

COMP 321 Lecture 12: Substitution/Products Lab

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Fall 2026

1 Substitution

Consider a language with the following expressions and typing rules:

$$e ::= x \mid \text{Num}(k) \mid \text{Plus}(e_1, e_2) \mid \text{Let}(e_1, x.e_2)$$

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \quad \frac{}{\Gamma \vdash \text{Num}(k) : \text{int}} \quad \frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash \text{Plus}(e_1, e_2) : \text{int}} \quad \frac{\Gamma \vdash e : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{Let}(e_1, x.e_2) : \tau_2}$$

Consider the following definition of substitution (like your `subst_exp` from Homework 3):

$$\begin{aligned} x_0[e_0/x_0] &= e_0 \\ y[e_0/x_0] &= y \text{ if } y \neq x_0 \\ \text{Num}(k)[e_0/x_0] &= \text{Num}(k) \\ \text{Plus}(e_1, e_2)[e_0/x_0] &= \text{Plus}(e_1[e_0/x_0], e_2[e_0/x_0]) \\ \text{Let}(e_1, y.e_2)[e_0/x_0] &= \text{Let}(e_1[e_0/x_0], y.e_2[e_0/x_0]) \text{ if } x_0 \neq y \end{aligned}$$

Task 1. Prove the substitution lemma:

For all $\Gamma, \Gamma', x_0, e, e_0, \tau_0$ and τ , if $\Gamma, x_0 : \tau_0, \Gamma' \vdash e : \tau$ and $\Gamma \vdash e_0 : \tau_0$ then $\Gamma, \Gamma' \vdash e[e_0/x_0] : \tau$.

The proof is by induction on $\Gamma, x_0 : \tau_0, \Gamma' \vdash e : \tau$, so you should start with one case for each typing rule:

- Case for

$$\frac{x : \tau \in (\Gamma, x_0 : \tau_0, \Gamma')}{\Gamma, x_0 : \tau_0, \Gamma' \vdash x : \tau}$$

We need to show that $\Gamma, \Gamma' \vdash x[e_0/x_0] : \tau$. There are two cases depending on whether $x = x_0$ or not.

In the first case, if $x = x_0$, then $\tau = \tau_0$. This is because $x : \tau \in (\Gamma, x_0 : \tau_0, \Gamma')$ means that the x entry in $(\Gamma, x_0 : \tau_0, \Gamma')$ contains the type τ , but we assume that variables occur at most once in a context, so the x_0 entry containing τ implies $\tau = \tau_0$. So we need to show that $\Gamma, \Gamma' \vdash x_0[e_0/x_0] : \tau_0$. By the above definition of substitution, this means showing that $\Gamma, \Gamma' \vdash e_0 : \tau_0$. This is almost the assumption that $\Gamma \vdash e_0 : \tau_0$, except we add some extra variables Γ' to the context. Technically we would need to prove a weakening lemma:

For all Γ' , if $\Gamma \vdash e : \tau$ then $\Gamma, \Gamma' \vdash e : \tau$.

This is proved by induction on $\Gamma \vdash e : \tau$ in a manner similar to the proof we are doing now.

In the second case where $x \neq x_0$, we need to show that $\Gamma, \Gamma' \vdash x[e_0/x_0] : \tau$. By definition, this is $\Gamma, \Gamma' \vdash x : \tau$. Since $x \neq x_0$ and $x : \tau \in (\Gamma, x_0 : \tau_0, \Gamma')$, we still have $x : \tau \in (\Gamma, \Gamma')$ (x is not the variable that was removed), so we can derive

$$\frac{x : \tau \in (\Gamma, \Gamma')}{\Gamma, \Gamma' \vdash x : \tau}$$

- Case for

$$\overline{\Gamma, x_0 : \tau_0, \Gamma' \vdash \text{Num}(k) : \text{int}}$$

We need to show that $\Gamma, \Gamma' \vdash \text{Num}(k) : \text{int}$, which is true by the typing rule for Num.

- Case for

$$\frac{\Gamma, x_0 : \tau_0, \Gamma' \vdash e_1 : \text{int} \quad \Gamma, x_0 : \tau_0, \Gamma' \vdash e_2 : \text{int}}{\Gamma, x_0 : \tau_0, \Gamma' \vdash \text{Plus}(e_1, e_2) : \text{int}}$$

We need to show that $\Gamma, \Gamma' \vdash (\text{Plus}(e_1, e_2))[e_0/x_0] : \text{int}$.

