive HoTT [Shulman,Schreiber]

Dependent type theory with modalities ∫A, ♭A, #A



Cohesive HoTT [Shulman,Schreiber]

Dependent type theory with modalities ∫A, ♭A, ♯A

$$\int \dashv \flat \dashv \sharp$$

$$\begin{array}{c} \flat A \vdash B \\ \hline\hline A \vdash \sharp B \end{array}$$

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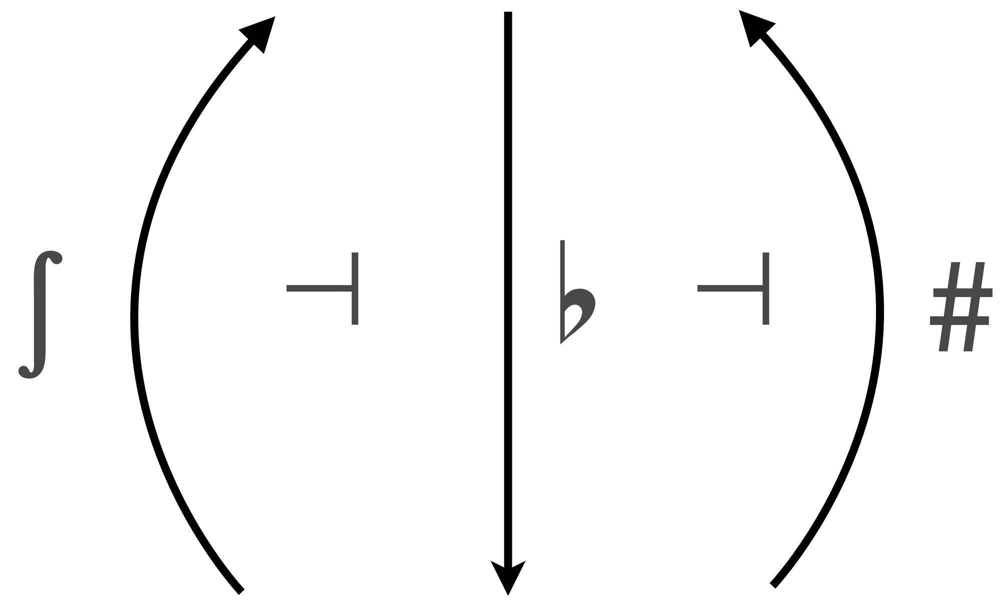
Dependent type theory with modalities ∫A, ♭A, #A



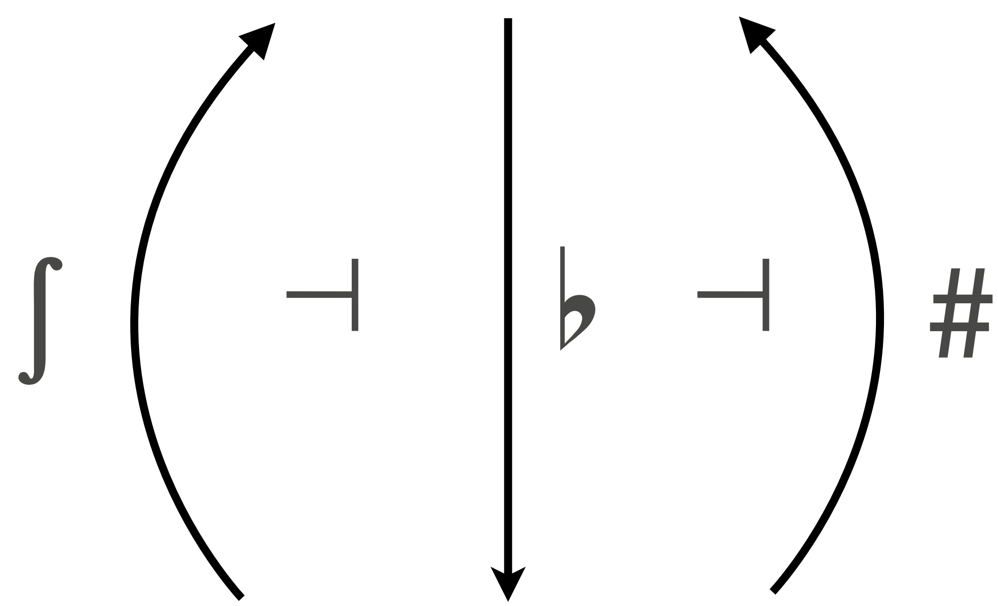
$$\frac{\flat A \vdash B}{A \vdash \# B}$$

monad

◇

Cohesive HoTT [Shulman,Schreiber]

Dependent type theory with modalities $\int A$, $\flat A$, $\sharp A$



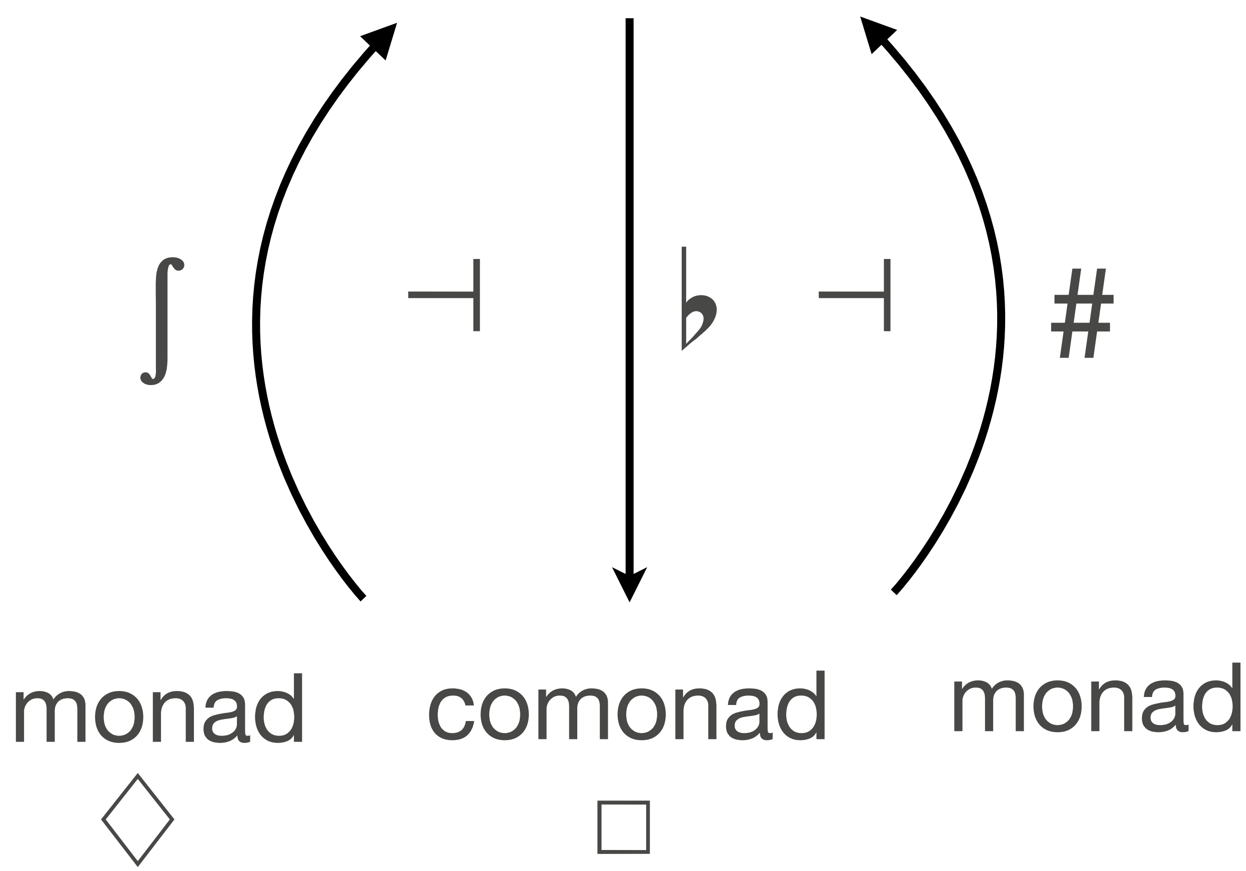
$$\frac{\flat A \vdash B}{A \vdash \sharp B}$$

