

# COMP 212 Spring 2022

## Lab 4

The goal for this lab is to practice asymptotic analysis.

### 1 Analysis

#### 1.1 Simple Fibonacci

The *Fibonacci sequence* is a sequence of numbers defined by the rule that the next number in the sequence is the sum of the previous two:

1, 1, 2, 3, 5, 8, 13, 21, ...

This can be implemented by a recursive function with two recursive calls:

```
fun fib (n : int) : int =
  case n of
    ~1 => 0
  | 0 => 1
  | 1 => 1
  | _ => fib (n - 1) + fib (n - 2)
```

We're going to add the harmless but slightly strange base case defining the negative first element of the sequence to the definition as well; you'll see why later in lab, but just go with it for now.

Like `evenP` in lab last week, this function has *three* useful cases—zero one and  $2 + n$  (we won't count the negative one case). The new thing about this function is that it makes recursive calls not just on  $n - 2$  but also on  $n - 1$ .

Because of these two recursive calls, the recurrence for the work looks like this:

$$\begin{aligned}W_{\text{fib}}(0) &= k_0 \\W_{\text{fib}}(1) &= k_1 \\W_{\text{fib}}(n) &= k_2 + W_{\text{fib}}(n - 1) + W_{\text{fib}}(n - 2) \text{ for non-zero } n\end{aligned}$$

This is not so helpful, since it says that the time to compute the  $n^{\text{th}}$  Fibonacci  $n$  is the  $n^{\text{th}}$  Fibonacci number!

However, if we can get an *upper bound* for this recurrence as follows:

$$\begin{aligned} W_{\text{fib}}(0) &= k_0 \\ W_{\text{fib}}(1) &= k_1 \\ W_{\text{fib}}(n) &\leq k_2 + 2W_{\text{fib}}(n-1) \text{ for non-zero } n \end{aligned}$$

Because  $W_{\text{fib}}(n)$  is *monotonically increasing* (it's never smaller on bigger inputs), we can pretend that it's two recursive calls on  $n-1$ .

If you write it out, you can see that the closed form of this recurrence is

$$W_{\text{fib}}(n) = k_0 + k_1 + k_2 * 2^{n+1} - 1$$

To see this, you can write the recursion out as a tree. `fib` does `k2` work at each recursive call, so we can label each node with `k2`. Each node has two children, because each call makes two recursive calls.

```

      k2
     /  \
    k2  k2
   /  \ /  \
  k2 k2 k2 k2
  ...

```

The  $k_2$  is uniform, so factor it out

```

      1
     /  \
    1   1
   /  \ /  \
  1  1 1  1  4
  ...

```

We want to count the number of nodes in this tree. The total has the form  $1+2+4+8+16+\dots$ . The reason is that the tree has  $n$  levels, because the recurrence recurs on  $n-1$ , and the  $i^{\text{th}}$  level has  $2^i$  work. Thus, the total amount of work is

$$\sum_{i=1}^n 2^i$$

If you look it up, the closed form of this sum is  $2^{n+1} - 1$  (cf. how many binary numbers are there with  $n$  bits).

Once you've written out the closed form, it's clear that this recurrence is  $O(2^n)$ , just by forgetting the constants.

## 1.2 Fast Fibonacci

In this problem, you will show that you can compute Fibonacci more efficiently. The key insight is that one of the recursive calls can be reused each time:

To compute	We need
<code>fib n</code>	<code>fib (n-1)</code> and <code>fib (n-2)</code>
<code>fib (n-1)</code>	<code>fib (n-2)</code> and <code>fib (n-3)</code>
<code>fib (n-2)</code>	<code>fib (n-3)</code> and <code>fib (n-4)</code>

So we really don't need two recursive calls, if we reuse the same computation of `fib n` the two times we use it. To implement this, you must *generalize* the problem so that we compute both `fib n` and `fib (n-1)`.

### 1.2.1 Programming

**Task 1.1** Implement a function

```
fastfib : int -> int * int
```

such that for all nats  $n$ ,  $\text{fastfib } n \cong (\text{fib}(n - 1), \text{fib } n)$

**Have the course staff check your code for fastfib before proceeding.**

### 1.2.2 Analysis

**Task 1.2** Write a recurrence for the work of fastfib,

$W_{\text{fastfib}}$ .

**Task 1.3** Compute and informally justify the closed form for your recurrence.

**Task 1.4** Give a tight big- $O$  bound for this closed form.

### 1.2.3 Proof

**Task 1.5** Prove that your code meets the spec, which is to say:

**Theorem 1.** *For all natural numbers  $n$ ,  $\text{fastfib } n \cong (\text{fib } (n - 1) , \text{fib } n)$*

Use the template on the following page. Have the course staff check your work once you finish.

**Theorem 2.** *For all natural numbers  $n$ ,  $\text{fastfib } n \cong (\text{fib } (n - 1) , \text{fib } n)$*

The proof is by induction on  $m$ .

- **Case for 0**

To show:

Proof:

- **Case for  $1 + k$**

Inductive hypothesis:

To show:

Proof:

Have the course staff check your code, analysis, and proof for `fastfib` before proceeding.

## 2 Merge

**Task 2.1** Write a function

```
merge : int list * int list -> int list
```

that merges two sorted lists into one sorted list. You should assume that your input lists are sorted in increasing order, and the list you return should also be in increasing order.

**Task 2.2** Write a recurrence relation for the work of `merge`, in terms of the lengths of `l1` and `l2`. What is the  $O$  of this recurrence?

**Task 2.3** Prove the following correctness theorem about `merge`:

**Theorem 3.** *For all lists of integers `l1` and `l2`, if `l1` and `l2` are both sorted in increasing order, then `merge (l1, l2)` is sorted in increasing order.*

*Hint:* In your proof, you will need a lemma about how the contents of `merge(l1,l2)` relates to `l1` and `l2`. You should state this lemma, and convince yourself it is true, but you don't need to prove it formally.