By the inductive hypotheses, the substitution into the arguments preserves their types, so we have $\Gamma, \Gamma' \vdash e_1[e_0/x_0] : \text{int}$ and $\Gamma, \Gamma' \vdash e_2[e_0/x_0] : \text{int}$.

We can apply the typing rule for Plus to derive

$$\frac{\Gamma, \Gamma' \vdash e_1[e_0/x_0] : \text{int} \quad \Gamma, \Gamma' \vdash e_2[e_0/x_0] : \text{int}}{\Gamma, \Gamma' \vdash \text{Plus}(e_1[e_0/x_0], e_2[e_0/x_0]) : \text{int}}$$

By the definition of substitution,

$$(\text{Plus}(e_1, e_2))[e_0/x_0] = \text{Plus}(e_1[e_0/x_0], e_2[e_0/x_0])$$

so this is what we needed to show.

- Case for

$$\frac{\Gamma, x_0 : \tau_0, \Gamma' \vdash e_1 : \tau_1 \quad \Gamma, x_0 : \tau_0, \Gamma', x : \tau_1 \vdash e_2 : \tau_2}{\Gamma, x_0 : \tau_0, \Gamma' \vdash \text{Let}(e_1, x.e_2) : \tau_2}$$

We need to show that $\Gamma, \Gamma' \vdash (\text{Let}(e_1, x.e_2))[e_0/x_0] : \text{int}$. Without loss of generality, we can assume $x \neq x_0$, because otherwise we could rename the variable x and its occurrences in e_2 to some unused variable name.

By the inductive hypotheses, the substitution into the arguments preserves their types. First, we have $\Gamma, \Gamma' \vdash e_1[e_0/x_0] : \tau_1$ (since $\Gamma, x_0 : \tau_0, \Gamma' \vdash e_1 : \tau_1$ and $\Gamma \vdash e_0 : \tau_0$).

Next, we have $\Gamma, (\Gamma', x : \tau_1) \vdash e_2[e_0/x_0] : \tau_2$ (since $\Gamma, x_0 : \tau_0, \Gamma', x : \tau_1 \vdash e_2 : \tau_2$ and $\Gamma \vdash e_0 : \tau_0$). Here, we apply the inductive hypothesis with the universally quantified Γ' argument of the theorem taken to be $(\Gamma', x : \tau_1)$ (in terms of the Γ' assumed for the proof). This produces the $x : \tau_1$ in the output of the inductive hypothesis.

We can apply the typing rule for Let to derive

$$\frac{\Gamma, \Gamma' \vdash e_1[e_0/x_0] : \tau_1 \quad (\Gamma, \Gamma'), x : \tau_1 \vdash e_2[e_0/x_0] : \tau_2}{\Gamma, \Gamma' \vdash \text{Let}(e_1[e_0/x_0], e_2[e_0/x_0]) : \text{int}}$$

By the definition of substitution,

$$(\text{Let}(e_1, x.e_2))[e_0/x_0] = \text{Let}(e_1[e_0/x_0], x.e_2[e_0/x_0])$$

so this is what we needed to show.

State and prove any lemmas that you need.

Have us check your work!

Task 2. Do the case of the type preservation theorem

If $\cdot \vdash e : \tau$ and $e \mapsto e'$ then $\cdot \vdash e' : \tau$.

for the following rule

$$\frac{e_1 \text{ done}}{\text{Let}(e_1, x.e_2) \mapsto e_2[e_1/x]}$$

and show how the above substitution lemma is used.

Proof. Assume $\cdot \vdash \text{Let}(e_1, x.e_2) : \tau_2$. By inversion, $\cdot \vdash e_1 : \tau_1$ and $x : \tau_1 \vdash e_2 : \tau_2$.

Applying the substitution theorem with $\Gamma = \cdot$, $\Gamma' = \cdot$, $x_0 = x$, $e_0 = e_1$, and $e = e_2$, $\tau_0 = \tau_1$, and $\tau = \tau_2$, we have $x : \tau_1 \vdash e_2 : \tau_2$ and $\cdot \vdash e_1 : \tau_1$ so $\cdot \vdash e_2[e_1/x] : \tau_2$. $\square$