monad $\Diamond$

comonad $\Box$

Cohesive HoTT [Shulman,Schreiber]

Dependent type theory with modalities ∫A, ♭A, #A



$$\frac{\flat A \vdash B}{A \vdash \# B}$$

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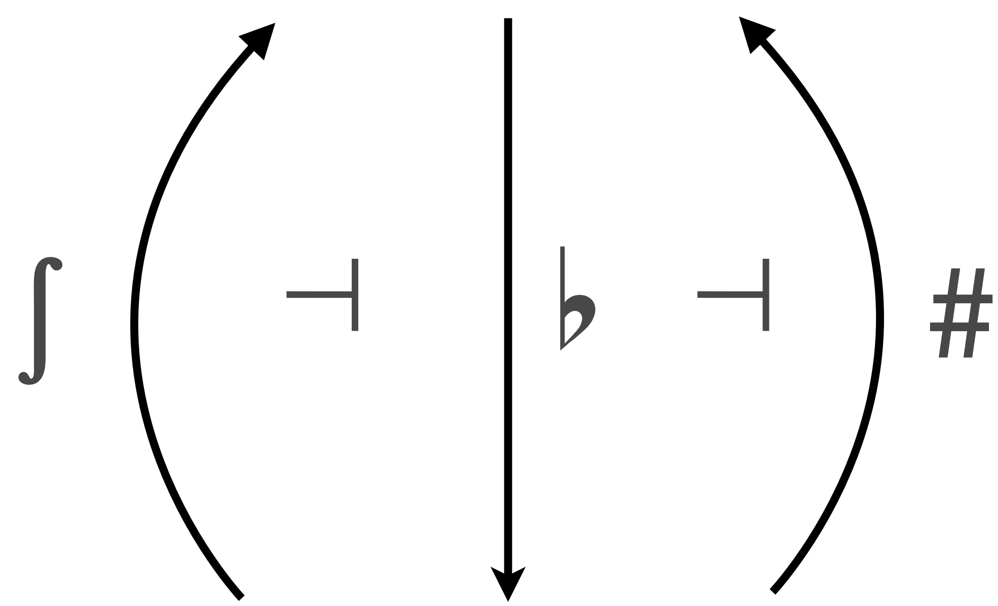
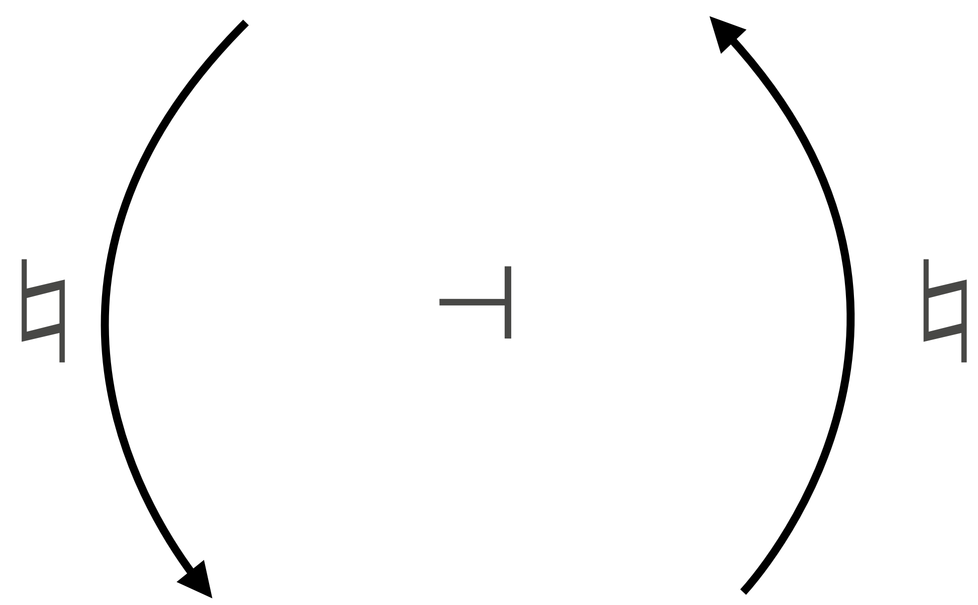


$$\frac{\flat A \vdash B}{A \vdash \# B}$$

(idempotent) monad comonad monad

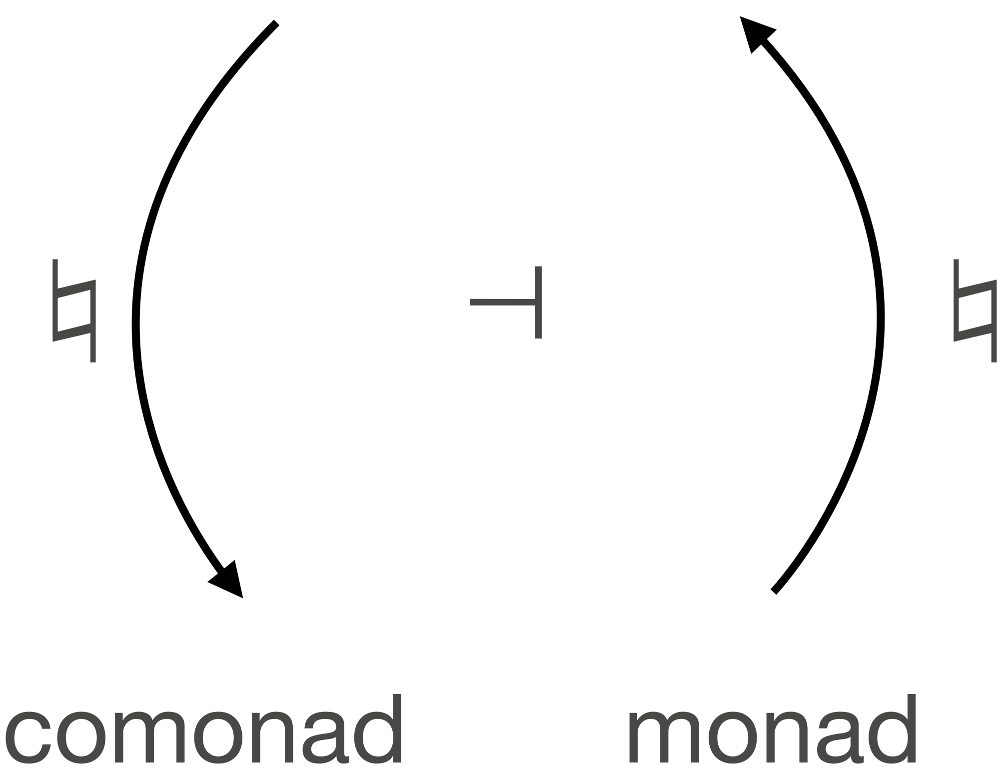
$\flat\flat A \cong \flat A$ ◇ □

$\mathbb{S}$-Cohesion [Finster,L.,Morehouse,Riley]



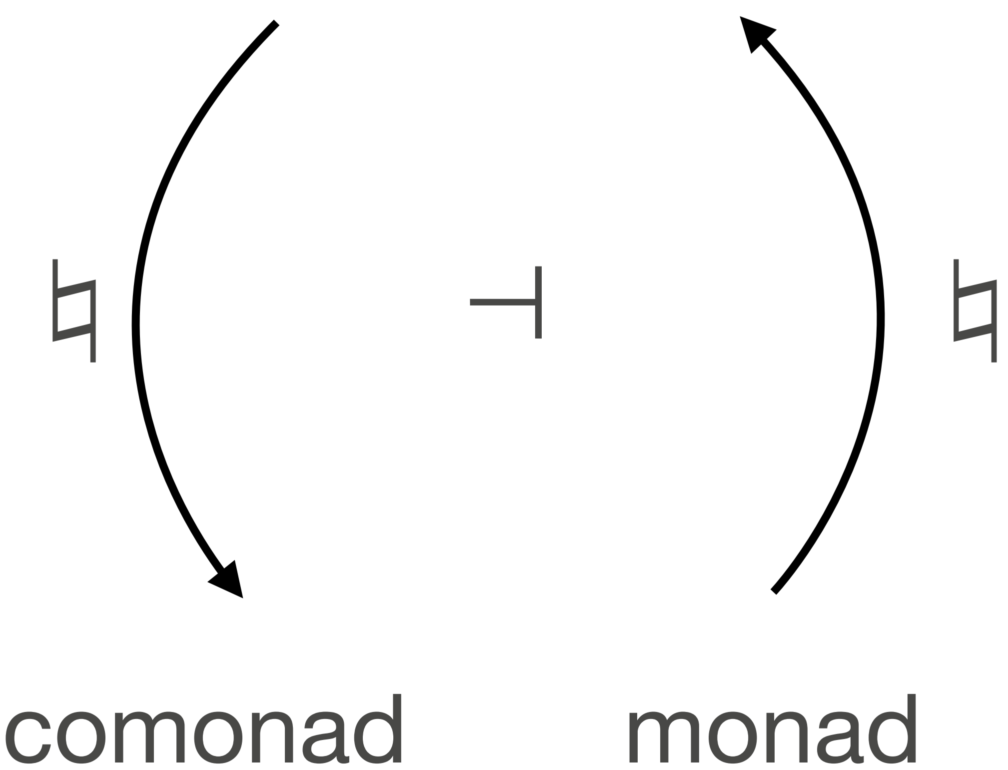
comonad monad

$\mathbb{S}$-Cohesion [Finster,L.,Morehouse,Riley]



$A \vdash \natural A \vdash A$

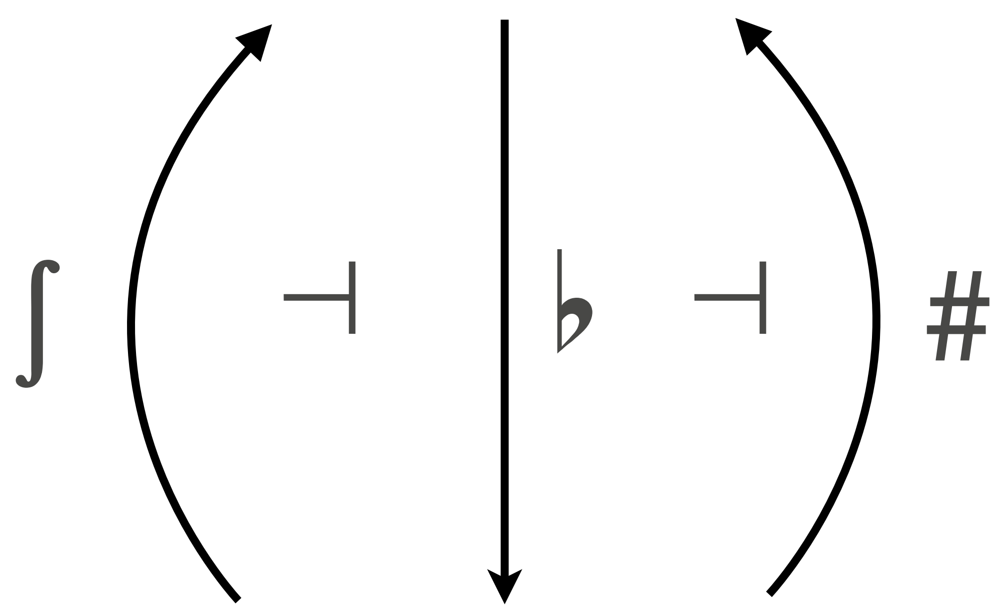
$\mathbb{S}$-Cohesion [Finster,L.,Morehouse,Riley]



$$A \vdash \natural A \vdash A \qquad \natural(A \wedge B) \cong \natural A \times \natural B$$

Differential Cohesion

[Schreiber; Gross,L.,New,Paykin,Riley,Shulman,Wellen]



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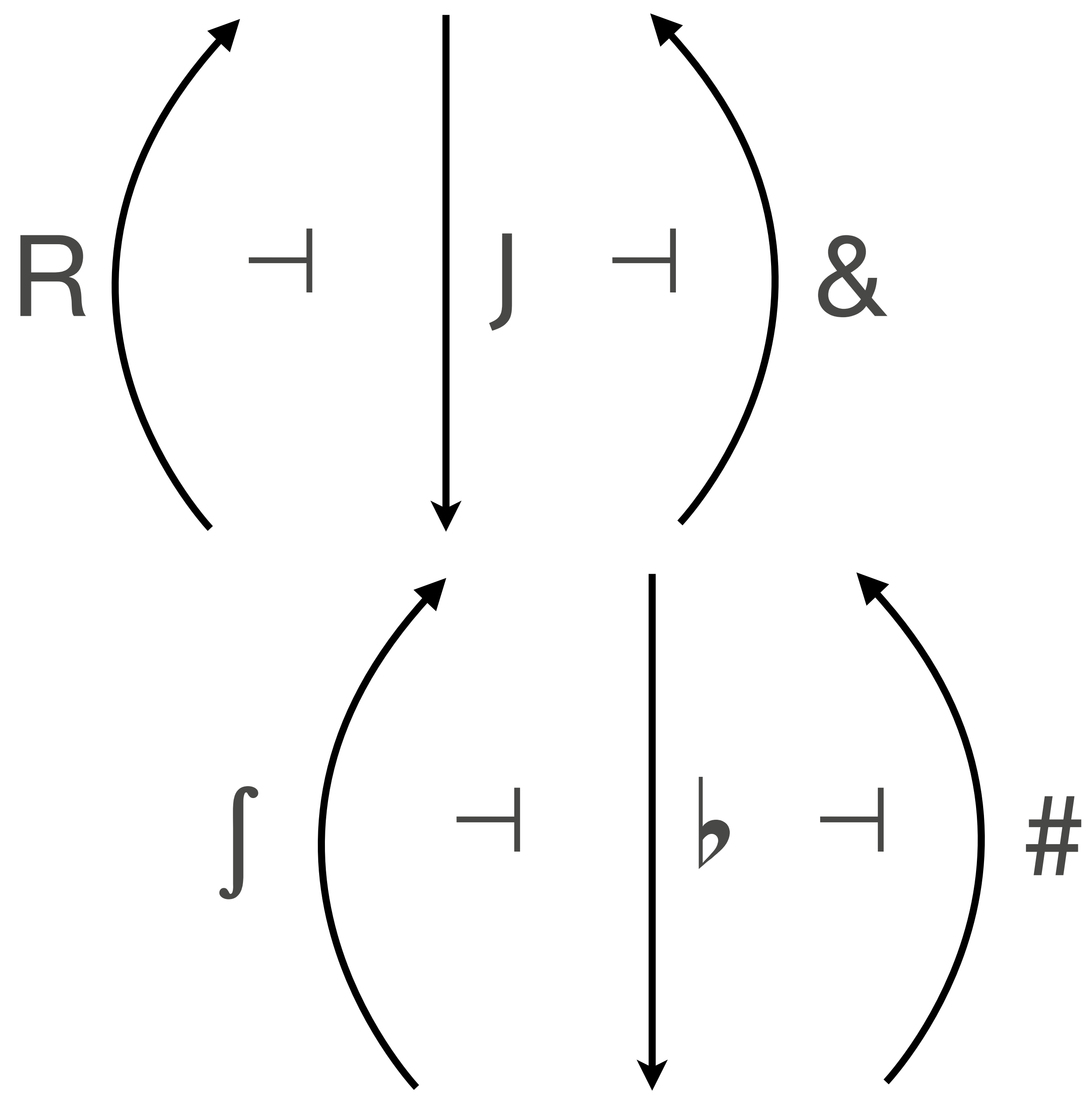

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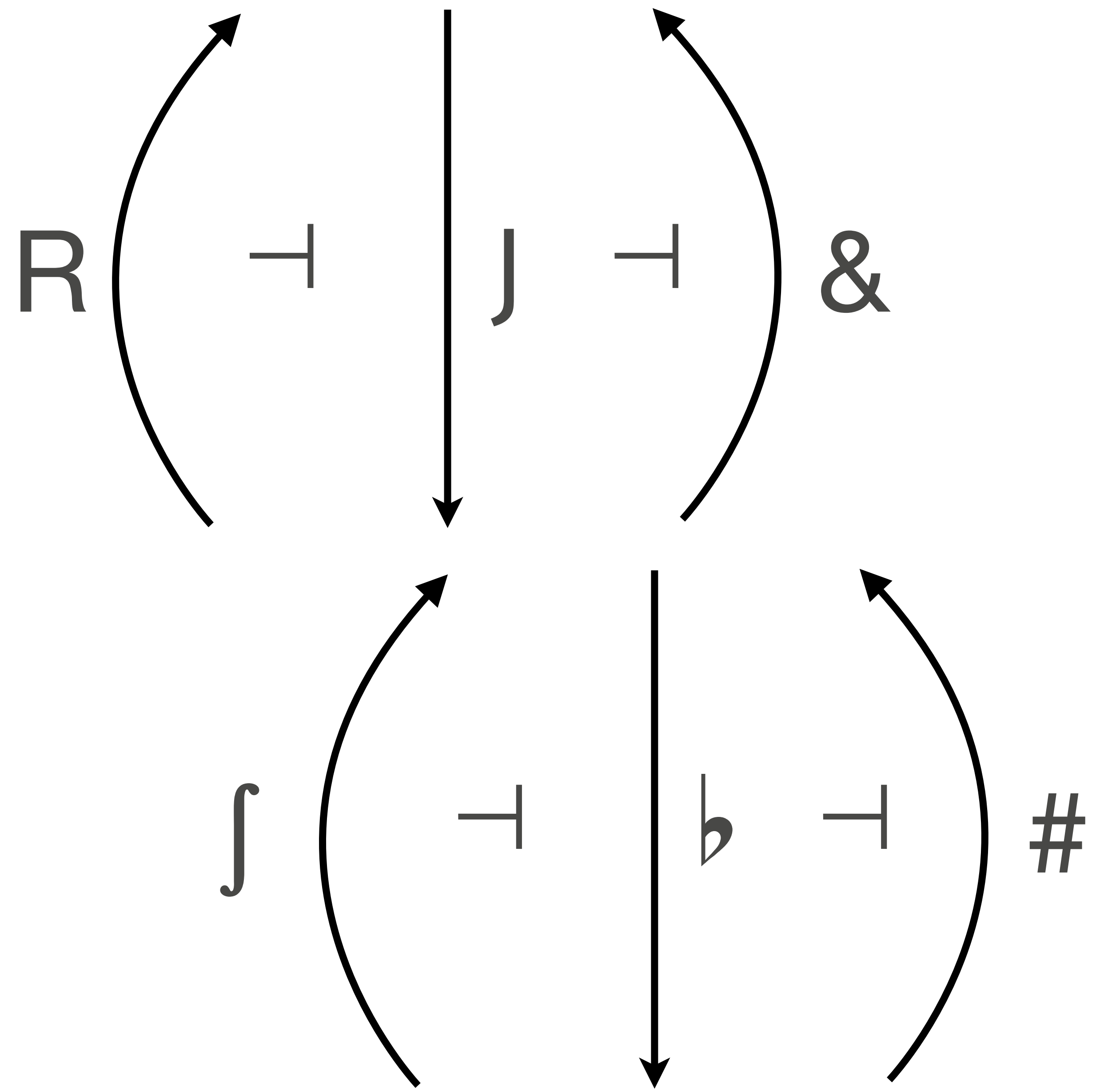

HoTT/UF Saturday!

What are the common patterns in substructural and modal logics?

S4 □

$$\boxed{\Gamma; \Delta \vdash C}$$

$$\frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C}$$

$$\frac{\Gamma\ ;\ \cdot\ \vdash A}{\Gamma\ ; \Delta \vdash \Box A} \qquad \frac{\Gamma, A\ ; \Delta, \Box A \vdash C}{\Gamma; \Delta, \Box A \vdash C}$$

S4 □

$$\boxed{\Gamma; \Delta \vdash C}$$

morally □Γ× Δ → C

$$\frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C}$$

$$\frac{\Gamma\,;\ \cdot\ \vdash A}{\Gamma\,;\ \Delta \vdash \Box A} \qquad \frac{\Gamma, A\,;\ \Delta, \Box A \vdash C}{\Gamma; \Delta, \Box A \vdash C}$$

S4 $\Box$

$$\boxed{\Gamma; \Delta \vdash C}$$

morally $\Box\Gamma \times \Delta \to C$

$$\frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C}$$

$$\frac{\Gamma ; \cdot \vdash A}{\Gamma ; \Delta \vdash \Box A} \qquad \frac{\Gamma, A ; \Delta, \Box A \vdash C}{\Gamma; \Delta, \Box A \vdash C}$$

context is all
boxed formulae
(up to weakening)

S4 □

$$\boxed{\Gamma; \Delta \vdash C}$$

morally □Γ× Δ → C

$$\frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C}$$

because □A → A

$$\frac{\Gamma ; \cdot \vdash A}{\Gamma ; \Delta \vdash \Box A}$$

context is all boxed formulae (up to weakening)

$$\frac{\Gamma, A ; \Delta, \Box A \vdash C}{\Gamma; \Delta, \Box A \vdash C}$$

Linear Logic !

cartesian/ structural → Γ ; **linear** → Δ

$$\frac{\Gamma, A;\ \Delta, A \vdash C}{\Gamma, A;\ \Delta \vdash C}$$

$$\frac{\Gamma\ ;\ \cdot\ \vdash A}{\Gamma\ ;\ \cdot\ \vdash\ !A} \qquad \frac{\Gamma, A\ ;\ \Delta \vdash C}{\Gamma;\ \Delta, !A \vdash C}$$

context is all !'ed formulae (no weakening)

Linear Logic ⊗

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

Linear Logic ⊗

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

context "is" a ⊗

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

Linear Logic ⊗

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

context "is" a ⊗

$$\frac{\Gamma_0 \equiv \Gamma, \Delta \qquad \Gamma \vdash A \qquad \Delta \vdash B}{\Gamma_0 \vdash A \otimes B}$$

context is a ⊗, up to some structural rules

Types inherit properties

$$\dfrac{\dfrac{A,B \equiv B,A \qquad B \vdash B \quad A \vdash A}{A,B \vdash B \otimes A}}{A \otimes B \vdash B \otimes A}$$

Types inherit properties

$$\dfrac{\dfrac{\dfrac{\dfrac{A,B;\ \cdot \vdash A \otimes B}{A,B\ ;\ \cdot \vdash\ !(A \otimes B)}}{A\ ;\ !B \vdash\ !(A \otimes B)}}{\cdot\ ;\ !A,\ !B \vdash\ !(A \otimes B)}}{\cdot\ ;\ !A \otimes !B \vdash\ !(A \otimes B)}$$

Pattern for □ ! ⊗

- Operation on contexts, with explicit or admissible structural properties
- Type constructor that "internalizes" the context operation, inherits the structural properties

This paper

A framework that abstracts the common aspects of many intuitionistic substructural and modal logics

- Products and left-adjoints (⊗,F) all one connective
- Negatives and right adjoints (⊸,U) another
- Cut elimination for all instances at once
- Equational theory: differ by structural rules
- Categorical semantics

Examples

- Non-assoc, ordered, linear, affine, relevant, cartesian, bunched products and implications
- N-linear variables [Reed,Abel,McBride]
- Monoidal, lax, non- left adjoints
- Non-strong, strong, $\Box$-strong monads
- Cohesion

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Logical adequacy: sequent is provable iff its encoding is

Closely Related Work

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- Display logic, Lambek calculus, resource semantics

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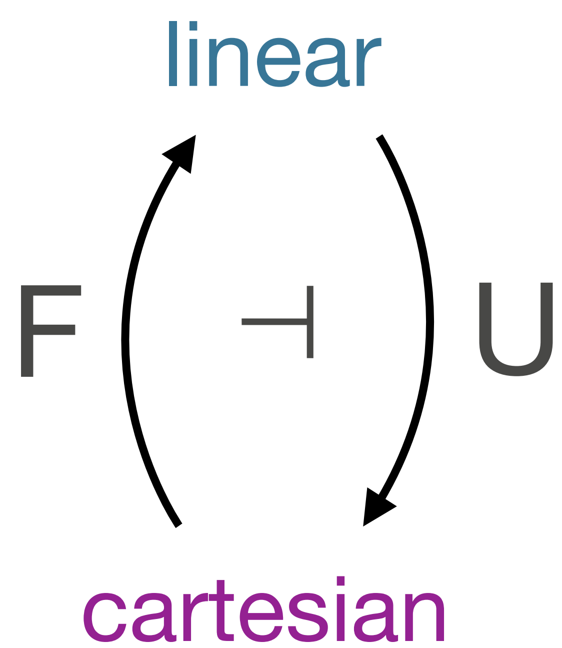

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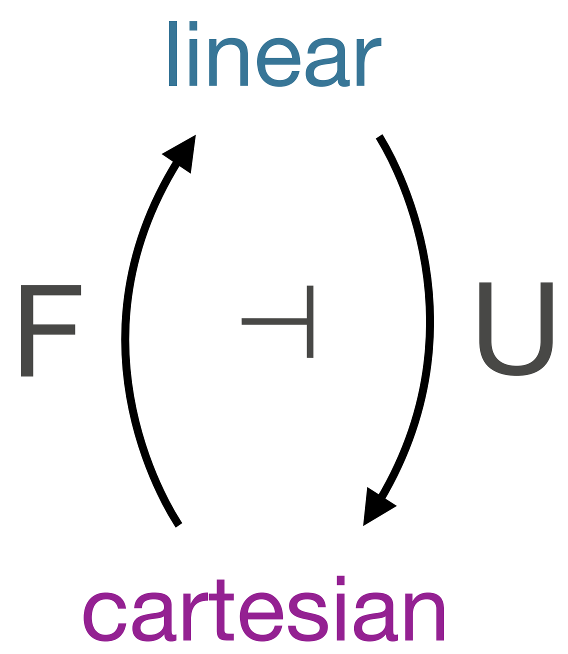

$$A ::= F\ C \mid A \otimes B \mid \ldots$$

$$C ::= U\ A \mid C \times D \mid \ldots$$

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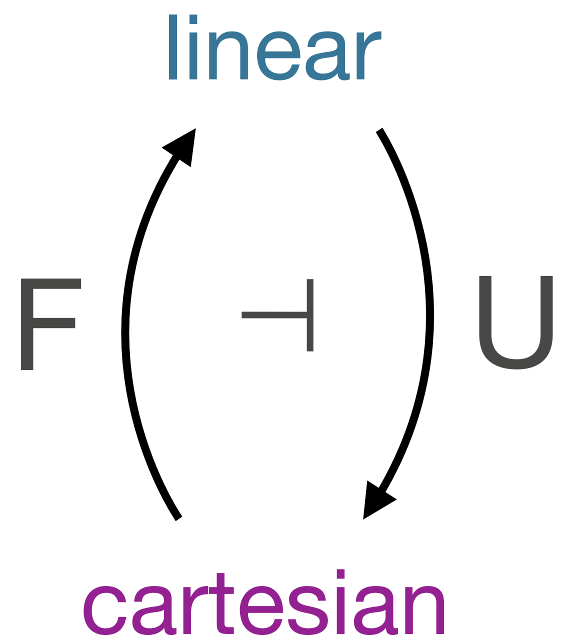

$$A ::= F\ C \mid A \otimes B \mid \ldots$$

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$$!A := FU\ A$$

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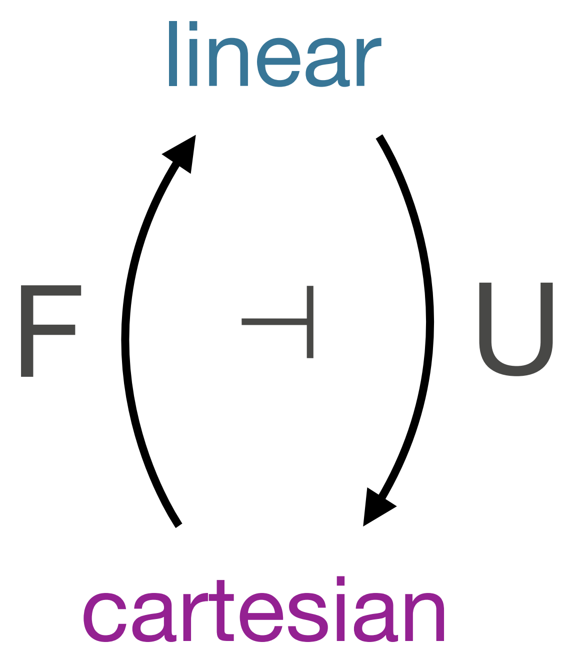

$$A ::= F\,C \mid A \otimes B \mid \ldots$$

$$C ::= U\,A \mid C \times D \mid \ldots$$

$$!A := FU\,A$$

- λ-calculus for Resource Separation [Atkey,2004]
- Adjoint logic [Reed,2009]

Technique

A substructural/modal typing judgement is an ordinary structural judgement, annotated with a **term** that describes the tree structure of the context

Sequent	$\Gamma \vdash_{\alpha} A$
Context descriptor	$\psi \vdash \alpha : p$

Sequent Calculus

$$\frac{x : P \in \Gamma \quad \beta \Rightarrow x}{\Gamma \vdash_\beta P}\ \mathsf{v} \qquad \frac{\Gamma, \Gamma', \Delta \vdash_{\beta[\alpha/x]} C}{\Gamma, x : \mathsf{F}_\alpha(\Delta), \Gamma' \vdash_\beta C}\ \mathsf{FL} \qquad \frac{\beta \Rightarrow \alpha[\gamma] \quad \Gamma \vdash_\gamma \Delta}{\Gamma \vdash_\beta \mathsf{F}_\alpha(\Delta)}\ \mathsf{FR}$$

$$\frac{x : \mathsf{U}_{x.\alpha}(\Delta \mid A) \in \Gamma \quad \beta \Rightarrow \beta'[\alpha[\gamma]/z] \quad \Gamma \vdash_\gamma \Delta \quad \Gamma, z : A \vdash_{\beta'} C}{\Gamma \vdash_\beta C}\ \mathsf{UL} \qquad \frac{\Gamma, \Delta \vdash_{\alpha[\beta/x]} A}{\Gamma \vdash_\beta \mathsf{U}_{x.\alpha}(\Delta \mid A)}\ \mathsf{UR}$$

Mode Theory

Modes	p,q,…
Context Descriptors	$x_1{:}p_1, \ldots, x_n{:}p_n \vdash \alpha : q$
Structural Properties	$\alpha \Rightarrow \beta$

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- Types p,q are “modes” of types/contexts

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- “Transformations” $\alpha \Rightarrow \beta$ are structural properties

Mode Theory

Modes	p,q,…	
Context Descriptors	$x_1{:}p_1, \ldots, x_n{:}p_n \vdash \alpha : q$	cartesian/ structural
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- Types p,q are "modes" of types/contexts
- Terms α are descriptions of the context
- "Transformations" $\alpha \Rightarrow \beta$ are structural properties

$$a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X$$

$$a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X$$

- Non-associative logic: no equations/transformations

$$a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X$$

- Non-associative logic: no equations/transformations
- Ordered logic: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$

$$a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X$$

- Non-associative logic: no equations/transformations
- Ordered logic: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
- Linear logic: $a \otimes b = b \otimes a$

$$a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X$$

- Non-associative logic: no equations/transformations
- Ordered logic: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
- Linear logic: $a \otimes b = b \otimes a$
- Relevant logic: $a \Longrightarrow a \otimes a$

$$a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X$$

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$$a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X$$

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- Relevant logic: $a \Longrightarrow a \otimes a$
- Affine logic: $a \Longrightarrow 1$
- BI: two function symbols $\ast$ and $\wedge$: $(a \ast b) \wedge (c \ast d)$

$a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X$

- Non-associative logic: no equations/transformations
- Ordered logic: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
- Linear logic: $a \otimes b = b \otimes a$
- Relevant logic: $a \Longrightarrow a \otimes a$
- Affine logic: $a \Longrightarrow 1$
- BI: two function symbols $\ast$ and $\wedge$: $(a \ast b) \wedge (c \ast d)$
- Modalities: unary function symbols: $\mathbf{r}(a) \otimes \mathbf{r}(b) \otimes c \otimes d$

cartesian

$$a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X$$

- Non-associative logic: no equations/transformations
- Ordered logic: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
- Linear logic: $a \otimes b = b \otimes a$
- Relevant logic: $a \Longrightarrow a \otimes a$
- Affine logic: $a \Longrightarrow 1$
- BI: two function symbols $*$ and $\wedge$: $(a * b) \wedge (c * d)$
- Modalities: unary function symbols: $\mathbf{r}(a) \otimes \mathbf{r}(b) \otimes c \otimes d$

Weakening over weakening

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$$\frac{\Gamma \vdash_\alpha B}{\Gamma, x{:}A \vdash_\alpha B}$$

Weakening over weakening

$$\frac{\Gamma \vdash_{\alpha} B}{\Gamma, x{:}A \vdash_{\alpha} B}$$

$$\frac{a{:}A, b{:}B, c{:}C, d{:}D \vdash_{(a \otimes b) \otimes (c \otimes d)} X}{a{:}A, b{:}B, c{:}C, d{:}D, e{:}E \vdash_{(a \otimes b) \otimes (c \otimes d)} X}$$

Structural Rules (Admiss)

$$\frac{\Gamma \vdash_{\alpha} B}{\Gamma, x{:}A \vdash_{\alpha} B}$$

Structural Rules (Admiss)

$$\frac{\Gamma \vdash_\alpha B}{\Gamma, x{:}A \vdash_\alpha B} \qquad \frac{\Gamma, y{:}B, x{:}A \vdash_\alpha B}{\Gamma, x{:}A, y{:}B \vdash_\alpha C}$$

Structural Rules (Admiss)

$$\frac{\Gamma \vdash_{\alpha} A \qquad \Gamma, x{:}A \vdash_{\beta} B}{\Gamma \vdash_{\beta[\alpha/x]} B}$$

$$\frac{\Gamma \vdash_{\alpha} B}{\Gamma, x{:}A \vdash_{\alpha} B} \qquad \frac{\Gamma, y{:}B, x{:}A \vdash_{\alpha} B}{\Gamma, x{:}A, y{:}B \vdash_{\alpha} C}$$

Structural Rules (Admiss)

$$\frac{\Gamma \vdash_{\alpha} A \qquad \Gamma, x{:}A \vdash_{\beta} B}{\Gamma \vdash_{\beta[\alpha/x]} B}$$

$$\frac{}{\Gamma, x{:}A \vdash_{x} A}$$

$$\frac{\Gamma \vdash_{\alpha} B}{\Gamma, x{:}A \vdash_{\alpha} B}$$

$$\frac{\Gamma, y{:}B, x{:}A \vdash_{\alpha} B}{\Gamma, x{:}A, y{:}B \vdash_{\alpha} C}$$

Structural Rules (Admiss)

$$\frac{\Gamma \vdash_\alpha A \qquad \Gamma, x{:}A \vdash_\beta B}{\Gamma \vdash_{\beta[\alpha/x]} B}$$

$$\frac{}{\Gamma, x{:}A \vdash_x A}$$

$$\frac{\Gamma \vdash_\alpha B}{\Gamma, x{:}A \vdash_\alpha B}$$

$$\frac{\Gamma, y{:}B, x{:}A \vdash_\alpha B}{\Gamma, x{:}A, y{:}B \vdash_\alpha C}$$

Hypothesis

$$\frac{\beta \Rrightarrow x \qquad x : P \in \Gamma}{\Gamma \vdash_{\beta} P}$$

Hypothesis

up to whatever structural properties you've asserted, the context is just x (typically x⊗y ⇒ x)

$$\frac{\beta \Rightarrow x \qquad x : P \in \Gamma}{\Gamma \vdash_{\beta} P}$$

F types

$$F_\alpha \ (x_1{:}A_1, \ldots, x_n{:}A_n)$$

F types

the “product” of $A_1 \ldots A_n$ structured according to α

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e.g. $A \otimes B := F_{(x \otimes y)}\,(x{:}A,\ y{:}B)$

F types

the "product" of $A_1 \ldots A_n$ structured according to α

$$F_\alpha\ (x_1{:}A_1, \ldots, x_n{:}A_n)$$

e.g. $A \otimes B := F_{(x \otimes y)}\ (x{:}A,\ y{:}B)$

$\flat A := F_{(r\ x)}\ (x{:}A)$

F Left

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C}$$

$$\frac{\Gamma, A \,;\, \Delta \vdash C}{\Gamma ;\, \Delta, !A \vdash C}$$

$$\frac{\Gamma, A \,;\, \Delta, \Box A \vdash C}{\Gamma ;\, \Delta, \Box A \vdash C}$$

$$\frac{\Gamma[A, B] \vdash C}{\Gamma[A * B] \vdash C}$$

F Left

$$\frac{\Gamma, \Delta \vdash_{\beta[\alpha/x]} B}{\Gamma,\ x{:}F_{\alpha}(\Delta) \vdash_{\beta} B}$$

F Left

remember where in the tree Δ variables occur

$$\frac{\Gamma, \Delta \vdash_{\beta[\alpha/x]} B}{\Gamma,\ x{:}F_{\alpha}(\Delta) \vdash_{\beta} B}$$

F Right

$$\frac{\Gamma_0 \equiv \Gamma, \Delta \qquad \Gamma \vdash A \qquad \Delta \vdash B}{\Gamma_0 \vdash A \otimes B}$$

$$\frac{\Gamma_0 \equiv (\Gamma; \cdot) \qquad \Gamma ; \cdot \vdash A}{\Gamma_0 \vdash !A}$$

F Right

$$\frac{\beta \Rightarrow \alpha[\gamma] \qquad \Gamma \vdash_{\gamma} \Delta}{\Gamma \vdash_{\beta} F_{\alpha}\, \Delta}$$

Exchange

$$x{:}A \otimes B \vdash_x B \otimes A$$

Exchange

$$x : F_{(y \otimes z)}(y{:}A, z{:}B) \vdash_x F_{(z' \otimes y')}(z'{:}B, y'{:}A)$$

$$x{:}A \otimes B \vdash_x B \otimes A$$

Exchange

$$\frac{y{:}A, z{:}B \vdash_{y\otimes z} F_{(z'\otimes y')}(z'{:}B, y'{:}A)}{x : F_{(y\otimes z)}(y{:}A, z{:}B) \vdash_{x} F_{(z'\otimes y')}(z'{:}B, y'{:}A)}$$

$$x{:}A\otimes B \vdash_{x} B\otimes A$$

Exchange

$$\frac{\overline{\quad y{:}A, z{:}B \vdash_{y\otimes z} F_{(z'\otimes y')}(z'{:}B, y'{:}A)\quad}}{x : F_{(y\otimes z)}(y{:}A, z{:}B) \vdash_{x} F_{(z'\otimes y')}(z'{:}B, y'{:}A)}$$

$$x{:}A\otimes B \vdash_{x} B\otimes A$$

Exchange

$$y \otimes z \Rightarrow (z' \otimes y')[?]$$

$$\dfrac{\overline{\quad y{:}A, z{:}B \vdash_{y \otimes z} F_{(z' \otimes y')}(z'{:}B, y'{:}A) \quad}}{x : F_{(y \otimes z)}(y{:}A, z{:}B) \vdash_{x} F_{(z' \otimes y')}(z'{:}B, y'{:}A)}$$

$$x{:}A \otimes B \vdash_{x} B \otimes A$$

Exchange

$$\dfrac{\overline{\quad y{:}A, z{:}B \vdash_{y \otimes z} F_{(z' \otimes y')}(z'{:}B, y'{:}A) \quad}}{x : F_{(y \otimes z)}(y{:}A, z{:}B) \vdash_x F_{(z' \otimes y')}(z'{:}B, y'{:}A)}$$

$$x{:}A \otimes B \vdash_x B \otimes A$$

Exchange

$$y \otimes z \Rightarrow (z' \otimes y')[z/z', y/y']$$

$$\dfrac{\dfrac{}{y{:}A, z{:}B \vdash_{y \otimes z} F_{(z' \otimes y')}(z'{:}B, y'{:}A)}}{x : F_{(y \otimes z)}(y{:}A, z{:}B) \vdash_x F_{(z' \otimes y')}(z'{:}B, y'{:}A)}$$

$$x{:}A \otimes B \vdash_x B \otimes A$$

Exchange

$$\dfrac{\overline{y{:}A, z{:}B \vdash_{y \otimes z} F_{(z' \otimes y')}(z'{:}B, y'{:}A)}}{x : F_{(y \otimes z)}(y{:}A, z{:}B) \vdash_x F_{(z' \otimes y')}(z'{:}B, y'{:}A)}$$

$$x{:}A \otimes B \vdash_x B \otimes A$$

Exchange

$$y \otimes z \Rightarrow z \otimes y$$

$$\dfrac{\dfrac{}{y{:}A, z{:}B \vdash_{y \otimes z} F_{(z' \otimes y')}(z'{:}B, y'{:}A)}}{x : F_{(y \otimes z)}(y{:}A, z{:}B) \vdash_{x} F_{(z' \otimes y')}(z'{:}B, y'{:}A)}$$

$$x{:}A \otimes B \vdash_{x} B \otimes A$$

Exchange

$$y \otimes z \Rightarrow z \otimes y$$

$$\dfrac{\dfrac{y{:}A \vdash_{y} A}{y{:}A, z{:}B \vdash_{y \otimes z} F_{(z' \otimes y')}(z'{:}B, y'{:}A)}}{x : F_{(y \otimes z)}(y{:}A, z{:}B) \vdash_{x} F_{(z' \otimes y')}(z'{:}B, y'{:}A)}$$

$$x{:}A \otimes B \vdash_{x} B \otimes A$$

Exchange

$$y \otimes z \Rightarrow z \otimes y$$

$$z{:}B \vdash_z B$$

$$y{:}A \vdash_y A$$

$$\frac{\dfrac{y{:}A,z{:}B \vdash_{y\otimes z} F_{(z'\otimes y')}(z'{:}B,y'{:}A)}{}}{x : F_{(y\otimes z)}(y{:}A,z{:}B) \vdash_x F_{(z'\otimes y')}(z'{:}B,y'{:}A)}$$

$$x{:}A\otimes B \vdash_x B\otimes A$$

Right Adjoints

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{\Delta \vdash A \quad \Gamma[B] \vdash C}{\Gamma[A \multimap B, \Delta] \vdash C}$$

$$\frac{\Gamma; \cdot \vdash_{\ell} A}{\Gamma \vdash_{c} U\,A} \qquad \frac{\Gamma; \Delta, A \vdash_{\ell} B}{\Gamma, U\,A; \Delta \vdash_{\ell} B}$$

Right Adjoints

$$\frac{\phi, c{:}q \vdash \alpha : p \qquad \Delta\ \mathrm{ctx}_{\phi} \qquad A\ \mathrm{type}_{p}}{U_{c.\alpha}(\Delta | A)\ \mathrm{type}_{q}}$$

$$A \multimap B := U_{c.(c \otimes y)}(y{:}A | B)$$

Right Adjoints

$$\frac{\phi, c{:}q \vdash \alpha : p \qquad \Delta\ \mathrm{ctx}_{\phi} \qquad A\ \mathrm{type}_p}{U_{c.\alpha}(\Delta | A)\ \mathrm{type}_q}$$

$$A \multimap B := U_{c.(c \otimes y)}(y{:}A | B)$$

$$A \setminus B := U_{c.(y \otimes c)}(y{:}A | B)$$

$$A / B := U_{c.(c \otimes y)}(y{:}A | B)$$

Right Adjoints

$$\frac{\phi, c{:}q \vdash \alpha : p \qquad \Delta\ \mathrm{ctx}_{\phi} \qquad A\ \mathrm{type}_{p}}{U_{c.\alpha}(\Delta | A)\ \mathrm{type}_{q}}$$

$$\frac{\Gamma, \Delta \vdash_{\alpha[\beta/c]} A}{\Gamma \vdash_{\beta} U_{c.\alpha}(\Delta | A)}$$

Right Adjoints

$$\dfrac{}{\dfrac{}{x{:}X \vdash_x U_{c\,\otimes\,a}(a{:}A|Y)}}$$

$$A \multimap Y$$

Right Adjoints

$$\frac{x{:}X, a{:}A \vdash_{x \otimes a} Y}{x{:}X \vdash_{x} U_{c \otimes a}(a{:}A | Y)}$$

$$A \multimap Y$$

Right Adjoints

$$
\begin{array}{c}
z{:}F_{x \otimes a}(x{:}X, a{:}A) \vdash_z Y \\
\hline\hline
x{:}X, a{:}A \vdash_{x \otimes a} Y \\
\hline\hline
x{:}X \vdash_x U_{c \otimes a}(a{:}A \mid Y)
\end{array}
$$

$$A \multimap Y$$

Right Adjoints

$$X \otimes A$$

$$\frac{\frac{z{:}F_{x \otimes a}(x{:}X, a{:}A) \vdash_z Y}{x{:}X, a{:}A \vdash_{x \otimes a} Y}}{x{:}X \vdash_x U_{c \otimes a}(a{:}A \mid Y)}$$

$$A \multimap Y$$

Examples

- Non-assoc, ordered, linear, affine, relevant, cartesian, bunched products and implications
- N-linear variables [Reed,Abel,McBride]
- Monoidal, lax, non- left adjoints
- Non-strong, strong, $\Box$-strong monads
- Spatial type theory

Logical adequacy: sequent is provable iff its encoding is

Bifibrations

cartesian 2-multicategories

D derivations

$\downarrow \pi$

M modes

- locally discrete fibration (action of structural rules)
- $F_\alpha \Delta$ makes this into an opfibration
- $U_\alpha(\Delta|A)$ makes this into a fibration

Sound/Complete: Syntax forms a bifibration and can be interpreted in any

Equational Theory

- βη for F and U
- equations governing action of 2-cells: when do two terms differ by placement of structural properties?

$$\frac{A,B \equiv B,A \quad B \vdash B \quad A \vdash A}{A,B \vdash B \times A}$$

$$\frac{A,B \equiv A,B,A,B \quad A,B \vdash B \quad A,B \vdash A}{A,B \vdash B \times A}$$

This paper

A framework that abstracts the common aspects of many intuitionistic substructural and modal logics

- Products and left-adjoints ($\otimes$,F) all one connective
- Negatives and right adjoints ($\multimap$,U) another
- Cut elimination for all instances at once
- Equational theory: differ by structural rules
- Categorical semantics